Blow up for the L^2 critical gKdV equation

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Introduction

We consider the L^2 critical (gKdV) equation

$$\text{(gKdV)} \quad \begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Recall the following important facts:

• The Cauchy problem is locally well-posed in H^1
[Kenig-Ponce-Vega, 92] ([Kato, 83])

• Mass and energy conservation

$$M_0 = \int u^2(t), \qquad E_0 = \frac{1}{2}\int u_x^2(t) - \frac{1}{6}\int u^6(t)$$

• Scaling invariance ($\lambda > 0$)

$$u^\lambda(t,x) = \frac{1}{\lambda^{\frac{1}{2}}} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right), \quad \|u^\lambda\|_{L^2} = \|u\|_{L^2}, \quad E(u^\lambda) = \frac{1}{\lambda^2} E(u)$$

- Solitons are special solutions defined by $(\lambda > 0,\ x_0 \in \mathbb{R})$

$$R^{\lambda,x_0}(t,x) = \frac{1}{\lambda^{\frac{1}{2}}} Q\left(\frac{1}{\lambda}(x - x_0) - \frac{1}{\lambda^3} t\right)$$

$$Q(x) = \left(\frac{3}{\cosh^2(2x)}\right)^{1/4}, \quad Q'' - Q + Q^5 = 0, \quad E(Q) = 0$$

- Global existence for "small" L^2 norm: [Weinstein, 83]

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \Rightarrow \text{the solution is global in } H^1$$

Main questions of this talk:

- Blow up problem for initial data:

$$u_0 \in H^1, \quad \|Q\|_{L^2} \le \|u_0\|_{L^2} \le \|Q\|_{L^2} + \alpha_0, \quad \alpha_0 \ll 1$$

- Classification of all possible behaviors for $\|u_0 - Q\|_{H^1} \ll 1$

First results on blow up for L^2 critical gKdV

[YM-Merle, 00-02]

Assume

$$u_0 \in H^1, \quad \|Q\|_{L^2} \leq \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0, \quad \alpha_0 \ll 1$$

Then:

(i) Blow up in finite or infinite time if $E_0 < 0$.
No information on the blow up regime.

(ii) Assuming blow up, Q is the universal blow up profile.

(iii) Blow up in finite time if $E_0 < 0$ **and** $\int_{x>1} x^6 u_0^2(x)dx < \infty$.
Moreover, for a sequence $t_n \to T$,

$$\|u_x(t_n)\|_{L^2} \leq \frac{C(u_0)}{T - t_n}$$

(iv) Global existence for minimal mass initial data with decay.

Blow up for L^2 critical NLS

$$(\text{NLS}) \quad \begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{N}} u = 0, \\ u_{|t=0} = u_0 \end{cases} \qquad (t,x) \in [0,T) \times \mathbb{R}^N$$

$$\Delta Q_{\mathrm{NLS}} - Q_{\mathrm{NLS}} + Q_{\mathrm{NLS}}^{1+\frac{4}{N}} = 0, \qquad Q_{\mathrm{NLS}} > 0 \text{ even}$$

- [Merle, 93]

The only H^1 blow up solution of (NLS) with minimal mass $\|u_0\|_{L^2} = \|Q_{\mathrm{NLS}}\|_{L^2}$ is (up to symmetries)

$$S_{\mathrm{NLS}}(t,x) = \frac{1}{t^{N/2}} e^{-i\left(\frac{|x|^2}{4t} - \frac{1}{t}\right)} Q_{\mathrm{NLS}}\left(\frac{x}{t}\right)$$

- Existence of unstable nontrivial $\frac{1}{(T-t)}$ blow up solutions.

[Bourgain-Wang, 98], [Krieger-Schlag, 09], [Merle-Raphaël-Szeftel, 11]

“log-log” blow up for (NLS)

- [Landman-Papanicolaou-Sulem-Sulem, 88], etc.

log-log conjecture

- [Perelman, 01]

Construction of a large class of log-log blow up solutions close to Q_{NLS}.

- [Merle-Raphaël, 03-06]

(i) Construction of an open set in H^1 of log-log blow up solutions close to Q_{NLS} (including all H^1 data with $E_0 \leq 0$ close to Q_{NLS})

$$\|\nabla u_{\mathrm{NLS}}(t)\|_{L^2} \sim C^* \sqrt{\frac{\log|\log(T-t)|}{T-t}}$$

(ii) Quantization of the focused mass at the blow up point $x(T)$:

$$|u_{\mathrm{NLS}}(t)|^2 \rightharpoonup \|Q_{\mathrm{NLS}}\|_{L^2}^2 \delta_{x=x(T)} + |u^*|^2, \quad u^* \in L^2.$$

Statement of new results for critical gKdV

[YM-Merle-Raphaël, 12]

Define ($\alpha_0 \ll 1$)

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 \text{ with } \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{x>1} x^{10}\varepsilon_0^2(x)dx < 1 \right\}$$

THM 1 (Negative or zero energy data close to Q)

Let $u_0 \in \mathcal{A}$. If $E(u_0) \le 0$ and $u(t)$ is not a soliton, then $u(t)$ blows up in finite time T with

$$\|u_x(t)\|_{L^2} \underset{t\sim T}{\sim} \frac{\|Q'\|_{L^2}}{\ell_0(T-t)} \quad \text{for } \ell_0(u_0) > 0$$

$$u(t) - \frac{1}{\lambda^{\frac{1}{2}}(t)} Q\left(\frac{. - x(t)}{\lambda(t)}\right) \underset{t\to T}{\to} u^* \quad \text{in} \quad L^2$$

$$\lambda(t) \underset{t\sim T}{\sim} \ell_0(T-t), \quad x(t) \underset{t\sim T}{\sim} \frac{1}{\ell_0^2(T-t)}$$

See [Rodnianski-Sterbenz, 10], [Raphaël-Rodnianski, 12], [Merle-Raphaël-Rodnianski, 11]

THM 2 (Existence and uniqueness of minimal mass blow up sol.)

(i) There exists a solution $S \in \mathcal{C}((0,+\infty), H^1)$ with minimal mass

$$\|S(t)\|_{L^2} = \|Q\|_{L^2}$$

such that

$$\|S_x(t)\|_{L^2} \sim \frac{\|Q'\|_{L^2}}{t} \quad \text{as } t \downarrow 0,$$

$$S(t) - \frac{1}{t^{\frac{1}{2}}} Q\left(\frac{. + \frac{1}{t} + \bar{c}t}{t}\right) \to 0 \quad \text{in} \quad L^2 \quad \text{as} \quad t \downarrow 0,$$

where $\bar{c}$ is a universal constant.

(ii) Let $u(t)$ be a solution with minimal mass which blows up in finite time. Then, $u = S$ up to invariances.

THM 3 (Classification and universality of $S(t)$)

Let $0 < \alpha_0 \ll \alpha^* \ll 1$. Only three scenarios are possible for $u_0 \in \mathcal{A}$

(Blow up) $u(t)$ blows up in finite time with blow up rate $\frac{1}{T-t}$.

(Soliton) $u(t)$ is global, bounded and locally converges to a soliton as $t \to +\infty$.

(Exit) there exists $t^* > 0$ such that $u(t)$ exits at $t = t^*$ the L^2 neighborhood of size α^* of the family of solitons.
Moreover, for some τ^*, $u(t^*)$ is L^2 close (related to α_0) to $S(\tau^*)$ (up to symmetries).

Consequence: **Assume that** $S(t)$ scatters at $+\infty$. Then, the (Exit) scenario implies scattering.

Classification results for NLKG, NLW [Nakanishi-Schlag, 10], [Krieger-Nakanishi-Schlag, 10] ([Duyckaerts-Kenig-Merle, 06-09])

Stable manifold: [Krieger-Schlag, 05], [Beceanu, 07]

Blow up rates for initial data with slow decay $u_0 \notin \mathcal{A}$

THM 4 (Unstable blow up rates)

There exist blow up solutions with the following blow up rates:

(i) Blow up in finite time: for any $\nu > \frac{11}{13}$,

$$\|u_x(t)\|_{L^2} \sim t^{-\nu} \quad \text{as} \quad t \to 0^+.$$

(ii) Blow up in infinite time:

$$\|u_x(t)\|_{L^2} \sim e^t \quad \text{as} \quad t \to +\infty.$$

For any $\nu > 0$,

$$\|u_x(t)\|_{L^2} \sim t^{\nu} \quad \text{as} \quad t \to +\infty.$$

Moreover, such solutions can be taken arbitrarily close to solitons.

See [Krieger-Schlag-Tataru, 08], [Bejenaru-Tataru, 09], [Donninger-Krieger, 12], [Perelman, 12]

Formal derivation of the dynamics in $\mathcal{A}$

$$u(t,x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} Q_{b(t)}\left(\frac{x - x(t)}{\lambda(t)}\right), \quad Q_b = Q + bP$$

$$u_t = -\frac{\lambda_t}{\lambda}(\Lambda Q_b)^\lambda - \frac{x_t}{\lambda}(Q_b')^\lambda + b_t P^\lambda, \quad \Lambda Q_b = \frac{1}{2} Q_b + y(Q_b)_y,$$

$$\Rightarrow \quad -\lambda^2 \lambda_t \Lambda Q_b + (Q_b'' - \lambda^2 x_t Q_b + Q_b^5)' + \lambda^3 b_t P = 0$$

Fix $\lambda^2 x_t = 1$ and $-\lambda^2 \lambda_t = b$. At first order in b,

$$b\,\Lambda Q + b\,(LP)' + \lambda^3 b_t P + O(b^2) = 0$$

where $LP = -P'' + P - 5Q^4 P$. We fix

$$(LP)' = -\Lambda Q \quad \text{and} \quad \lambda^3 b_t = -2b^2$$

Combining the equations of λ_t and b_t, one gets

$$\frac{d}{dt}\left(\frac{b}{\lambda^2}\right) = \frac{1}{\lambda^2}\left(b_t - 2\frac{\lambda_t}{\lambda}b\right) = 0$$

and

$$-\lambda_t = \frac{b}{\lambda^2} = \ell_0 \quad \text{(scaling law)}$$

Three scenarios:

- $\underline{\ell_0 > 0}$:

$$\lambda_t = -\ell_0 < 0 \quad \Rightarrow \quad \text{blow up and } \lambda(t) = \ell_0(T - t)$$

  Example: $E_0 < 0$ but also $E_0 = 0$ (rigidity argument)

- $\underline{\ell_0 = 0}$:

$$\lambda(t) = \text{Cte} \quad \Rightarrow \quad \text{soliton}$$

- $\underline{\ell_0 < 0}$:

$$\lambda_t = -\ell_0 > 0 \quad \Rightarrow \quad \text{defocusing and then (Exit)}$$

Full ansatz - control of the remainder term

We decompose the solution $u(t,x)$ as

$$u(t,x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} Q_{b(t)}\left(\frac{x - x(t)}{\lambda(t)}\right) + \frac{1}{\lambda^{\frac{1}{2}}(t)} \varepsilon\left(t, \frac{x - x(t)}{\lambda(t)}\right)$$

where (b, λ, x) are adjusted to obtain orthogonality conditions on ε.

The function $\varepsilon(s,y)$ and (b,λ,x) are governed by

$$\varepsilon_s - (L\varepsilon)_y = \left(\tfrac{\lambda_s}{\lambda} + b\right)\Lambda Q + \left(\tfrac{x_s}{\lambda} - 1\right) Q' + \tfrac{\lambda_s}{\lambda}\Lambda\varepsilon + O(b^2 + |b_s| + |\varepsilon|^2)$$

and $\int \varepsilon Q = \int \varepsilon \Lambda Q = \int \varepsilon y \Lambda Q = 0$ (s is the rescaled time $\frac{ds}{dt} = \frac{1}{\lambda^3}$)

The uniform control of some norm of ε is a fundamental point in all the regimes to justify the dynamics of the parameters.

Tools for a simplified linear model (with orthogonality)

$$\varepsilon_s - (L\varepsilon)_y = \alpha(s)\Lambda Q + \beta(s)Q'$$

- Energy conservation at the level of ε:

$$\forall s, \quad (L\varepsilon(s), \varepsilon(s)) = \mathrm{Cte}$$

- Monotonicity argument: for $A \gg 1$,

$$\frac{d}{ds}\int_{``y>-A``}(\varepsilon_y^2 + \varepsilon^2 - 5Q^4\varepsilon^2)(s,y)dy \leq e^{-\frac{A}{10}}\|\varepsilon(s)\|_{H^1}^2$$

- Viriel type argument (under orthogonality conditions):

$$-\frac{d}{ds}\int y\varepsilon^2 = H(\varepsilon,\varepsilon) \geq \mu_0\|\varepsilon(s)\|_{H^1}^2$$

Main estimate on ε

Definition of a Liapunov functional for $\varepsilon(s)$

$$\mathcal{F}(s) \sim \int \left[\varepsilon_y^2\psi_1 + \varepsilon^2\psi_2 - 5Q^4\varepsilon^2\psi_1\right](s,y)dy$$

where
- $\psi_1(y) = 0$ for $y < -A$, $\psi_1(y) = 1$ for $y > -\frac{1}{2}A$,
- $\psi_2(y) = 0$ for $y < -A$, $\psi_2(y) = 1 + y$ for $y > -\frac{1}{2}A$.

$\mathcal{F}(t)$ is a mixed **energy monotonicity** and **Viriel** quantity

PROP. *Under a suitable assumption on space decay of $\varepsilon(s,y)$ on the right (which requires decay on the initial data), it holds*

$$\frac{d}{ds}\left(\frac{\mathcal{F}}{\lambda^2}\right) + \frac{\|\varepsilon\|^2_{H^1_{\mathrm{loc}}}}{\lambda^2} \lesssim \frac{b^4}{\lambda^2}$$

The blue term is reminiscent of the "Kato smoothing effect".
The term $\frac{b^4}{\lambda^2}$ is due to the equation of Q_b (order b only).

Full estimates

- Control of $\frac{b}{\lambda^2}$

$$\left|\frac{b(t_2)}{\lambda^2(t_2)} - \frac{b(t_1)}{\lambda^2(t_1)}\right| \lesssim \frac{b^2(t_1)}{\lambda^2(t_1)} + \frac{b^2(t_2)}{\lambda^2(t_2)} + \frac{\mathcal{F}(t_1)}{\lambda^2(t_1)}$$

- Equation of λ

$$\left|\lambda^2\lambda_t + b\right| \lesssim \|\varepsilon(t)\|^2_{H^1_{\mathrm{loc}}} + |b|^2$$

- Control of ε

$$\frac{\mathcal{F}(t_2)}{\lambda^2(t_2)} + \int_{t_1}^{t_2} \frac{\|\varepsilon(t)\|^2_{H^1_{\mathrm{loc}}}}{\lambda^5} dt \lesssim \frac{\mathcal{F}(t_1)}{\lambda^2(t_1)} + \frac{b^3(t_1)}{\lambda^2(t_1)} + \frac{b^3(t_2)}{\lambda^2(t_2)}$$

Analysis of the **(Exit)** case

Definition of the L^2 **(Exit)** time (α^* small but fixed) :

$$t^* = \sup\{0 < t < T, \text{ such that } \forall t' \in [0,t],\ u(t) \in \mathcal{T}_{\alpha^*}\}$$

where $\mathcal{T}_{\alpha^*}$ is an L^2 tube around the family of solitons:

$$\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 \quad \text{with} \quad \inf_{\lambda_0 > 0,\ x_0 \in \mathbf{R}} \left\| u - \frac{1}{\lambda_0^{\frac{1}{2}}} Q\left(\frac{\cdot - x_0}{\lambda_0}\right) \right\|_{L^2} < \alpha^* \right\}$$

New and general approach to:

1. Construct the minimal mass solution S

2. Prove universality of the **(Exit)** case and a "no-return lemma" based on the properties of S

Existence of a minimal mass solution

Choose a sequence of well-prepared initial data, for example:

$$u_n(0) = Q_{b_n(0)}, \;\; b_n(0) = -\frac{1}{n}, \;\; \|u_n(0)\|_{L^2} - \|Q\|_{L^2} \sim -\frac{c}{n}, \;\; \varepsilon_n(0) = 0$$

(Blowup) and **(Soliton)** are not possible $\Rightarrow$ **(Exit)** regime

$$u_n(t,x) = \frac{1}{\lambda_n^{\frac{1}{2}}(t)} (Q_{b_n(t)} + \varepsilon_n) \left(t, \frac{x - x_n(t)}{\lambda_n(t)} \right)$$

$$(\lambda_n)_t \sim -b_n(0), \quad \lambda_n(t) \sim 1 - b_n(0)t, \quad b_n(t) = b_n(0)\lambda_n^2(t).$$

At the (Exit) time t_n^*: $b_n(t_n^*) = -\alpha^*$, $\lambda_n^2(t_n^*) \sim \frac{b_n(t_n^*)}{b_n(0)} \sim n\alpha^*$ (defocalisation)

Renormalize the solution at t_n^* :

$$v_n(\tau,x)=\lambda_n^{\frac{1}{2}}(t_n^*)u_n(t_n^*+\tau\lambda_n^3(t_n^*),\lambda_n(t_n^*)x+x_n(t_n^*)).$$

$$v_n(\tau,x)=\frac{1}{\lambda_{v_n}^{\frac{1}{2}}(\tau)}(Q_{b_{v_n}}+\varepsilon_{v_n})\left(\tau,\frac{x-x_{v_n(\tau)}}{\lambda_{v_n(\tau)}}\right)$$

$$\begin{aligned}\lambda_{v_n}(\tau)&\sim\frac{1}{\lambda_n(t_n^*)}\left[1-b_n(0)(t_n^*+\tau\lambda_n^3(t_n^*))\right]\\&\sim\frac{1}{\lambda_n(t_n^*)}\left[\lambda_n(t_n^*)-\tau b_n(0)\lambda_n^3(t_n^*)\right]=1-\tau b_n(t_n^*)=1+\tau\alpha^*.\end{aligned}$$

Mass, energy conservation and $\varepsilon_n(0)=0\Rightarrow\sup_\tau\|\varepsilon_{v_n}\|_{H^1}\leq\delta(\alpha^*)$.

Extract a weak limit $v_n(0)\rightharpoonup v(0)$ in H^1 weak such that the corresponding solution $v(\tau)$ blows up backwards at $\tau^*\sim-\frac{1}{\alpha^*}$. Moreover, $\|v(0)\|_{L^2}\leq\|Q\|_{L^2}$ by weak limit and blow up yields

$$\|v(0)\|_{L^2}=\|Q\|_{L^2}.$$

Description of the *general* (Exit) scenario

PROP Let $(u_n(0))$ be a sequence in H^1 satisfying:

1. $u_n(0) \in \mathcal{A}$;
2. $\|u_n(0) - Q\|_{H^1} \leq \frac{1}{n}$;
3. the solution u_n satisfies the **(Exit)** scenario

Then, there exists $\tau^* = \tau^*(\alpha^*)$ such that

$$\lambda_n^{\frac{1}{2}}(t_n^*) u_n\left(t_n^*, \lambda_n(t_n^*) \cdot + x_n(t_n^*)\right) \to \lambda_S^{\frac{1}{2}}(\tau^*) S\left(\tau^*, \lambda_S(\tau^*) \cdot + x_S(\tau^*)\right)$$

in L^2 as $n \to +\infty$.

The idea of the proof is similar as before, except that the H^1 bound is lost for general (not well-prepared) initial data.

The uniqueness of S is decisive.