Arnold diffusion, weak KAM theory I

Ascona

1.

1. Motivation
2. Arnold Conjecture
3. Growth of Sobolev norms
4. Diffusion Mechanism
5. weak KAM theory
6. Hamilton-Jacobi equ.
6. Lax-Oleinik operator

2.

1. Motivation $q \in \mathbb{T}^n$

$H: T^*\mathbb{T}^n \to \mathbb{R}$ Hamiltonian.

$(q, p)$

$$\begin{cases} \dot q = -\partial_p H \\ \dot p = \partial_q H \end{cases} \qquad S_E = \{H = E\}$$

$\Phi_t$ - time $t$ flow

Ergodic Hypothesis (EH)

Is $\Phi_t|_{S_E}$ typically ergodic?

3.

KAM No

$H_\varepsilon(q,p) = H_0(p) + \varepsilon H_1(q,p)$

KAM Theorem. All but $\sim\sqrt{\varepsilon}$ set of initial conditions on $\mathbb{T}^n \times B^n$ have quasiperiodic orbits

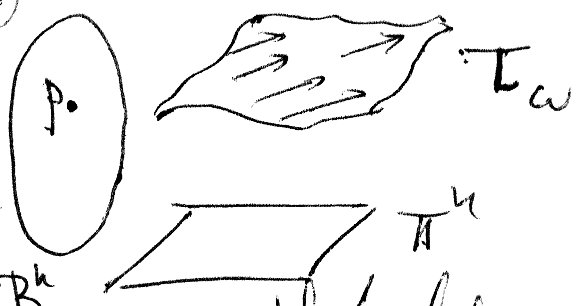
$n=2$ No diffusion

Hamilton-Jacobi $H_\varepsilon(q, \omega + \varepsilon\nabla u(q)) = \alpha_\omega$

$T_\omega = \mathrm{Graph}_\omega = \{(q, \omega + \varepsilon\nabla u(q)) : q \in \mathbb{T}^n\}$

$\omega(p_*) = \partial_p H_0(p)$

4.




n - smooth/analytic

QuasiEH: Does $\Phi_t|_{S_E}$ typ have a dense orbit?

$H_0$ convex in $p$, e.g. $H_0(p) = \frac{p^2}{2}$

## Arnold '63, '64 $\leftrightarrow$ Conjecture (5)

For a generic $\varepsilon h$
$\exists\ (q,p)(t),\ t>0$ st.

weak form
$$\sup |p(t)-p(0)| > O(1) \text{ indep of } \varepsilon$$

strong form
$$\bigcup_{t>0} p(t)\ \text{~~bands~~ "shadows"}\ B^n$$

## Thm 1 [Bernard-K-Zhang] (2)

$\forall n>2$ weak form holds.
For a generic $\varepsilon h$ $\exists (q,p)(t),\ t\geq 0$
$$\sup|p(t)-p(0)| > \ell(H_1) > 0 \text{ indep of } \varepsilon$$

## Thm 2 [BKZ, KZ1, KZ2]

$n=3$ strong form holds $\forall \gamma>0$
For a generic $\varepsilon h$ $\exists (q,p)(t),\ t\geq 0$
$$\bigcup_t p(t) \text{ is } \gamma\text{-dense}$$

(7-8)

## Related problems: (6)

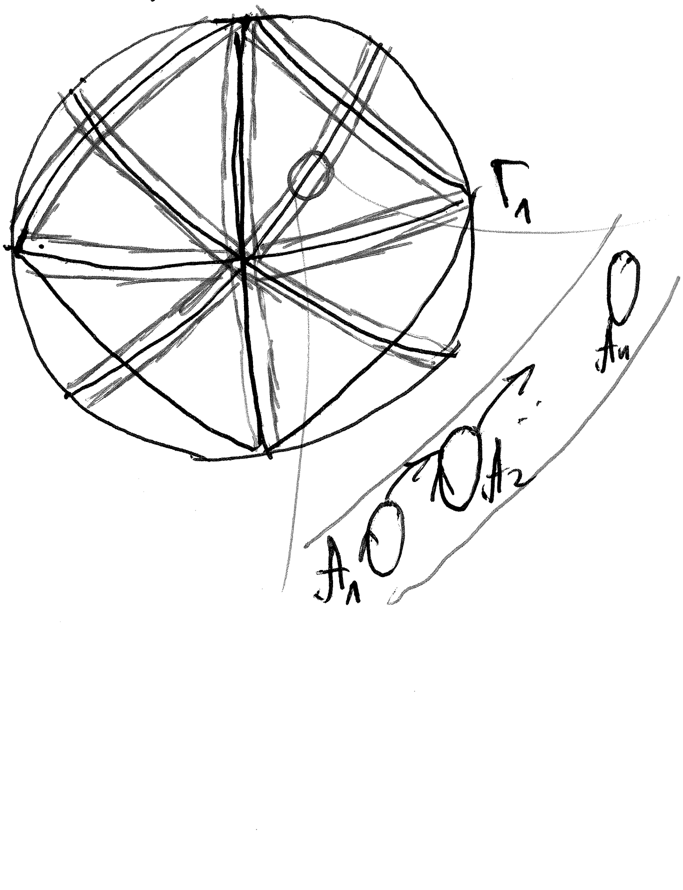
periodic NLS
$$i u_t + \Delta u + \lambda u|u|^{p-2} = 0$$
$$u(0)=\varphi \in H^s(\mathbb{T}^d)$$
$$u(x,t) = \sum_k e^{ikx}\hat{a}_k(t)$$
$\lambda=0$ $|a_k(t)| = |a_k(0)|\ \forall t$

Bourgain: $\exists$? ~~the~~ solution $u(t)$ whose higher Sobolev norms grow?

I-team Yes
Guardia-K

## Diffusion Mechanism

Resonant web
$$S_{1/2} = \left\{ \frac{|p|^2}{2} + \varepsilon H_1(q,p) = \frac{1}{2} \right\} \approx \mathbb{S}^2$$
$|p| \approx 1$ $\quad \Gamma_k = \{ k^{av}(p) = 0\}$



## Diffusion Mechanism

(2)

- Find enough many (quasi) periodic solutions $u_\omega : \mathbb{T}^n \to \mathbb{R}$
- Find heteroclinic orbits
- Shadow a chain

$$H_\varepsilon(q, \omega_i + \nabla u_{\omega_i}(q)) = E$$

## Large Gap Problem

(3)

- If $\{u_{\omega_i}\}$ are smooth/analytic (KAM like) $\mathrm{dist}_p(u_{\omega_i}, u_{\omega_{i+1}}) \gtrsim \sqrt{\varepsilon}$
- Heteroclinics have $\mathrm{diam}_p \sim \varepsilon$

## Weak KAM theory

(5)

$H$ - convex, superlinear

Thm A (Lions-Papanicolaou-Varadhan, Fathi, Bernard)

$u : \mathbb{T}^n \to \mathbb{R}$ is subsolution of (HJ) if Lipschitz and

$$H(q, \omega + \nabla u(q)) \le \alpha \quad \text{for a.e. } x$$

$\Leftrightarrow$ viscosity

$\exists$ $C^1$ subsolution

Bernard $\exists\, C^{1,1}$ sub-solution

Typically $C^{1,1}$-sub-solutions are dense

$$\exists!\, \alpha \;\; T_\omega u_\omega = u_\omega + \alpha$$

(6)

Thm B $\exists!$ nonempty compact $\tilde{A}(\omega) \subset \mathbb{T}^n \times \mathbb{R}^n$

- $\tilde{A}(\omega)$ is $\Phi_t$-invariant
- $\tilde{A}(\omega) \subset \mathrm{Graph}_\omega = \{(q, \omega + \nabla u(q))\}$
- Away from $\pi \tilde{A}(\omega)$ $H(q, \omega + \nabla u(q)) < \alpha$.

! Usually Cantor sets

Lax-Oleinik operator

$L(q, \dot{q})$ - Legendre transform of $H_\varepsilon(q,p)$

$$T_\omega u(q) = \min_{\substack{\gamma(0)=q' \\ \gamma(1)=q}} u(q') + \int_0^1 L(\gamma(\tau), \dot\gamma(\tau)) - \omega \cdot \dot\gamma(\tau)\, d\tau$$

$T_\omega : C^0 \circlearrowleft$ contracting