Normal form and Quasi–periodic solutions for the non–linear Schrödinger equation

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Nonlinear Schrödinger equation

Consider the Nonlinear Schrödinger equation on the torus $\mathbb{T}^n$.

$$iu_t - \Delta u = F(|u|^2)u \tag{1}$$

where $u := u(t,\varphi)$, $\varphi \in \mathbb{T}^n$,
$F(y)$ is an analytic function, $F(0) = 0$
Note that we have no explicit space dependence.
This means that we have constants of motion due to translation invariance.

A good model is the CUBIC NLS:

$$iu_t - \Delta u = F(|u|^2)u = |u|^2 u \qquad (2)$$

with $q \in \mathbb{N}$. Or more in general:

$$iu_t - \Delta u = |u|^{2q} u$$

Quasi-periodic solutions

Main result, with C. Procesi

Consider the cubic NLS. For all $m \in \mathbb{N}$, there exist Cantor families of small quasi–periodic solutions of Equation (1) with m frequencies $\omega_1, \dots, \omega_m$.
We also prove the existence of an reducible elliptic normal form close to the solution.

m is arbitrarily large but finite
The solutions exist for all ω in a positive measure Cantor set. A quasi–periodic solution is a solution $u(t, \varphi)$ of Equation (1) such that

$$u(t, \varphi) = U(\omega t, \varphi)$$

where $\omega \in \mathbb{R}^n$ and $U : \mathbb{T}^m \times \mathbb{T}^n \to \mathbb{C}$. The solutions we find are analytic.

Main problems

Our equation $iu_t - \Delta u = |u|^2 u$ does not have external parameters.

COMPLETELY RESONANT SYSTEM. For the linear equation

$$iu_t - \Delta u = 0$$

all the bounded solutions are periodic of period 2π.

$$u(t, \varphi) = \sum_k u_k e^{\mathrm{i}(k \cdot \varphi + |k|^2 t)}$$

quasi-periodic solutions are due to the Non-Linearity

Main problems

Even if you add external parameters to avoid the resonance problem.

$$\mathrm{i}u_t - \Delta u + V(x)u = |u|^2 u$$

- DEGENERACY: the eigenvalues of $i\partial_t - \Delta$ are highly degenerate (the multiplicity of the eigenvalues grows to infinity!)
- SMALL DIVISORS: The spectrum of the linear part $i\partial_t - \Delta$ accumulates to zero on the space of quasi–periodic functions.

We do not expect quasi-periodic solutions to be typical

In the case of $\mathbb{T}^2$, Colliander-Keel-Staffilani-Takaoka-Tao, Invent.(2010) use unstable solutions to prove diffusion.

There is no a-priori reason why the solutions should have an integrable elliptic normal form close to them.

Some literature

non-resonant PDEs in one dimension

Kuksin, Craig, Wayne, Pöschel...

resonant PDEs in one dimension

- Kuksin, Pöschel, Annals (96). (cubic NLS)
- Geng (quintic NLS)
- Magistrelli, P. (NLS of degree 7)

non-resonant PDEs on $\mathbb{T}^n$(with outer parameters)

- Bourgain, Annals Studies (2005): NLS on $\mathbb{T}^n$
- Geng-You, CMP (2005): smoothing NLS on $\mathbb{T}^n$, existence and stability.
- Eliasson-Kuksin, Annals (2010): NLS on $\mathbb{T}^n$, existence and stability.
- Xu – P. (2011) NLS on $\mathbb{T}^n$, existence and stability (non–linearities which do not depend on the space variable)

resonant PDEs on $\mathbb{T}^n$

- Bourgain, Annals (96) cubic NLS on $\mathbb{T}^2$ with two frequencies.
- Gentile-P., CMP (2009) periodic solutions on $\mathbb{T}^n$.
- Berti-P.: periodic solutions for NLS on Lie groups
- Geng-You-Xu Adv. Math.(11): quasi–periodic solutions on $\mathbb{T}^2$
- Wang(2009-2011) quasi-periodic solutions general analytic NLS

- C. Procesi, P. CMP (2012) (Normal form for the general analytic NLS)
- Nguyeng Bich V., C. Procesi, P. Preprint (non-degeneracy of the normal form)
- C. Procesi, P. (in preparation) (quasi–periodic solutions)

Our result not only gives existence of solutions but also an integrable elliptic normal form close to the solutions

The plan

The construction of quasi–periodic solutions is performed in three steps:

The plan

1. Construction of integrable normal forms (applying Birkhoff normal form)
2. Proof of non–degeneracy of the normal form (algebraic argument)
3. The KAM algorithm and quasi–Töpliz property.

Dynamical systems approach

Passing to the Fourier representation

$$u(t,\varphi) := \sum_{k\in\mathbb{Z}^n} u_k(t)e^{\mathrm{i}(k,\varphi)} ,$$

$$|u|^2_{a,p} = \sum_{k\in\mathbb{Z}^n} |u_k|^2 e^{2a|k|}|k|^{2p} < \infty$$

Eq. (1) can be written as an

infinite dimensional Hamiltonian dynamical system:

$$H = \sum_{k\in\mathbb{Z}^n} |k|^2 u_k\bar u_k + \sum_{k_i\in\mathbb{Z}^n : k_1+k_3=k_2+k_4} u_{k_1}\bar u_{k_2} u_{k_3}\bar u_{k_4} \qquad (3)$$

with respect to the complex symplectic form $i\sum_k du_k\wedge d\bar u_k$.

Dynamical systems approach

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with respect to the complex symplectic form $i\sum_k du_k \wedge d\bar{u}_k$.

The system has the constants of motion:

$$L = \sum_{k \in \mathbb{Z}^n} u_k \bar{u}_k \,, \quad M = \sum_{k \in \mathbb{Z}^n} k u_k \bar{u}_k$$

the fact that M is preserved will be crucial to the proof!

Birkhoff Normal Form

$$H = K(u,\bar u) + H^{(4)}(u,\bar u)\,, \quad K(u,\bar u) = \sum_k |k|^2 u_k \bar u_k$$

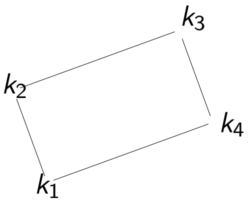
where $H^{(4)}$ is a polynomial of degree 4 and the linear frequencies (in our case $|k|^2$) are all rational.
With a sympletic change of variables we reduce the Hamiltonian H to

$$H_{Birk} = K(u,\bar u) + H^{(4)}_{res}(u,\bar u) + H^{(6)}$$

where $H^{(6)}$ is small while $H^{(4)}_{res}$ Poisson commutes with K.

One step of Birkhoff normal form produces

$$H_{Birk} = \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{\substack{k_i \in \mathbb{Z}^n : k_1 + k_3 = k_2 + k_4 \\ |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2}} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} + H^{(6)} \quad (4)$$



Even if we ignore the term $H^{(6)}$, this equation is still very complicated but Has a lot of invariant subspaces where the equation is significantly easier!

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Given a set $S \subset \mathbb{Z}^n$ consider the subspace

$$U_S := \{u = \{u_k\}_{k\in\mathbb{Z}^n} : \quad u_k = 0\,,\ \text{if } k \notin S\}$$

For generic choices of S the space U_S is invariant for the dynamics of

$$K + H^{(4)}_{Res} = \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{\substack{k_i\in\mathbb{Z}^n : k_1+k_3=k_2+k_4 \\ |k_1|^2+|k_3|^2=|k_2|^2+|k_4|^2}} u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4}$$

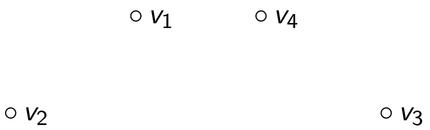
the Hamiltonian $K + H^{(4)}_{Res}$ restricted on U_S is

$$\sum_{k\in S} |k|^2 |u_k|^2 - \sum_{k\in S} |u_k|^4$$

Invariant subspaces

Genericity condition on S:

There are no triples $k_1, k_2, k_3 \in S$ that form a right angle



One can also construct invariant subspaces U_S where the dynamics is more complicated

Colliander–Keel–Staffilani–Takaoka–Tao

Grebert-Thomann

U_S is not invariant for the NLS Hamiltonian

$$K + H^{(4)}_{Res} + H^{(6)}$$

so the dynamics restricted to U_S can give information on the solution only for finite times.
We have to study the dynamics of the normal modes u_k with $k \notin S$.

Elliptic/action-angle variables.

Let us now partition

$$\mathbb{Z}^n = S \cup S^c, \quad S := (v_1, \dots, v_m).$$

where:

Let us now set

$$u_k := z_k \text{ for } k \in S^c, \qquad u_{v_i} := \sqrt{\xi_i + y_i} e^{\mathrm{i} x_i} \text{ for } v_i \in S,$$

this puts the tangential sites in action angle variables

$$y := \{y_1, \dots, y_m\}, \quad x := x_1, \dots, x_m$$

the ξ are parameters.

$$\xi \in A_{\varepsilon^2} := \{\xi \, : \, \frac{1}{2}\varepsilon^2 \le \xi_i \le \varepsilon^2\},$$

Let us now set

$$u_k := z_k \text{ for } k \in S^c\,, \qquad u_{v_i} := \sqrt{\xi_i + y_i}\, e^{\mathrm{i}x_i} \text{ for } v_i \in S,$$

For all $r \le \varepsilon/2$ this is a well known analytic and symplectic change of variables in the domain

$$D_{a,p}(s,r) = D(s,r) :=$$

$$\{x, y, z \ :\ |Im(x)| < s\,,\ |y| \le r^2\,,\ \|z\|_{a,p} \le r\} \subset \mathbb{T}^m_s \times \mathbb{C}^m \times \ell^{(a,p)} \times \ell^{(a,p)}.$$

$$||z||^2_{a,p} := |z_0|^2 + \sum_{k\in S^c} |z_k|^2 e^{2a|k|} |k|^{2p}$$

The *normal form Hamiltonian*

substitute

$$u_k := z_k \text{ for } k \in S^c\,, \qquad u_{v_i} := \sqrt{\xi_i + y_i}\, e^{\mathrm{i}x_i} \text{ for } v_i \in S,$$

in

$$\sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{\substack{k_i\in\mathbb{Z}^n : k_1+k_3=k_2+k_4 \\ |k_1|^2+|k_3|^2=|k_2|^2+|k_4|^2}} u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4} + H^{(6)}$$

We impose some simple constraints

After normalizing the NLS Hamiltonian is $N + P$ where P is small and the leading term is:

$$N := \sum_{1 \le i \le m} (|v_i|^2 - 2\xi_i) y_i + \sum_{k \in S^c} |k|^2 |z_k|^2 \qquad (5)$$

$$+ \mathcal{Q}(x, z)$$

set $\omega_i := |v_i|^2 - 2\xi_i$. $\mathcal{Q}(x, z)$ is a quadratic form in the normal variables z

N has the quasi-periodic solutions

$$x = x_0 + \omega t\,, \quad y = 0\,, \quad z = 0$$

The *normal form Hamiltonian*

$$\mathcal{Q}(x,z) = 4 \sum^{*}_{\substack{1\le i\neq j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}e^{\mathrm{i}(x_i-x_j)} z_h\bar{z}_k + \tag{6}$$

$$2 \sum^{**}_{\substack{1\le i<j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}e^{-\mathrm{i}(x_i+x_j)} z_h z_k + 2 \sum^{**}_{\substack{1\le i<j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}e^{\mathrm{i}(x_i+x_j)} \bar{z}_h\bar{z}_k.$$

The constraints $\sum^*, \sum^{**}$ mean

that the terms are *resonant with the quadratic part K*, that is:

Definition

- Here $\sum^*$ denotes that (h, k, v_i, v_j) give a rectangle:

  $$\{(h, k, v_i, v_j) \,|\, h + v_i = k + v_j, \;\; |h|^2 + |v_i|^2 = |k|^2 + |v_j|^2\}.$$

  We say $h \in H_{i,j}, \;\; k \in H_{j,i}$.
- $\sum^{**}$ means that (h, v_i, k, v_j) give a rectangle:

  $$\{(h, v_i, k, v_j) \,|\, h + k = v_i + v_j, \;\; |h|^2 + |k|^2 = |v_i|^2 + |v_j|^2\}.$$

  We say $h, k \in S_{i,j}$

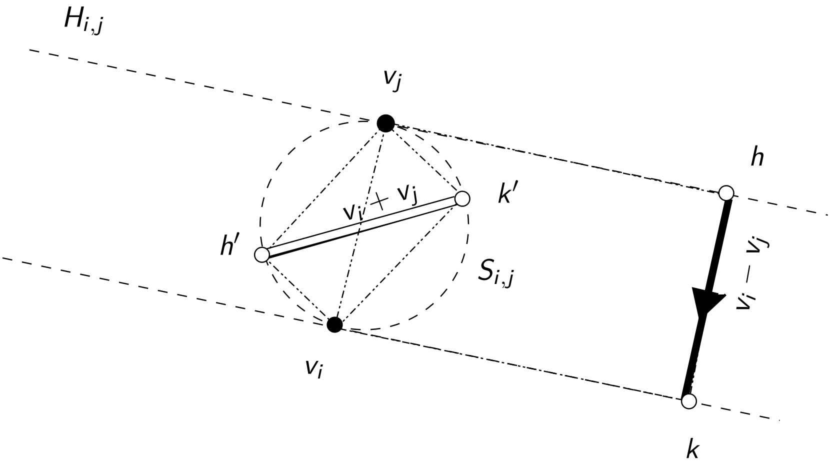


Figure: The plane $H_{i,j}$ and the sphere $S_{i,j}$. The points h, k, v_i, v_j form the vertices of a rectangle. Same for the points h', v_i, k', v_j

The Hamilton equations associated to N are linear with non-constant coefficients:

$$\mathrm{i}\dot{z} - Q^{+}(\omega t) z + Q^{-}(\omega t)\bar{z} = 0$$

$$Q = \begin{vmatrix} Q^{+} & Q^{-} \\ -\bar{Q}^{-} & -\bar{Q}^{+} \end{vmatrix}$$

is an infinite matrix.
Can we reduce to constant coefficients? Can we diagonalize?
The answer is YES but the proof requires subtle arguments.

An idea of the method:

Consider a matrix of the form

$$D + \varepsilon Q$$

Where D is diagonal. If D has distinct eigenvalues then one may diagonalize $D + \varepsilon Q$ by a perturbation scheme. If D has multiple eigenvalues we can only block diagonalize on the eigenspaces of distinct eigenvalues. To complete the diagonalization we need information on Q.
In finite dimension: $I + \varepsilon Q$
a sufficient condition Q has distinct eigenvalues

In our case direct inspection shows that we have from the start a block diagonal matrix such that on each block D is proportional to the identity.
We cannot rely on perturbation theory we must study the matrix Q very attentively!

A first theorem for generic choices of $S = \{v_1, \dots, v_m\}$.

Theorem

- *For generic v_i's the quadratic Hamiltonian $\mathcal{Q}(x, w)$ is an infinite sum of independent (decoupled) terms each depending on a finite number of variables (at most $n+1$ variables z_j together with their conjugates $\bar{z}_j$).*
- *One can exhibit an explicit symplectic change of variables which integrates N, namely makes all the angles disappear from $\mathcal{Q}(x, w)$.*

Theorem

There exists a map

$$S^c \ni k \to L(k) \in \mathbb{Z}^m\,, \quad |L(k)| < 2n$$

such that the analytic symplectic change of variables:

$$z_k = e^{-\mathrm{i}L(k).x} z'_k,\ y = y' + \sum_{k\in S^c} L(k)|z'_k|^2,\ x = x'.$$

reduces N to costant coefficients

$$N = (\omega(\xi), y') + \sum_{k\in S^c} (|k|^2 + \sum_i L_i(k)|v_i|^2)|z'_k|^2 + \tilde{\mathcal{Q}}(w')\,, \qquad (7)$$

The final Theorem and goal for the normal form

For the cubic NLS:

Theorem

- for generic values of the parameters ξ (outside some algebraic hyper surface) we can find a further symplectic change of coordinates so that
- N is *diagonal* (possibly with some complex terms)
- N is *non degenerate* in the sense that it satisfies the first and second Melnikov conditions.
- there exists a positive measure region of the parameters ξ in which N is elliptic (all real eigenvalues).

$$H_{\rm fin} = (\omega(\xi), y) + \sum_{k \in S^c} \Omega_k |z_k|^2 + P(\xi, x, y, z, \bar z) \qquad (8)$$

$$\omega_i = |v_i|^2 - 2\xi_i$$

$$\Omega_k = |k|^2 + \sum_i L_i(k)|v_i|^2 + \theta_k(\xi)\,, \quad \forall k \in S^c$$

The $L_i(k)$ are integers

$$\theta_k(\xi) \in \{\theta^{(1)}(\xi), \dots, \theta^{(K)}(\xi)\}\,, \quad K := K(n,m), \qquad (9)$$

list different analytic homoeogeneous functions of ξ.

Non-Degeneracy

The Melnikov resonances:

$$(\omega(\xi), \nu) = 0\,, \quad (\omega(\xi), \nu) + \Omega_k(\xi) = 0\,, \tag{10}$$

$$(\omega(\xi), \nu) + \Omega_k(\xi) + \sigma\Omega_h(\xi) = 0$$

hold on a zero measure subset of the parameters ξ.
In order to prove this we must restrict to those indexes ν, h, k which satisfy momentum conservation.

Genericity condition: Resonance polynomials

Definition

Given a list $\mathcal{R} := \{P_1(y), \dots, P_N(y)\}$ of non–zero polynomials in k vector variables y_i, we say that a list of vectors $S = \{v_1, \dots, v_m\}, v_i \in \mathbb{C}^n$ is GENERIC relative to $\mathcal{R}$ if, for any list $A = \{u_1, \dots, u_k\}$ such that $u_i \in S$, $\forall i$, the evaluation of the resonance polynomials at $y_i = u_i$ is non–zero.

If m is finite this condition is equivalent to requiring that S (considered as a point in $\mathbb{C}^{nm}$) does not belong to the algebraic variety where at least one of the resonance polynomials is zero.

Some remarks

There is no a-priori reason why this change of variables should exist. If one does not impose *good* genericity conditions then this is false.

This change of variables that reduces N to constant coefficients exists for all analytic NLS

$$iu_t - \Delta u = F(|u|^2)u$$

provided that F does not explicitly depends on φ.

Problem is proving the non-degeneracy!

We can proceed in the same way also when S is an infinite set.

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Kam theorem: the cubic NLS

Under the hypotheses of the previous theorem

$$H_{\mathrm{fin}} = (\omega(\xi), y) + \sum_{k \in S^c} \Omega_k(\xi)|z_k|^2 + P(\xi, x, y, z, \bar{z})$$

Theorem

There exists a Cantor set $\mathcal{C}$, such that: $\forall \xi \in \mathcal{C}$ there exists an analytic sympectic change of variables under which the Hamiltonian H_{fin} becomes

$$(\omega^\infty(\xi), y) + \sum_{k \in S^c} \Omega_k^\infty(\xi)|z_k|^2 + P^\infty(\xi, x, y, z, \bar{z})$$

with $X_{P^\infty}|_{y=0, z=0} = 0$.

The KAM algorithm is a rapidly convergent iterative scheme which produces a sequence of changes of variables

$$H^{(p)} = (\omega^{(p)}(\xi), y) + \sum_{k \in S^c} \Omega_k^{(p)}(\xi)|z_k|^2 + P^{(p)}(\xi, x, y, z, \bar{z}),$$

with $X_{P^{(p)}}|_{y=0,z=0} \to 0$.
The main point is to impose the Melnikov conditions:

$$|(\omega^{(p)}(\xi), \nu)| \geq \frac{\gamma}{|\nu|^\tau}, \quad |(\omega^{(p)}(\xi), \nu) + \Omega_k^{(p)}(\xi)| \geq \frac{\gamma}{|\nu|^\tau}$$

$$|(\omega^{(p)}(\xi), \nu) + \Omega_k^{(p)}(\xi) \pm \Omega_h^{(p)}(\xi)| \geq \frac{\gamma}{|\nu|^\tau},$$

The last condition is quite tricky to verify!

The main idea is to prove some asymptotic for the normal frequencies.
One would like something like

$$\Omega_k^{(p)} = |k|^2 + c^{(p)}(k) + O(|k|^{-\delta})$$

where $c^{(p)}(k)$ assumes a finite number of values (possibly growing with p).
At step zero ok:

$$\Omega_k = |k|^2 + \sum_i L_i(k)|v_i|^2 + \theta_k(\xi)$$

To prove the asymptotics for all steps

we use the properties of quasi-Töplitz functions introduced in Xu-P. (similar to the Töplitz-Lipschitz functions of Eliasson-Kuksin (2010))

we use the fact that our equation has no explicit dependence of the space variables so that the TOTAL MOMENTUM is preserved.

The quasi-Töplitz functions are closed with respect to:

Poisson Brackets

solving the Homological equation

For a quadratic function

$$\sum_k \Omega_k |z_k|^2$$

this means that for all N sufficiently large and for $|k| > N$

$$\Omega_k = |k|^2 + c_N(k) + O(1/N)$$

c_N assumes a finite (N dependent) number of values.

In Xu-P. we use the conservation of momentum to define the quasi-Töplitz functions.
This restriction has been removed in Berti-Biasco-P. for the case of the one Derivative non-linear wave equation.

$$\mathrm{y}_{tt} - \mathrm{y}_{\mathrm{xx}} + \mathrm{m}\mathrm{y} = g(\mathrm{x}, \mathrm{y}, \mathrm{y}_{\mathrm{x}}, \mathrm{y}_t), \quad \mathrm{x} \in \mathbb{T},$$

where $\mathtt{m} > 0$
Note this is not an Hamiltonian equation.

The reason why we restrict to the cubic case is that we do not know in general how to prove full non–degeneracy namely

$$(\omega(\xi), \nu) + \Omega_k(\xi) - \Omega_h(\xi) = 0,$$

holds true on a proper algebraic hypersurface for all non–trivial choices of $\nu \in \mathbb{Z}^m$ $h, k \in S^c$ (recall that $\mathbb{Z}^n = S \cup S^c$). In the cubic case we need subtle arguments combining algebra and combinatorics

Open problems:

1. May we impose the non–degeneracy conditions for all values of $q \in \mathbb{N}$? This is a possibly very difficult problem in algebra .... (results on $\mathbb{T}^1$ and $\mathbb{T}^2$)

2. What can we say on the stability with weaker non–degeneracy conditions?

The *normal form Hamiltonian*

$$Q(x,z,\bar z) = 4 \sum^{*}_{\substack{1\le i\ne j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}\, e^{\mathrm{i}(x_i-x_j)} z_h \bar z_k +$$

$$2 \sum^{**}_{\substack{1\le i<j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}\, e^{-\mathrm{i}(x_i+x_j)} z_h z_k + 2 \sum^{**}_{\substack{1\le i<j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}\, e^{\mathrm{i}(x_i+x_j)} \bar z_h \bar z_k .$$

Geometric graph

Definition

We construct the graph Γ_S with vertices all the points of $\mathbb{Z}^n$ by connecting with an edge all the Fourier indexes which contribute non–trivially to $\mathcal{Q}(x, z)$.

We want to study the connected components of the graph Γ_S, since they describe the blocks of Q!

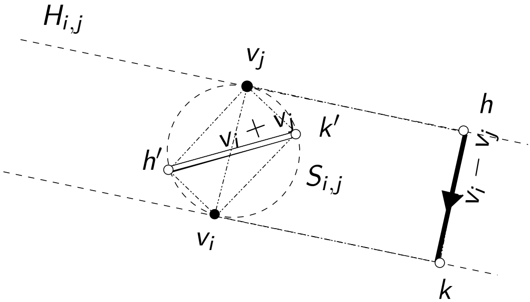

Figure: The plane $H_{i,j}$ and the sphere $S_{i,j}$. The points h, k, v_i, v_j form the vertices of a rectangle. Same for the points h', v_i, k', v_j

We can construct a graph which represents the matrix of Q by connecting all the h, k as above by an edge.

A component as solution of a system of equations

A tree in the graph with e edges and $e+1$ vertices is obtained by solving a system of $e(n+1)$ linear and quadratic equations ($n+1$ for each edge), in $(e+1)n$ variables (the coordinates of the $e+1$ vertices).

- We can expect that if $e(n+1) > (e+1)n \iff e > n$ these equations may be incompatible.
- So we expect no tree with $e > n$ edges.

A component as solution of a system of equations

The equations depend on the parameters v_i so the compatibility conditions are expressed by polynomial equations on the v_i which for us are the *resonances*.

We meet a substantial difficulty. Certain special systems of equations (Corresponding to trees with $e > n$ edges) are never incompatible!

They have as solutions the vectors v_i and we have to make sure that no other big component appears.

It is relatively easy to give a uniform bound
(depending on m and n)
on the dimension of the blocks of Q (a proof is for instance in Gentile P. (CMP 2009)).

Proving optimal bounds is much more subtle!

Proposition

For generic choices of S the connected components of Γ_S have at most $n+1$ vertices.

Dynamical consequences

$$Q(x,z,\bar z) = 4 \sum^{*}_{\substack{1\le i\ne j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}\, e^{\mathrm{i}(x_i-x_j)} z_h \bar z_k +$$

$$2 \sum^{**}_{\substack{1\le i<j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}\, e^{-\mathrm{i}(x_i+x_j)} z_h z_k + 2 \sum^{**}_{\substack{1\le i<j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}\, e^{\mathrm{i}(x_i+x_j)} \bar z_h \bar z_k =$$

$$(z, A(\xi,x)\bar z) + (z, B(\xi,x) z) + (\bar z, \bar B(\xi,x)\bar z),$$

where A is composed of blocks of dimension $\le n+1$; the blocks are described by a finite number of matrices B is a finite matrix.

Reduction

The change of variables which reduces to constant coefficients is: very simple

$$z_k = e^{-\mathrm{i}L(k).x} z_k', \;\; y = y' + \sum_{k\in S^c} L(k)|z_k'|^2, \;\; x = x'. \qquad (11)$$

where $L(k) \in \mathbb{Z}^m$ and $|L(k)| \leq 2n+2$.

We obtain the normal form Hamiltonian:

$$N = (\omega, y') + \sum_{k \in S^c} \tilde{\Omega}_k(\xi)|z'_k|^2 + Q(x = 0, z', \bar{z}')$$

where $\tilde{\Omega}_k(\xi) = |k|^2 + (\omega, L(k))$.

This is a list of uncoupled finite dimensional systems!

Let γ be a connected component of Γ_S we have the quadratic Hamiltonian

$$\sum_{k \in \gamma} \tilde{\Omega}_k(\xi)|z'_k|^2 + Q_\gamma(x = 0, z', \bar{z}')$$

We then apply the standard theory of quadratic Hamiltonians to diagonalize the matrices.

Write

$$\sum_{k\in\gamma}\tilde{\Omega}_k(\xi)|z'_k|^2 + Q_\gamma(x=0,z',\bar{z}') = \frac{1}{2}(w, JM_\gamma w)$$

J is the symplectic matrix and $w = z, \bar{z}$.

We get the elliptic normal form if M is diagonable with real eigenvalues.

It turns out that

$$M_\gamma = \textit{scalar matrix} + M'_\gamma$$

M'_γ is in a finite list of matrices.