Global dynamics of energy critical focusing nonlinear wave equations

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In this lecture we will discuss the energy critical nonlinear wave equation in 3 space dimensions.

We start by a review of the linear wave equation

$$(\mathrm{LW}) \qquad \begin{cases} \partial_t^2 w - \Delta w = h \\ w|_{t=0} = w_0 \\ \partial_t w|_{t=0} = w_1 \end{cases}$$

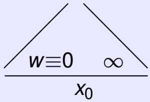
We write the solution:

$$w(t) = S(t)(w_0, w_1) + D(t)(h)\,,$$

where $S(t)$ denotes the solution of the homogeneous problem $(h = 0)$ and $D(t)$ the solution of the inhomogeneous one $((w_0, w_1) = (0, 0))$.

One of the main properties of the linear wave equation is the finite speed of propagation:

If $\mathrm{supp}(w_0, w_1) \cap \overline{B(x_0, a)} = \phi$, $\mathrm{supp}\, h \cap \left(\bigcup_{0 \le t \le a} B(x_0, a - t) \times \{t\} \right) = \phi$,
then $w \equiv 0$ on $\bigcup_{0 \le t \le a} B(x_0, a - t) \times \{t\}$.



An important estimate (Strichartz estimate) is:

$$\|w\|_{L^8_{x,t}} \le C \left\{ \|(w_0, w_1)\|_{\dot{H}^1 \times L^2} + \|D^{1/2} h\|_{L^{4/3}_{x,t}} \right\} .$$

The energy critical nonlinear wave equation, in the focusing case is:

$$(\text{NLW}) \qquad \begin{cases} \partial_t^2 u - \Delta u = u^5 \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^3), \quad x \in \mathbb{R}^3,\ t \in \mathbb{R} \\ \partial_t u|_{t=0} = u_1 \in L^2(\mathbb{R}^3) \end{cases}$$

The defocusing case has $-u^5$.

(NLW) is called energy critical because $\frac{1}{\lambda^{1/2}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right)$ is also a solution and this leaves unchanged the $\dot{H}^1 \times L^2$ norm.

Small data theory for (NLW): If $\|(u_0,u_1)\|_{\dot{H}^1\times L^2}$ is small $\exists!$ solution u, defined for all time, such that $u \in C((-\infty,+\infty);\dot{H}^1 \times L^2) \cap L^8_{xt}$, which scatters i.e.

$$\|(u(t),\partial_t u(t)) - S(t)(u_0^{\pm},u_1^{\pm})\|_{\dot{H}^1\times L^2} \xrightarrow[t\to\pm\infty]{} 0\,.$$

Moreover, for any data $(u_0,u_1) \in \dot{H}^1 \times L^2$, we have short time existence and hence there exists a maximal interval of existence $I = (-T_-(u), T_+(u))$.

The energy $E(u) = \frac{1}{2}\int|\nabla u(t)|^2 + \frac{1}{2}\int|\partial_t u(t)|^2 - \frac{1}{6}\int|u(t)|^6$ is constant for $t \in I$. In the defocusing case, $-\frac{1}{6}$ becomes $1/6$.

In the defocusing case work of Struwe, Grillakis, Shatah-Struwe, Bahouri-Shatah (80's-90's) proves that for any $(u_0, u_1) \in \dot{H}^1 \times L^2$, the solution exists globally and scatters.

In the focusing case this fails. Levine (74) showed that if $E(u_0, u_1) \leq 0$, then $T_-, T_+ < \infty$. (This is done by obstruction). Recently, Krieger-Schag-Tataru 09 constructed solutions for which $T_+ < \infty$. Also, in the focusing case, the elliptic equation admits a non-negative solution W (ground-state), which solves $\Delta u + u^5 = 0$.

This elliptic equation has been much studied in connection with the Yamabe problem in differential geometry. W has the explicit form

$$W(x) = \frac{1}{(1 + W^{2/3})^{1/2}}$$

W is the unique non-negative solution of the elliptic equation (Gidas-Ni-Nirenberg 79) and the only $\dot{H}^1$ solution (Pohozaev 65). W is a global in time solution of (NLW), which we call a soliton. It does not scatter to a linear solution "non-dispersive" solution. Recently (2012) Donninger-Krieger have constructed global in time solutions, which are bounded in $\dot{H}^1 \times L^2$, are radial, and don't scatter to either a linear solution or to W.

We now recall some results for (NLW) in the last few years.

Thm 1: (KM 08) If $E(u) < E(W)$ then:

i) If $\|\nabla u_0\| < \|\nabla W\|$, we have global existence, scattering

ii) If $\|\nabla u_0\| > \|\nabla W\|$, we have $T_+, T_- < \infty$.

The case $\|\nabla u_0\| = \|\nabla W\|$ is impossible.

A strengthening of this result is:

Thm 2: (DKM 09) If $\sup_{0<t<T_+} \|\nabla u(t)\|^2 + \frac{1}{2}\|\partial_t u(t)\|^2 < \|\nabla W\|^2$ (or $\sup_{0<t<1} \|\nabla u(t)\|^2 + \varepsilon\|\partial_t u(t)\|^2 < \|\nabla W\|^2$ in the radial case) we have global existence and scattering.

The next result deals with the case $E(u) = E(W)$.

Thm 3: (DM 08) There exist W_-, W_+ radial, with $E(W_-) = E(W_+) = E(W)$ s.t. if $E(u) = E(W)$, then:

- **i)** If $\|\nabla u_0\| < \|\nabla W\|$, then u is globally defined, and u scatters to linear solution at $\pm\infty$, or $u = W_-$, which has: W_- scatters at $-\infty$ to W and at $+\infty$ to a linear solution.
- **ii)** If $\|\nabla u_0\| = \|\nabla W\|$, $u = W$.
- **iii)** If $\|\nabla u_0\| > \|\nabla W\|$, then, either $T_+, T_- < \infty$, or $u = W_+$, which has: W_+ scatters at $-\infty$ to W and $T_+(W_+) < \infty$. (DKM 11, KNS 11).

Next we turn to the existence of type II blow-up solutions, i.e. s.t. $T_+ < \infty$ and $\sup\limits_{0<t<T_+} \|\nabla u(t)\| + \|\partial_t u(t)\| < \infty$.

Thm 4: (Krieger-Schlag-Tataru 09) $\forall \eta_0 > 0 \, \exists$ radial solution s.t. $T_+ = 1$, $\sup\limits_{0<t<1} \|\nabla u(t)\| + \|\partial_t u(t)\| < \infty$, $\sup\limits_{0<t<1} \|\nabla u(t)\| \leq \|\nabla W\| + \eta_0$ and

$$(u(t), \partial_t u(t)) = \left(\frac{1}{\lambda(t)^{1/2}} W\left(\frac{x}{\lambda(t)}\right), 0 \right) + \eta(x,t),$$

with η continuous in $\dot{H}^1 \times L^2$ up to $t = 1$ and $\lambda(t) = (1-t)^{1+\nu}$, $\nu > 1/2$. (It is believed that $\nu > 0$ works).

We next show that this is a “universal” phenomenon:

Thm 5: (DKM 09, 10) Assume that u is a solution so that $T_+ = 1$, $\sup_{0<t<1} \|\nabla u(t)\| + \|\partial_t u(t)\| < \infty$. (Type II solution)

i) Assume that u is radial and

$$\sup_{0<t<T_+} \|\nabla u(t)\| \leq \|\nabla W\| + \eta_0\,, \quad \eta_0 \text{ small } > 0\,.$$

The $\exists\,(v_0, v_1) \in \dot{H}^1 \times L^2$, $\lambda(t) > 0$, $i_0 \in \{\pm 1\}$ s.t.

$$(u(t), \partial_t u(t)) = (v_0, v_1) + \left(\frac{i_0}{\lambda(t)^{1/2}} W\left(\frac{x}{\lambda(t)}\right), 0\right) + o(1) \text{ in } \dot{H}^1 \times L^2$$

where $\lambda(t) = o(1-t)$.

ii) Non-radial case. Assume that

$$\sup_{0<t<T_+} \|\nabla u(t)\|^2 + \frac{1}{2}\|\partial_t u(t)\|^2 \leq \|\nabla W\|^2 + \eta_0\,, \quad \eta_0 \text{ small.}$$

Then, after rotation and translation of $\mathbb{R}^3$, $\exists\,(v_0, v_1) \in \dot{H}^1 \times L^2$, $i_0 \in \{\pm 1\}$, ℓ small, $x(t) \in \mathbb{R}^3$, $\lambda(t) > 0$ s.t.

$$\begin{aligned}(u(t), \partial_t u(t)) \quad &= \quad (v_0, v_1) + \left(\frac{i_0}{\lambda(t)^{1/2}} W_\ell \left(\frac{x - x(t)}{\lambda(t)}, 0 \right), \right. \\ &\left. \frac{i_0}{\lambda(t)^{3/2}} \partial_t\, W_\ell \left(\frac{x - x(t)}{\lambda(t)}, 0 \right) \right) + o(1) \text{ in } \dot{H}^1 \times L^2\,,\end{aligned}$$

where $\lambda(t) = o(1-t)\,,\ \lim_{t\uparrow 1} \frac{x(t)}{1-t} = \ell \vec{e}_1\,,\ \vec{e}_1 = (1,0,0)\,,\ |\ell| \leq C\eta_0^{1/4}\,,$

and $W_\ell(x,t) = W\left(\frac{x_1 - t\ell}{\sqrt{1-\ell^2}}, x_2, x_3 \right)$ is the Lorentz transform of W.

Remark: Note that $(3/4)^{1/4}(1-t)^{-1/2}$ is a solution. Using this and finite speed of propagation it is easy to construct type I solutions, i.e. $T_+ = 1$ and $\lim_{t\uparrow 1}\|(u(t), \partial_t u(t))\|_{\dot{H}^1\times L^2} = +\infty$. Note that type I and type II solutions need not be mutually exclusive.

Thm 5 (DKM 11) W_+ (from Theorem 3) is type I.

Next, I will turn to the main new topic in this lecture, namely soliton resolution for radial solutions of (NL).

For a long time there has been a widespread belief that global in time solutions of dispersive equations, asymptotically in time, decouple into a sum of finitely many modulated solitons, a free radiation term and a term that goes to 0 at infinity. Such a result should hold for globally well-posed equations, or in general, with the additional condition that the solution does not blow-up. When blow-up may occur such decompositions are always expected to be unstable. So far the only cases where a result of this type has been proved is for the integrable KdV and NLS equations in *Id*.

For $\partial_t u + \partial_x^3 u + u\partial_x u = 0$, for data with regularity and decay, this has been established by Eckhaus and Schuur. Corresponding results for the other integrable KdV equation, the modified KdV $\partial_t u + \partial_x^3 u + u^2 \partial_x u = 0$ were also obtained by the same authors (Miura transform). Heuristic arguments for this conjecture, in the case of the cubic NLS in $1-d$, $i\partial_t u + \partial_x^2 u + |u|^2 u = 0$ in $1-d$, another integrable model, were given by Ablowitz-Segur 76 and Zakharov-Shabat 71.

These are all globally well-posed equations, for which one expects that these decompositions are stable, unlike in the case of equations where blow-up may occur.

For more general equations, so far, results have been found for data close to the soliton, in subcritical nonlinearities, due to several authors. (Buslaev-Perelman 92 for NLS with specific nonlinearities in *Id*, Soffer-Weinstein 90 in higher *d*, Martel-Merle for gKdV 2001 ...).

For corresponding results near the soliton, in the case of finite time blow-up, for critical problems, besides the ones of DKM mentioned earlier, there has been work of Martel-Merle gKdV 2002, Merle-Raphael 04,04 for the mass critial NLS, etc.

There have also been large solution results for critical equivariant wave maps into the sphere due to Christodoulou-Tahvildar-Zadeh, Shatah-T-Z and Struwe. They show convergence along some sequence of times converging to the blow-up time, locally in space, to a soliton (harmonic map).

In the finite time blow-up case, for the $1-d$ nonlinear wave equation, Merle-Zaag have obtained results of this kind through the use of a global Lyapunov function in self-similar variables.

In critical elliptic problems, such as the ones mentioned earlier, in domains excluding a small ball, considering radial solutions, there have been obtained results on decompositions into “towering bubbles” (the analog of a finite sum of modulated solitons), as the size of the ball goes to 0. (Musso-Pistoia 2006).

The first general results for radial solutions of (NLW), were for type II solutions, and held only for a sequence of times (DKM 11).

We now have the full soliton resolution for radial solutions of (NLW), in the two asymptotic regimes, finite time blow-up and global in time. (Work of Duyckaerts-K-Merle 12).

Theorem: Let u be a radial solution of (NLW). Then, one of the following holds:

a) Type I blow-up: $T_+ < \infty$ and

$$\lim_{t \uparrow T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} = \infty$$

b) Type II blow-up: $T_+ < \infty$ and $\exists\, (v_0, v_1) \in \dot{H}^1 \times L^2$

$$J \in \mathbb{N}\backslash\{0\} \quad \text{and} \quad \forall\, j \in \{1, \ldots, J\},\ i_j \in \{\pm 1\}$$

and a positive $\lambda_j(t)$ s.t.

$$\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t) \ll T_+ - t\,,$$

$$\text{and } (u(t), \partial_t u(t)) = (v_0, v_1) + \left(\sum_{j=1}^{J} \frac{i_j}{\lambda_j(t)^{1/2}} W\left(\frac{x}{\lambda_j(t)}\right), 0 \right) + o(1)$$

in $\dot{H}^1 \times L^2$.

c) $T_+ = +\infty$ and $\exists$ a solution v_L of the (LW), $J \in \mathbb{N}$ and for all $j \in \{1, \ldots, J\}$, $i_j \in \{\pm 1\}$ and a positive $\lambda_j(t)$ s.t.

$$\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t) \ll t\,,$$

and

$$(u(t), \partial_t u(t)) = (v_L(t), \partial_t v_L(t)) + \left(\sum_{j=1}^{J} \frac{i_j}{\lambda_j(t)^{1/2}} W\left(\frac{x}{\lambda_j(t)}\right), 0 \right) + o(1)$$

in $\dot{H}^1 \times L^2$.

Remark 1: When $T_+ < \infty$, a), b) imply that $\lim_{t \uparrow T^+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} = \ell$ exists, $\ell \in [\|\nabla W\|, +\infty]$, i.e. no mixed asymptotics. Also, solutions split into type I, type II. Note that by previous results, both type I, type II exist. We expect that solutions as in b) with $J > 1$ exist. For the $1 - d$ nonlinear wave equation situation mentioned earlier, this has been shown by Côte-Zaag 11, while in the elliptic setting this is in the work of Mussi-Pistoia mentioned earlier, also in the radial case.

Remark 2: When $T_+ = \infty$, c) in particular implies that $\sup_{t>0} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty$.

More precisely, $\lim_{t \uparrow \infty} \|(u(t), \partial_t u(t))\|^2_{\dot{H}^1 \times L^2} = \ell$, and $2\,E(u) \leq \ell \leq 3\,E(u)$. Also $J \leq E(u)/E(W)$. In this case we also expect that solutions with $J > 1$ exist.

Remark 3: It is known that the set S_1 of initial data such that the corresponding solution scatters to a linear solution is open. It is believed that the set S_2 of initial data leading to type I blow-up is also open. Our theorem gives a description of solutions whose data is in S_3, the complementary set to $S_1 \cup S_2$. We believe that from our Theorem one can show that S_3 is the boundary of $S_1 \cup S_2$. In particular we conjecture that the asymptotic behavior of solutions with data in S_3 is unstable.

Ideas for the proof (global case): The fundamental new ingredient of the proof is the following dispersive property that all radial solutions to (NLW) (other than 0 and $\pm W$ up to scaling) must have:

$$\exists\, R > 0\,,\ \eta > 0 \text{ s.t. for all } t \geq 0 \text{ or all } t \leq 0$$

$$(*) \qquad \int_{|x|>R+|t|} |\nabla_{x,t}\, u(x,t)|^2\, dx \geq \eta\,.$$

We establish this only using the behavior of u in outside regions, $|x| > R + |t|$, without using any global integral identity of virial (Pohozaev) type. In fact, this approach gives a new proof, without integral identities, of Pohozaev's result that $0, \pm W$ are the only radial $\dot{H}^1$ solutions of $\Delta u + u^5 = 0$ and also of the result of DKM 09, which characterizes "compact" radial solutions of (NLW) as $0, \pm W$.

Next, we show that a global radial solution must be bounded for at least one sequence of times going to infinity. This uses an adaptation of Levine's blow-up argument. Then we show that an expansion as in the conclusion in c) must hold on any sequence of times going to infinity along which the sequence is bounded. In order to show this we first show that if a solution is bounded for a sequence times, then the solution has linear behavior in the region outside a finite distance from the boundary of the light cone $|x| = |t|$. This constructs the free radiation term v_L.

Then we use the profile decomposition of Bahouri-Gérard (99). We combine this with the finite speed of propagation to see that $(*)$ (with $R > 0$) decouples the dynamics of different profiles in regions $|x| > R + |t|$. This is accomplished through a "perturbation theorem". If $\{tu\}$ is the sequence of times on which the solution is bounded, we apply the profile decomposition to $(u(tu), \partial_t u(tu)) - (v_L(tu), \partial_t v_L(tu))$ and use the above argument. Assuming that there is a non-zero profile which is not $\pm W$, using $(*)$ we can see that this profile sends an "energy charmel" into the future, which contradicts the fact that outside finite distance from the boundary of the light cone $u(tu) - v_L(tu)$ is small, or into the past, which eventually contradicts the uniform $\dot{H}^1 \times L^2$ bound on $(u(tu), \partial_t u(tu))$. Finally, once this is done, continuity arguments give the general statement.