The Bounded L^2 curvature "conjecture" in general relativity

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Cauchy Problem for EE

$(\mathcal{M}, \mathbf{g})$ Lorentzian, $\mathbf{R}$ curvature tensor of $\mathbf{g}$

Einstein Vacuum equations: $\mathbf{Ric}_{\alpha\beta} = 0$

Wave coordinates: $\square_{\mathbf{g}} x^\alpha = \frac{1}{\sqrt{|\mathbf{g}|}} \partial_\beta (\mathbf{g}^{\beta\gamma} \sqrt{|\mathbf{g}|} \partial_\gamma) x^\alpha = 0, \alpha = 0, 1, 2, 3$

$\square_{\mathbf{g}} \mathbf{g}_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(\mathbf{g}, \partial \mathbf{g}), \alpha, \beta = 0, 1, 2, 3,$ with $\mathcal{N}_{\alpha\beta}$ quadratic w.r.t $\partial \mathbf{g}$

Cauchy data: (Σ_0, g_0, k) where $\Sigma_0 = \{t = 0\}$, $\mathbf{g}(0, .) = g_0$, $\partial_t \mathbf{g}(0, .) = k$

Question: Under which regularity do we have local existence for EE?

Semilinear Wave Equations

$$\begin{cases} \Box\phi = \mathcal{N}(\phi, \partial\phi),\ (t,x) \in \mathbb{R}^{1+3} \\ \phi(0,.) = \phi_0 \in H^s(\mathbb{R}^3), \partial_t\phi(0,.) = \phi_1 \in H^{s-1}(\mathbb{R}^3) \end{cases}$$

where $\mathcal{N}$ is quadratic w.r.t $\partial\phi$

$$\|(\phi(t), \partial_t\phi(t))\|_{H^s\times H^{s-1}} \lesssim \|(\phi_0, \phi_1)\|_{H^s\times H^{s-1}} \exp\left(\int_0^t \|\partial\phi(\tau)\|_{L^\infty} d\tau\right)$$

Sobolev embedding in $\mathbb{R}^3$: WP for $s > 5/2$

Strichartz for $\Box\phi = 0 \Rightarrow$ WP for $s > 2$ (Ponce-Sideris)

Ill-posed for $s = 2$ in general (Cex of Lindblad)

If $\mathcal{N} = Q_{ij}$ with $Q_{ij}(\phi, \psi) = \partial_i\phi\partial_j\psi - \partial_i\psi\partial_j\phi$

Bilinear estimates for $Q_{ij} \Rightarrow$ WP for $s > 3/2$ (Klainerman-Machedon)

Quasilinear Wave Equations

$$\begin{cases} \Box_{\mathbf{g}(\phi)}\phi = \mathcal{N}(\phi, \partial\phi),\ (t,x) \in \mathbb{R}^{1+3} \\ \phi(0,.) = \phi_0 \in H^s(\mathbb{R}^3), \partial_t\phi(0,.) = \phi_1 \in H^{s-1}(\mathbb{R}^3) \end{cases}$$

Using Sobolev embedding: WP for $s > 5/2$

Strichartz for $\Box_{\mathbf{g}}\phi = 0$ requires $\mathbf{g} \in C^{1,1}$ (Smith)

Strichartz with loss enough: WP for s>2+1/4 (Bahouri-Chemin)

WP for s>2 (Klainerman-Rodnianski for EE, Smith-Tataru for general quasilinear wave equations)

Interesting geometrical hyperbolic equations satisfy the null structure

Goal: prove that EE are WP in H^2

Bounded L^2 curvature "conjecture"

Conjecture. Let (Σ_0, g_0, k) *with* $R \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$. *Then, EE are WP*

Motivations:

- First WP result for a quasilinear wave equation below $H^{2+\epsilon}$
- The assumptions $R \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$ are natural from the point of view of geometry
- Rather than a WP result, it can be viewed as a breakdown criterion. In particular, $\mathbf{R} \in L^2$ is a fundamental quantity controlling singularity formation
- Sharp result with respect to a "null scaling": the control of the Eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ requires $\mathbf{R} \in L^2$

Strategy of the proof

A *Recast the EE as a quasilinear Yang-Mills theory*

B *Prove appropriate bilinear estimates for solutions to* $\square_{\mathbf{g}}\phi = 0$

C *Construct a parametrix for* $\square_{\mathbf{g}}\phi = 0$*, and obtain the control of the parametrix and of its error term*

D *Prove a sharp* $L^4(\mathcal{M})$ *Strichartz estimate for the parametrix*

Goal: take inspiration from the proof of Klainerman-Machedon for the WP of Yang-Mills in $H^1(\mathbb{R}^3)$

Achieve Steps B, C and D only assuming L^2 bounds on $\mathbf{R}$

WP of Yang-Mills in $H^1(\mathbb{R}^3)$ (Klainerman-Machedon)

$$\Box \mathbf{A} + \nabla_{t,x}(\nabla_{t,x} \cdot \mathbf{A}) = [\mathbf{A}, \nabla_{t,x}\mathbf{A}] + \mathbf{A}^3, \quad \mathbf{A} = (A_0, A_1, A_2, A_3)$$

Gauge freedom. Choosing the Coulomb gauge $\partial_i A_i = 0$:

$$\Box \mathbf{A} + \nabla_{t,x}(\partial_0 A_0) = [\mathbf{A}, \nabla_{t,x}\mathbf{A}] + \mathbf{A}^3$$

$\mathcal{P}$ = projector on divergence free vectofields:

$$\begin{aligned} \Delta(A_0) &= l.o.t \\ \Box(A_i) &= \left(\mathcal{P}\left(Q_{jl}(\nabla^{-1}A, A) + \nabla^{-1}(Q_{jl}(A, A))\right)\right)_i + l.o.t \end{aligned}$$

$$\|\boldsymbol{\partial} A\|_{L^\infty_t L^2(\mathbb{R}^3)} \lesssim \|Q_{jl}(\nabla^{-1}A, A)\|_{L^2_t L^2(\mathbb{R}^3)} + \|\nabla^{-1}(Q_{jl}(A, A))\|_{L^2_t L^2(\mathbb{R}^3)} + l.o.t.$$

Prove two bilinear estimates to conclude

Step A: EE as a quasilinear Yang-Mills theory

Let e_α an orthonormal frame on $\mathcal{M}$, i.e. $\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta}$

Let $(\mathbf{A}_\mu)_{\alpha\beta} := (\mathbf{A})_{\alpha\beta}(\partial_\mu) = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha)$

The definition of $\mathbf{R}$ yields:

$$\mathbf{R}(e_\alpha, e_\beta, \partial_\mu, \partial_\nu) = \partial_\mu(\mathbf{A}_\nu)_{\alpha\beta} - \partial_\nu(\mathbf{A}_\mu)_{\alpha\beta} + (\mathbf{A}_\nu)_\alpha{}^\lambda(\mathbf{A}_\mu)_{\lambda\beta} - (\mathbf{A}_\mu)_\alpha{}^\lambda(\mathbf{A}_\nu)_{\lambda\beta}$$

$\mathbf{D}^\mu \mathbf{R}_{\alpha\beta\mu\nu} = 0$ yields the tensorial wave equation:

$$(\square_{\mathbf{g}}\mathbf{A})_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu \mathbf{A}_\mu) = \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) + [\mathbf{A}^\mu, \mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu] + \mathbf{A}^3$$

In view of the Klainerman-Machedon proof, we need in particular a procedure to scalarize the tensorial wave equation and to project on divergence free vectorfields without destroying the null strucure

Scalarization and projection procedure

Scalarization: compute $[X, \Box_{\mathbf{g}}]$ for any vectorfield X and use it with $X = e_\alpha, \alpha = 0, 1, 2, 3$

Projection: compute $[\mathcal{P}, \Box_{\mathbf{g}}]$ where $\mathcal{P}$ = projector on divergence free vectofields

These commutators generate numerous dangerous terms which need to satisfy the null structure

We check this using the symmetries of $\mathbf{R}$, the Bianchi identities, the link between $\mathbf{A}$ and $\mathbf{R}$, the fact that A_0 is better than A_1, A_2, A_3, and the Coulomb gauge

The energy estimate for the wave equation

The proof reduces to the control of a scalar function ϕ satisfying

$$\Box_{\mathbf{g}}(\phi) = \text{null forms } + l.o.t$$

Let ϕ a scalar function and $Q_{\alpha\beta}$ its energy momentum tensor:

$$Q_{\alpha\beta} = Q_{\alpha\beta}[\phi] = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}\mathbf{g}_{\alpha\beta}\left(\mathbf{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

$$\int_{\Sigma_t} Q_{TT} = \int_{\Sigma_0} Q_{TT} + \int_{\mathcal{R}} \Box_{\mathbf{g}}\phi T(\phi) + \int_{\mathcal{R}} Q^{\alpha\beta}\mathbf{D}_\alpha T_\beta$$

Last term in RHS is dangerous $\Rightarrow$ needs to display the null structure and requires to prove the corresponding trilinear estimate

Step B: the bilinear estimates

We need to estimate the following null forms

$$\|Q_{ij}(\phi, A)\|_{L^2(\mathcal{M})} \text{ and } \|(-\Delta_g)^{-\frac{1}{2}}(Q_{ij}(\partial\phi, A))\|_{L^2(\mathcal{M})}$$

where ϕ is a scalar function ϕ satisfying

$$\square_{\mathbf{g}}(\phi) = \text{null forms } + l.o.t$$

To prove these bilinear estimates in a quasilinear setting:

- write ϕ by iterating the basic parametrix of step C (construction and control of the parametrix)
- Rethink the proof of bilinear estimates in the quasilinear setting
- For the second type of bilinear estimate, rely on the structure of Q_{ij} and a sharp $L^4(\mathcal{M})$ Strichartz estimate for the parametrix

Step C: construction and control of the parametrix

$$S(t,x) = \sum_{\pm} \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u_\pm(t,x,\omega)} f_\pm(\lambda\omega)\lambda^2 d\lambda d\omega$$

where $\mathbf{g}^{\alpha\beta}\partial_\alpha u_\pm \partial_\beta u_\pm = 0$ on $\mathcal{M}$ such that $u_\pm(0,x,\omega) \sim x.\omega$ when $|x| \to +\infty$ on Σ_0

Construction: for any (ϕ_0, ϕ_1) there exists $f_\pm$ such that

$S(0,.) = \phi_0$, $\mathbf{D}_T S(0,.) = \phi_1$ and $\|\lambda f_\pm\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)}$

$$E(t,x) = \square_{\mathbf{g}} S(t,x) = i \sum_{\pm} \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u_\pm(t,x,\omega)} \square_{\mathbf{g}} u_\pm(t,x,\omega) f_\pm(\lambda\omega)\lambda^3 d\lambda d\omega$$

Control of the error term: $\|Ef\|_{L^2(\mathcal{M})} \lesssim \|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)}$

Step C: construction and control of the parametrix

- *Goal: Achieve Step* **C** *only assuming* L^2 *bounds on* **R**. *This requires to exploit the full structure of Einstein equations*
- *The regularity in* ω *of* u *obtained in Step* **C** *is limited*
- *A careful choice of* $u(0, x, \omega)$ *(related to the mean curvature flow) allows us to "squeeze" as much regularity in* x *and* ω *as possible*
- $\mathbf{R} \in L^2$ *is minimal to obtain a lower bound on the radius of injectivity of level surfaces of the phase* u
- *Step* **C** *requires* L^2 *bounds for Fourier integral operators, and in turn several integration by parts. Classical proofs (*TT^* *and* T^*T *arguments) would fail by far*