KAM TYPE THEOREMS FOR NON-LINEAR HAMILTONIAN PDE'S

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ABSTRACT. We concentrate on higher space dimensions with periodic boundary conditions and the basic model is the non-linear Schrödinger equation. We discuss some questions which are mostely open and we illustratete these problems by a list of partial results. We've tried to cite most people who has made important contributions to the field, but the list does not claim to be in any sense exhaustive.

This is a survey lecture given at the conference in "Non-linear hamiltonian PDE's" in Ascona, July 1-6, 2012.

1. Quasi-periodic solutions and KAM-tori

We consider the d-dimensional nonlinear Schrödinger equation

$$(NLS) \qquad \frac{1}{i}\dot{u} = \Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u})$$

under the periodic boundary condition $x \in \mathbb{T}^d$. The convolution potential $V : \mathbb{T}^d \to \mathbb{C}$ is analytic with real Fourier coefficients $\hat{V}(a),\ a \in \mathbb{Z}^d$. F is an analytic function in $\Re u$, $\Im u$ and x.

When $\varepsilon = 0$, then

$$(*) \qquad u(t, x) = \sum_{a \in \mathcal{A}} \hat{u}(a) e^{i(|a|^2 + \hat{V}(a))t} e^{i\langle a, x\rangle}$$

is a quasi-periodic solution of (NLS) with frequencies

$$\omega_a = |a|^2 + \hat{V}(a)$$

– here $\mathcal{A}$ is a finite subset of $\mathbb{Z}^d$ and the amplitudes $\hat{u}(a) \neq 0,\ a \in \mathcal{A}$.

When $\varepsilon \neq 0$?

Date: August 27, 2012.

1.1. **A Hamiltonian system.** Let

$$\mathcal{Y}_s = H_s(\mathbb{T}^d, \mathbb{C}) \times H_s(\mathbb{T}^d, \mathbb{C}) \approx H_s(\mathbb{T}^d, \mathbb{C}^2) = \{(u, v = \bar{u})\},$$

$s > \frac{d}{2}$, and consider the symplectic 2-form

$$((u, v), (\tilde{u}, \tilde{v})) \mapsto \int_{\mathbb{T}^d} u\tilde{v} - \tilde{u}v dx.$$

By contraction on the first variable, the symplectic form defines a mapping $\mathcal{Y}_s \to \mathcal{Y}_s^*$ with inverse

$$J_\Omega : \mathcal{Y}_s^* \to \mathcal{Y}_{-s}.$$

For any smooth function H on $\mathcal{Y}_s$, $J_\Omega dH$ is the Hamiltonian"vector field" associated to H.

Let

$$H(u, v) = i \int_{\mathbb{T}^d} \nabla u \nabla v + ... + \varepsilon F(x, u, v) dx.$$

Then the Hamiltonian vectorfield associated to the 1-form $dH_{(u,v)}$ is

$$\begin{cases} i(\Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial v}(x, u, v) \\ -i(\Delta v + \overline{V(x)} * v + \varepsilon \frac{\partial F}{\partial u}(x, u, v)) \end{cases}$$

which preserves the subspace $\{(u, \bar{u})\}$. On this subspace the Hamiltonian equations becomes

$$\dot{u} = i(\Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u})$$

and it's complex conjugate.

1.2. **The Fourier transform.** By the Fourier transform

$$H^s(\mathbb{T}^d, \mathbb{C}^2) \ni (u, v) \longleftrightarrow (\hat{u}, \hat{v}) \in l_s^2(\mathbb{Z}^d, \mathbb{C}^2)$$

the symplectic form becomes

$$\sum_{a \in \mathbb{Z}^d} d\hat{u}(a) \wedge d\hat{v}(a)$$

and the Hamiltonian becomes

$$H = i[\sum_{a \in \mathbb{Z}^d} (|a|^2 + \hat{V}(a))\hat{u}(a)\hat{v}(a) \ + \epsilon F].$$

1.3. **Action angle variables.** Define for $a \in \mathcal{A} \subset \mathbb{Z}^d$

$$\begin{cases} \hat{u}(a) = \sqrt{2(r_a^0 + r_a)} e^{i\varphi_a} \\ \hat{v}(a) = \sqrt{2(r_a^0 + r_a)} e^{-i\varphi_a}, \end{cases}$$

and for $a \in \mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A}$

$$\begin{cases} u_a = \hat{u}(a) \\ v_a = \hat{v}(a). \end{cases}$$

The symplectic form now becomes

$$\sum_{a \in \mathcal{A}} d\varphi_a \wedge dr_a + \sum_{a \in \mathcal{L}} du_a \wedge dv_a$$

and the Hamiltonian becomes

$$H = \sum_{a \in \mathcal{A}} \omega_a r_a \ + \ \sum_{a \in \mathcal{L}} \Omega_a u_a v_a + \epsilon F,$$

$$\Omega_a = |a|^2 + \hat{V}(a).$$

H is areal holomorphic on some complex domain $\mathcal{D}(\rho, \sigma)$:

$$\|(\varphi, r, u, v)\|_{\rho,\sigma} = \frac{1}{\rho} |\Im \varphi| + \frac{1}{\sigma^2} |r| + \frac{1}{\sigma} \|(u, v)\|_s < 1.$$

Then $H + \varepsilon F$ is analytic and the Hamiltonian equations have a well-defined local flow.

When $\varepsilon = 0$, this system has an invariant torus

$$\Gamma = \mathbb{T}^d \times \{r = 0\} \times \{u = v = 0\}$$

with induced flow $\varphi \mapsto \varphi + t\omega$. This corresponds to the quasi-periodic solution $(*)$ of the (linear) Schrödinger equation when

$$\hat{u}_a^0 = \sqrt{2 r_a^0} e^{i\varphi_a}, \quad a \in \mathcal{A}.$$

The ω_a's are the *(basic or tangential) frequencies* and the Ω_a's are the *normal frequencies* of the torus (solution).

1.4. **External parameters.** This is a standard form for the perturbation theory of lower-dimensional (isotropic) tori with one exception: it is strongly degenerate. We therefore need external parameters to control the basic frequencies and the simplest choice is to let the basic frequencies (i.e. the potential itself) be our free parameters. The parameters will belong to a set

$$U \subset \{\omega \in \mathbb{R}^{\mathcal{A}} : |\omega| \leq C\}$$

and the Hamiltonian H_ω depend on ω.

1.5. **Results.** The normal frequencies will be assumed to verify

$$\begin{array}{ll} |\Omega_a|,\ |\Omega_a+\Omega_b|\ge C' & \forall\, a,b\in\mathcal{L}\,, \\ |\Omega_a-\Omega_b|\ge C' & \forall\, a,b\in\mathcal{L}, |a|\neq|b|. \end{array}$$

This will be fulfilled, for example, if $\mathcal{A}$ is sufficiently "large", or if V is small and $\mathcal{A}\ni 0$.

Theorem-QP. *Under the above assumptions, for ε sufficiently small there exist a subset $U'_\varepsilon\subset U$, which is large in the sense that*

$$\lim_{\varepsilon\to 0}\mathrm{Leb}\,(U\setminus U'_\varepsilon)=0,$$

and for each $\omega\in U'$, a real holomorphic symplectic diffeomorphism

$$\Phi:\mathcal{D}(\frac{\rho}{2},\frac{\sigma}{2})\to\mathcal{D}(\rho,\sigma)$$

and a vector $\omega'\in\mathbb{R}^{\mathcal{A}}$ such that

$$(H_{\omega'}+\varepsilon F)\circ\Phi(\varphi,r,u,v)=<\omega,r>+\mathcal{O}^2(r-r^0,u,v)$$

Moreover

$$\|\Phi-id\|_{\frac{\rho}{2},\frac{\sigma}{2}}\lesssim\varepsilon,$$

and the mapping $\omega\mapsto\omega'(\omega)$ verifies

$$|\omega'-\mathrm{id}|_{\mathcal{C}^1(U')}\lesssim\varepsilon.$$

Then $(H_{\omega'}+\varepsilon F)\circ\Phi$ has an invariant torus $\Gamma(r^0)$ with induced flow $\varphi\mapsto\varphi+t\omega$. The linearized equation of $(H_{\omega'}+\varepsilon F)\circ\Phi$ on this torus becomes, in the (u,v) direction

$$(**)\qquad \frac{d}{dt}\begin{pmatrix}\hat u\\ \hat v\end{pmatrix}=\begin{pmatrix}+i & 0\\ 0 & -i\end{pmatrix}\left[\begin{pmatrix}\Omega & 0\\ 0 & \Omega\end{pmatrix}+\varepsilon A(\varphi)\right]\begin{pmatrix}\hat u\\ \hat v\end{pmatrix}.$$

Theorem-KAM. *Under the above assumptions, for ε sufficiently small there exist a subset $U'_\varepsilon\subset U$, which is large in the sense that*

$$\lim_{\varepsilon\to 0}\mathrm{Leb}\,(U\setminus U'_\varepsilon)=0,$$

and for each $\omega\in U'$, a real holomorphic symplectic diffeomorphism

$$\Phi=\Phi_\omega:\mathcal{D}(\frac{\rho}{2},\frac{\sigma}{2})\to\mathcal{D}(\rho,\sigma)$$

and a vector $\omega'\in\mathbb{R}^{\mathcal{A}}$ such that $(H_{\omega'}+\varepsilon F)\circ\Phi$ equals

$$<\omega,r>+<u,Qv>+\mathcal{O}^3(r-r^0,u,v)$$

where and Q is a Hermitian and block-diagonal matrix with finite-dimensional blocks.

Moreover

$$\|\Phi - id\|_{\frac{\rho}{2},\frac{\sigma}{2}} \lesssim \varepsilon,$$

and the mapping $\omega \mapsto \omega'(\omega)$ *verifies*

$$|\omega' - \mathrm{id}|_{\mathcal{C}^1(U')} \lesssim \varepsilon.$$

Now the linearized equation of $(H_{\omega'} + \varepsilon F) \circ \Phi$ on the torus $\Gamma(r^0)$ becomes, in the (u, v) direction

$$(**) \qquad \frac{d}{dt}\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} +i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} \Omega + \varepsilon Q & 0 \\ 0 & \Omega + \varepsilon Q \end{pmatrix}\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}.$$

Since Q is Hermitian and block diagonal the eigenvalues of equation are purely imaginary

$$\pm i\Omega'_a = \Omega_a + \mathcal{O}(\varepsilon), \quad a \in \mathcal{L}$$

– the normal frequencies of the perturbed torus.

1.6. **Reducibility.** Hence, what the KAM-theorem achieves more than the QP-theorem is the *reducibility* of the variational equation of the torus. Reducibility is a common phenomena for finite-dimensional linear quasi-periodic co-cycles near constant coefficients, but in infinite dimension the results are more sparse.

Consider

$$(QPS)_\omega \qquad \frac{1}{i}\dot{u} = \Delta u + \varepsilon V(t\omega, x)u$$

where $V : \mathbb{T}^n \times \mathbb{T}^d \to \mathbb{R}$ is real analytic.

Theorem-RED. *For* ε *sufficiently small there exist a subset* $U'_\varepsilon \subset U$*, which is large in the sense that*

$$\lim_{\varepsilon \to 0} \mathrm{Leb}\,(U \setminus U'_\varepsilon) = 0,$$

and for each $\omega \in U'$*, a real holomorphic mapping*

$$\Phi : \{|\Im\varphi| < \frac{\rho}{2}\} \to \mathcal{B}(H^0(\mathbb{T}^d))$$

and a bounded Hermitian operator Q *on* $H^0(\mathbb{T}^d)$ *such that* $v(t) = \Phi(t\omega)u(t)$ *solves*

$$\frac{1}{i}\dot{v} = \Delta v + \varepsilon Q v$$

if, and only if, $u(t)$ *solves* $(QPS)_\omega$.

Moreover Q *is "block diagonal" and*

$$\|Q\|_{H^s,H^s} = \|Q\|_{H^0,H^0}\,, \quad \forall s$$

and

$$\|\Phi(\varphi) - id\|_{H^s,H^s} \lesssim \varepsilon, \quad \forall \varphi, \ \forall s.$$

1.7. **Some references.** In finite dimension contributions during the last fifty years, starting with the seminal works of Kolmogorov, Arnold and Moser, have produced a fairly complete picture of the perturbative situation for non-linear Hamiltonian systems. Results for isotropic tori was claimed by Melnikov in the late 60's but the first proof was provided in Eliasson '85.

For PDE's in one space dimension ($d = 1$) the theory is also fairly well developed by Kuksin '89, Craig, Wayne, Pöschel and others – see the books by Kuksin and Craig and references therein.

In higher space dimension ($d \geq 2$) , Theorem-QP is due to Bourgain '04 (at least when F is idependent of x) and Theorem-KAM is due to Eliasson&Kuksin '10.

In $d \geq 2$ the results are more sparse than in $d = 1$. For example, for the non-linear wave equation

$$(NLW) \qquad \ddot{u} = \Delta u + V(x) * u + \varepsilon F(u)$$

Theorem-QP holds (Bourgain '04), but a statement like Theorem-KAM has not been proven.

The reducibility theory is also very well developped for quasi-periodic co-cycles in finite dimension (see for example Chavaudret '09 for a recent result) but for PDE's the situation is less well understood. Theorem-RED is due to Eliasson&Kuksin '09 and similar results for some versions of the harmonic oscillartor have been obtained by Bambusi&Graffi '01, Liu&Yuan '09 and Grebert&Thomann '10.

2. Variants and questions

2.1. **One external parameter.** In the proof of theorem KAM one needs conditions on the small divisors of the type (for example)

$$|\langle k, \omega\rangle + (\Omega_b - \Omega_a)| \geq \kappa \, |k|^{-\tau} \, e^{-(\log|a-b|)^2}, \quad \forall k \in \mathbb{Z}^n \setminus 0, \ \forall a, b \in \mathcal{L}$$

for the normal frequencies

$$\Omega_a = |a|^2 + \hat{V}(a)$$

and for (certain) perturbations of them. The role of the parameters is to assure such conditions.

In Theorem-QP/KAM the number of external parameters ω is the same as the dimension of the torus. In finite dimension it is known

(Eliasson '85) that one parameter

$$\omega = \lambda\omega_0, \quad \lambda \in \text{ interval } \subset \mathbb{R}$$

is sufficient, if ω_0 is a Diophantine vector. (Under an appropriate non-degeneracy condition one may even allow Ω to depend on ω.)

Is the same true in theorem QP and/or KAM? There are certain results to this effect: Geng&Ren '10 and Berti&Biasco '11 for NLW in one dimension; Berti&Bolle '12 for a version of the NLS any dimension –

$$\frac{1}{i}\dot{u} = \Delta u + V(x)u + \varepsilon F(t\omega, x, |u|^2)u + \varepsilon G(t\omega, x)$$

2.2. **Internal parameters - "non-degenerate" systems.** Let

$$H = H_2 + H_4 + \mathcal{O}(\varepsilon) = <\omega_0, r> + <u, \Omega v> + \frac{1}{2} <r, Mr> + \mathcal{O}(\varepsilon).$$

Then the change of variables

$$r \mapsto r^0 + r,$$

transforms H to

$$H = <\omega_0 + Mr^0, r> + <u, \Omega v> + \frac{1}{2} <r, Mr> + \mathcal{O}(\varepsilon).$$

If $\det M \neq 0$, then

$$r^0 \mapsto \omega = \omega_0 + Mr^0$$

is a local diffeomorphism.

2.3. **Internal parameters - Birkhoff normal form.** Let $H = H_2 + \mathcal{O}^3(r, u, v)$.

By a Birkhoff normal form, we can transform H to

$$H_2(r, uv) + H_4(r, uv) + \mathcal{O}^3(u, v) + \mathcal{O}^5(r, u, v) + h(\varphi, r, u, v),$$

where h is of order ≤ 4.

If there are no low-order resonances between the frequencies, then $h = 0$, and $H_2 + H_4$ will be (generically) non-degenerate.

But, frequently in PDE's there are low-order resonances between the frequencies, and h is non-zero and depend on φ.

Example (B. Grébert) Consider

$$(Beam) \qquad \ddot{u} = \Delta^2 u + mu + u^3, \quad m \in [1, 2].$$

For a.e. m

$$h(\varphi, r, u, v) = \sum_{\substack{a', b' \in \mathcal{A} \\ |a| = |a'|, \ |b| = |b'|}} (\dots)\sqrt{r_{a'}}e^{i\varphi_{a'}}\sqrt{r_{b'}}e^{i\varphi_{b'}}v_a v_b \ + \ \dots$$

In general, there is not much one can do now! However, if $\mathcal{A}$ has the property

$$a' \neq b' \in \mathcal{A} \Longrightarrow |a'| \neq |b'|,$$

then

$$a' = l(a), \quad b' = l(b)$$

and the "rotations"

$$u_a \mapsto e^{-il(a)} u_a, \quad v_a \mapsto e^{il(a)} v_a,$$

completed to a symplectic transformation in (φ, r, u, v) , transforms H to

$$H_2(r, uv) + H_4(r, uv) + h(r, u, v) + \mathcal{O}^3(u, v) + \mathcal{O}^5(u, v).$$

OBS! this transformation changes the normal frequencies.

The use of such "rotations" is a frequent tool in the theory of reducibility for linear quasi-periodic co-cycles since more than twenty years and it has recently been applied to PDE's: Geng&You '06, Beam; Geng&You '11, 2d-NLS; Procesi&Procesi '11, NLS.

By chosing different sets $\mathcal{A}$ one can violate integrability in different (controlled?) ways and perhaps find interesting new dynamics.

2.4. **Multiplicative potential.**

$$(NLS') \qquad \frac{1}{i}\dot{u} = \Delta u + V(x)u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u})$$

A theorem-KAM requires an analysis and a good control of the Fourier-support of the eigenfunction of the operator

$$-\Delta + V(x) : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d),$$

at least high up in the spectrum. This control is quite good in dimension $d = 1$ and there are results for this model: Chierchia&You '00 and Yuan '06 (1d-NLW).

But for $d \geq 2$ the situation is much more complicated and not well enough understood.

Berti&Bolle '11 proves quasi-periodic solutions for a model of NLS in any space dimension without any information on the eigenfunction... but no reducibility

2.5. **Almost periodic solution.** Perturbations of solutions

$$(*) \qquad u(t,x) = \sum_{a\in\mathcal{A}} \hat{u}(a) e^{i(|a|^2+\hat{V}(a))t} e^{i\langle a,x\rangle}$$

with

$$\#\mathcal{A} = \infty$$

and

$$|\hat{u}(a)| \geq C_1 e^{-C_2|a|}, \quad \forall a.$$

There exists only one result of this type, Bourgain '05, for a particular model of NLS. There are more general results but with much stronger decay like in Geng '12. Very slow decay is obtained in Pöschel '02 but with non-local non-linearities.

3. References

Eliasson, L. H.: Perturbations of stable invariant tori, Report No 3, Inst. Mittag–Leffler (1985), and Ann. Scuola Norm. Sup. Pisa, Cl. Sci., IV Ser. 15, 115–147 (1988)

Kuksin S. B.: Nearly integrable infinite-dimensional Hamiltonian Systems, Springer-Verlag, Berlin, 1993

Kuksin S. B.: Analysis of Hamiltonian PDEs, Oxford University Press, Oxford, 2000

Craig W.: Problèmes de Petits Diviseurs dans les Équations aux Dérivées Partielles, Société Mathématique de France, Panoramas et Synthéses, 9, 2000

Bourgain J.: Green's function estimates for lattice Schrödinger Operators and applications, Annals of Mathematical Studies, Princeton University Press, Princeton, 2004

Eliasson, L.H., Kuksin S.B.: KAM for the nonlinear Schrödinger equation, Ann. of Math. (2) 172, 371–435 (2010)

Chavaudret, C.: Strong almost reducibility for analytic and Gevrey quasi-periodic cocycles, arXiv0912.4814

Eliasson, L.H., Kuksin S.B.: On reducibility of Schrödinger equations with quasiperiodic in time potentials, Comm. Math. Phys. 286, 125–135 (2009)

Bambusi, D., Graffi, S.: Time quasi-periodic unbounded perturbations of quasi-periodic Schrödinger operators and KAM method, Commun. Math. Phys. 219, 465–480 (2001)

Liu, J., Yuan, X.: Spectrum for quantum Duffing oscillator and small divisor equation with large variable coefficient, Comm. Pure Appl. Math. 63, 1145–1172 (2010)

Grébert, B., Thomann, L.: KAM for the quantum harmonic oscillator, Commun. Math. Phys. 307, 383-427 (2011)
Geng J., Ren X.: Lower dimensional invariant tori with prescribed frequency for nonlinear wave equation, J. Differential Equations, 249, 2796–2821 (2010)
Berti M., Biasco L.: Branching of Cantor Manifolds of elliptic tori and applications to PDE's, Commun. Math. Phys. 305, 741–796 (2011)
Berti M., Bolle P.: Sobolev quasi periodic solutions of multidimensional wave equations with a multiplicative potential, arXiv:1202.2424
Geng J., You J.: KAM tori for higher dimensional beam equations with constant potentials, Nonlinearity 19, 2405–2423 (2006)
Geng J., Xu X., You J.: An infinite dimensional KAM theorem and its application to the 2 dimensional cubic Schrödinger equation, Adv. Math. 226, 5361–5402 (2011)
Procesi C., Procesi M.: A normal form for the non-linear Schrödinger equation, Commun. Math. Phys. 312, 501–557 (2012) arXive 1005:3838
Chiercha L., You J.:KAM tori for 1D non-linear wave equations with periodic boundary conditions, Commun. Math. Phys. 211, 497–525 (2000)
Yuan X.: Quasi-periodic solutions for completely resonant non-linear wave equations, J. Differential Equations 230, 213-274 (2006)
Feldman J., Knrrer H., Trubowitz G.: The perturbatively stable spectrum of a periodic Schrödinger operator, Invent. Math. 100, 259–300 (1990)
Feldman J., Knrrer H., Trubowitz G.: Perturbatively unstable eigenvalues of a periodic Schrödinger operator, Comment. Math. Helv. 66, 557–579 (1991)
Wang W.-M.: Eigenfunction localization for the 2D periodic Schrödinger operator, Int. Math. Res. Not. IMRN 8, 1804-1838 (2011)
Bourgain, J.: On invariant tori of full dimension for 1D periodic NLS. J. Functional Anal. 229, 62–94 (2005).
Pöschel, J.: On the construction of almost periodic solutions for a nonlinear Schrödinger equation. Ergodic Th. and Dynam. Systems 22, 1537–1549 (2002).
Geng J.: Invariant tori of full dimenision for a non-linear Schrödinger equation, J. Differential Equations 252, 1-34 (2012)

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