Global well - posedness and scattering for the mass - critical nonlinear Schrödinger equation

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The mass critical NLS

$$iu_t + \Delta u = \mu |u|^{\frac{4}{d}} u,$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d, t \in \mathbb{R}, \quad u(x,t) \in \mathbf{C},$$

Theorem 1. (D., 2009, 2010, 2011)

Given $u_0 \in L^2(\mathbf{R}^d)$, $\mu = +1$, the mass critical NLS is globally well - posed and scattering. When $\mu = -1$, the mass critical NLS is globally well - posed and scattering for $\|u_0\|_{L^2(\mathbf{R}^d)} < \|Q\|_{L^2(\mathbf{R}^d)}$. Q is the unique positive solution to the elliptic problem $\Delta Q + Q^{1+\frac{4}{d}} = Q$.

$e^{it}Q$ solves the mass - critical NLS for $\mu = -1$.

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Energy Critical Problem

$$\begin{aligned} iu_t + \Delta u &= \mu |u|^{\frac{4}{d-2}} u, \\ u(0,x) &\in \dot{H}^1. \end{aligned} \tag{1}$$

$d = 3$, $\mu = +1$, radial - Bourgain, nonradial (CKSTT)

$d = 3$, $\mu = -1$, radial - Kenig - Merle

$d = 4$, $\mu = +1$, radial Bourgain, nonradial Ryckman - Visan

$d = 4$ $\mu = -1$, radial - Kenig - Merle

$d \geq 5$, $\mu = +1$, radial Tao, nonradial Visan.

$d \geq 5$, $\mu = -1$, radial Kenig - Merle, nonradial Killip - Visan.

Mass Critical Problem

$$\begin{aligned} iu_t + \Delta u &= \mu |u|^{\frac{4}{d}} u, \\ u(0,x) &\in L^2. \end{aligned} \tag{2}$$

$d = 1$, $\mu = +1$, D.

$d = 1$, $\mu = -1$, D.

$d = 2$, $\mu = +1$, radial Tao, Killip, Visan, nonradial D.

$d = 2$ $\mu = -1$, radial Tao, Killip, Visan, nonradial D.

$d \geq 3$, $\mu = +1$, radial Tao, Visan, Zhang. nonradial D.

$d \geq 3$, $\mu = -1$, radial Killip, Visan, Zhang. nonradial D.

Definition Well - posedness

The mass - critical NLS is said to be globally well - posed if a solution $u(t,x)$ exists for all time,

$$u(t,x) \in C_t^0(\mathbf{R}; L^2(\mathbf{R}^d)) \cap L_{t,loc}^{\frac{2(d+2)}{d}}(\mathbf{R}; L^{\frac{2(d+2)}{d}}(\mathbf{R}^d)),$$

a solution depends continuously on u_0 in the $L^2(\mathbf{R}^d)$ topology, and the solution is continuous in time. Such a solution is unique.

Definition Scattering

A global solution to the mass critical NLS is said to scatter if there exist $u_\pm \in L^2(\mathbf{R}^d)$ such that

$$\|u(t,x) - e^{it\Delta}u_+\|_{L^2(\mathbf{R}^d)} \to 0, \tag{3}$$

as $t \to +\infty$ and

$$\|u(t,x) - e^{it\Delta}u_-\|_{L^2(\mathbf{R}^d)} \to 0 \tag{4}$$

as $t \to -\infty$. Additionally we say a solution scatters forward in time if it satisfies (3) and backward in time if it satisfies (4).

Theorem

If u solves $iu_t + \Delta u = F$, (p, q), $(\tilde{p}, \tilde{q})$ are admissible pairs,

$$\|u\|_{L_t^p L_x^q(I \times \mathbf{R}^d)} \lesssim_{p,q,\tilde{p},\tilde{q}} \|u(0)\|_{L^2(\mathbf{R}^d)} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}(I \times \mathbf{R}^d)}. \tag{5}$$

A pair (p, q), is called admissible if $\frac{2}{p} = d(\frac{1}{2} - \frac{1}{q})$, $p \geq 2$ for $d \geq 3$, $p > 2$ for $d = 2$, $p \geq 4$ for $d = 1$.

Theorem

(Cazenave and Weissler) For $\mu = \pm 1$ the mass critical NLS is globally well - posed and scattering if $\|u_0\|_{L^2(\mathbf{R}^d)} < \epsilon(d)$, $\epsilon(d)$ sufficiently small.

The mass critical NLS is locally well - posed on some interval $[-T, T]$, $T(\|u_0\|_{H^1}) > 0$.

Theorem

(Weinstein) We have the Sobolev embedding

$$\|f\|_{L_x^{\frac{2(d+2)}{d}}(\mathbf{R}^d)}^{\frac{2(d+2)}{d}} \leq \frac{d}{2(d+2)} \frac{\|f\|_{L^2(\mathbf{R}^d)}^{4/d}}{\|Q\|_{L^2(\mathbf{R}^d)}^{4/d}} \|\nabla f\|_{L^2(\mathbf{R}^d)}^2 \tag{6}$$

Since

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx + \frac{\mu d}{2(d+2)} \int |u(t,x)|^{\frac{2(d+2)}{d}} dx,$$

the mass - critical problem is globally well - posed when $u(0) \in H^1$, $\mu = +1$, or when $u(0) \in H^1$, $\|u(0)\|_{L^2} < \|Q\|_{L^2}$, $\mu = -1$.

Theorem

A solution to the mass - critical NLS is globally well - posed and scattering if and only if a solution u to the mass - critical problem satisfies

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}((-\infty,\infty)\times\mathbf{R}^d)} < \infty. \tag{7}$$

Theorem

(Killip, Tao, Visan, Zhang) Theorem 1 *is true for* $d \geq 2$, $u(0)$ *radial.*

Define the function

$$A_\mu(m) = \sup\{\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbf{R}\times\mathbf{R}^d)} : \|u(t)\|_{L^2} = m,$$

$$\text{u solves the mass critical NLS } \}.$$

Theorem

(Killip, Tao, Visan, Zhang) $A_\mu(m)$ *is a continuous function of* m*.*
$\{m : A_\mu(m) = \infty\}$ *is a closed set.*

$\{m : A_\mu(m) = \infty\}$ has a minimal element m_0.

Theorem

(Killip, Tao, Visan, Zhang) Suppose $m_0 < \infty$ *for* $\mu = 1$, $m_0 < \|Q\|_{L^2}$ *when* $\mu = -1$. *Then there exist functions* $N(t) : [0, \infty) \to (0, \infty)$, $\xi(t), x(t) : [0, \infty) \to \mathbf{R}^d$ *such that for all* $\eta > 0$ *there exists* $C(\eta) < \infty$ *such that*

$$\int_{|x-x(t)| \geq \frac{C(\eta)}{N(t)}} |u(t,x)|^2 dx + \int_{|\xi - \xi(t)| \geq C(\eta) N(t)} |\hat{u}(t,\xi)|^2 d\xi < \eta, \tag{8}$$

$N(0) = 1$, $N(t) \leq 1$ *for* $t \geq 0$, $\xi(0) = x(0) = 0$, $|N'(t)|, |\xi'(t)| \lesssim N(t)^3$, *and*

$$\int_0^\infty \int |u(t,x)|^{\frac{2(d+2)}{d}} dxdt = +\infty. \tag{9}$$

Also for any compact $I \subset [0, \infty)$

$$\int_I N(t)^2 dt \lesssim \int_I \int |u(t,x)|^{\frac{2(d+2)}{d}} dxdt \lesssim 1 + \int_I N(t)^2 dt. \tag{10}$$

Theorem

(Killip, Tao, Visan, Zhang) $[0, \infty)$ *can be divided into intervals of local constancy that satisfy*

$$\int_{J_k} \int |u(t,x)|^{\frac{2(d+2)}{d}} dxdt = 1. \tag{11}$$

On these intervals

$$N(J_k) = \sup_{t \in J_k} N(t) \sim \inf_{t \in J_k} N(t) \sim \int_{J_k} N(t)^3 dt. \tag{12}$$

Remark: Possibly after modifying $C(\eta)$ by a constant we can transform

$$N(t) \mapsto \alpha(t) N(t), \tag{13}$$

for some $\epsilon > 0$,

$$\epsilon < \alpha(t) < \frac{1}{\epsilon}. \tag{14}$$

Study two situations separately.

$$\int_0^\infty N(t)^3 dt < \infty \tag{15}$$

This is called the rapid frequency cascade.

$$\int_0^\infty N(t)^3 dt = +\infty. \tag{16}$$

This is called the quasi - soliton.

Long time Strichartz estimates

Theorem

(D') (2009) Suppose $d > 2$ and $u(t,x)$ is a minimal mass blowup solution in the form of slide 10. Suppose $\int_J N(t)^3 dt = K$, J is a compact subset of $[0,\infty)$. Then

$$\|u_{|\xi-\xi(t)|>M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbf{R}^d)} \lesssim (\frac{K}{M})^{1/2} + \sigma_J(\frac{M}{2}), \tag{17}$$

where $\sigma_J(\frac{M}{2})$ is a frequency envelope that majorizes

$$\inf_{t\in J} \|u_{|\xi-\xi(t)|>\frac{M}{2}}\|_{L_x^2(\mathbf{R}^d)}. \tag{18}$$

A more technical version of this theorem was proved for dimensions $d = 1, 2$ by (D' 2010). In this case we use a norm constructed from the spaces of U_Δ^2 of (Hadac, Herr, and Koch), (Koch and Tataru).

Rapid Frequency Cascade

Suppose $\int_0^\infty N(t)^3 dt = K < \infty$. For $0 \leq s < 1 + \frac{4}{d}$,

$$\|u\|_{L_t^\infty \dot{H}^s([0,\infty)\times \mathbf{R}^d)} \lesssim K^s. \tag{19}$$

This implies $E(t) \equiv 0$, which is a contradiction.

Quasi - soliton

Suppose $\int_J N(t)^3 dt = K$.
In the defocusing case use the interaction Morawetz estimates of (Colliander, Keel, Staffilani, Takaoka, Tao) ($d = 3$), (Tao, Visan, Zhang) ($d \geq 4$), (Colliander, Grillakis, Tzirakis) ($d = 1, 2$), (Planchon, Vega) ($d \geq 1$).

$$N(t)^3 \lesssim \frac{d^2}{dt^2} \int |u(t,x)|^2 |x-y| |u(t,y)|^2 dxdy. \tag{20}$$

$$\begin{aligned} &\frac{d}{dt} \int |u(t,x)|^2 |x-y| |u(t,y)|^2 dxdy \\ &= 2 \int |u(t,y)|^2 Im[\bar{u}(t,x)\partial_j u(t,x)] \frac{(x-y)_j}{|x-y|} dxdy. \end{aligned} \tag{21}$$

Use the long time Strichartz estimates to estimate the error arising from truncating u to frequencies $\leq CK$. Have $K \lesssim o(K)$. Contradicts $K \nearrow \infty$ as $T \to \infty$.

In the focusing case have to construct an interaction Morawetz estimate. In the defocusing case the potential is $\frac{(x-y)_j}{|x-y|}$.

The error estimates apply equally well in the focusing and defocusing case for a potential $a_j(t, x-y)$ such that

1. $|a_j(t,x)| \lesssim 1$,
2. $|\nabla a_j(t,x)| \lesssim \frac{1}{|x|}$ for $d \geq 2$, $\|\nabla a(t,x)\|_{L^1(\mathbf{R})} \lesssim 1$,
3. $a_j(t,x) = -a_j(t,-x)$,
4. $\|\partial_t a_j(t,x)\|_{L^1(\mathbf{R}^2)} \lesssim 1$.

Construction of the interaction Morawetz estimate.

1. $N(t) \equiv 1$, u even, $d = 1$.

2. $N(t) \equiv 1$, $d = 1$.

3. $N(t) \equiv 1$.

4. $\int_0^{+\infty} N(t)^3 dt = +\infty$.

Construction of the interaction Morawetz estimate.

1. $N(t) \equiv 1$, u even, $d = 1$.

Take $\psi(0) = 0$, $\psi'(x) = \phi(x)$, ϕ even, $\phi \equiv 1$ on $[-R, R]$, ϕ supported on $[-2R, 2R]$.

$$\partial_t Im(\bar{u}\partial_j u) = -2\partial_k Re(\partial_k \bar{u}\partial_j u) + \frac{1}{2}\partial_j \partial_k^2(|u|^2) + \frac{2}{d+2}\partial_j(|u|^{\frac{2(d+2)}{d}}). \quad (22)$$

$$M(t) = \int \psi(x) Im[\bar{u}\partial_j u](t,x)dx. \quad (23)$$

$$\begin{aligned} \dot{M}(t) &= 2\int \phi(x)[|u_x|^2 - \frac{1}{3}|u|^6]dx - \frac{1}{2}\int \phi''(x)|u(t,x)|^2 dx \\ &\geq C(\|u\|_{L^2})\|u_x\|_{L^2}^2 - o_R(1) \gtrsim N(t)^2 = N(t)^3. \end{aligned} \quad (24)$$

$N(t) \equiv 1$, $d = 1$.

Let $\psi(0) = 0$, $\psi'(x) = \phi(x)$,

$$\phi(x-y) = \frac{1}{R}\int \chi(x-s)\chi(y-s)ds, \tag{25}$$

$\chi \equiv 1$ on $[-R, R]$, $\chi \equiv 0$ on $|x| \geq R+1$.

$$M(t) = \int \psi(x-y)|u(t,y)|^2 Im[\bar{u}\partial_x u](t,x)dxdy. \tag{26}$$

$$\begin{aligned}\dot{M}(t) = 2\int \psi'(x-y)|u(t,y)|^2[|u_x|^2 - \frac{1}{3}|u|^6]dx \\ -2\int \psi'(x-y)Im(\bar{u}\partial_x u)Im(\bar{u}\partial_x u)dxdy \\ -\frac{1}{2}\int \psi'''(x-y)|u(t,y)|^2|u(t,x)|^2dxdy.\end{aligned} \tag{27}$$

If we could get rid of $-2\int \psi'(x-y) Im(\bar{u}\partial_x u) Im(\bar{u}\partial_x u) dxdy$ we could proceed as in case 1. For any $\xi(s)$,

$$\begin{aligned} &\frac{1}{R}\int \chi(x-s)\chi(y-s)|u(t,y)|^2|u_x|^2 dxdy \\ &-\frac{1}{R}\int \chi(x-s)\chi(y-s) Im(\bar{u}\partial_y u) Im(\bar{u}\partial_x u) dxdy \end{aligned} \tag{28}$$

$$\begin{aligned} &= \frac{1}{R}\int \chi(x-s)\chi(y-s)|u(t,y)|^2|\partial_x(e^{-ix\cdot\xi(s)}u)|^2 dxdy \\ &-\frac{1}{R}\int \chi(x-s)\chi(y-s) Im(e^{ix\cdot\xi(s)}\bar{u}\partial_y(e^{-ix\cdot\xi(s)}u)) \\ &\qquad \times Im(e^{ix\cdot\xi(s)}\bar{u}\partial_x(e^{-ix\cdot\xi(s)}u)) dxdy. \end{aligned} \tag{29}$$

Therefore,

$$\dot{M}(t) \geq C(\|u\|_{L^2})\|u\|_{L^6_x}^6 - o_R(1) \gtrsim N(t)^2 = N(t)^3. \tag{30}$$

3. $N(t) \equiv 1$. In this case

$$M(t) = \int_{S^{d-1}} \int \int \psi((x-y)_\omega)|u(t,y)|^2 Im(\bar{u}\partial_\omega u)(t,x)dxdyd\omega. \quad (31)$$

4. $\int_0^\infty N(t)^3 dt = \infty$.

Here we take

$$M(t) = \int_{S^{d-1}} \int \int \psi((x-y)_\omega \tilde{N}(t))|u(t,y)|^2 Im(\bar{u}\partial_\omega u)(t,x)dxdyd\omega. \quad (32)$$

$\tilde{N}(t) \leq N(t)$, $\tilde{N}(t) \sim N(t)$,

$$\frac{|\tilde{N}'(t)|}{\tilde{N}(t)^3} \leq \frac{|N'(t)|}{N(t)^3}. \quad (33)$$

$$\int |\tilde{N}'(t)|dt \leq \delta(\|u\|_{L^2}) \int N(t)^3 dt. \quad (34)$$