Quasi-linear perturbations of Hamiltonian Klein-Gordon equations on spheres

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1. Quasi-linear perturbations of Hamiltonian systems

Let X be a Riemannian manifold, Δ the Laplace-Beltrami operator on X, $d\mu$ the Riemannian volume. On $L^2(X,\mathbb{R}^2)$ define the scalar product $\langle V, V'\rangle = \int_X V \cdot V' \, d\mu$ and the symplectic form $\omega(V,V') = \langle {}^t JV, V'\rangle$ with $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. For $m \in]0,+\infty[$, set

$$\Lambda_m = \sqrt{-\Delta + m^2}, \;\; G_0(V) = \frac{1}{2}\int_X V \cdot (\Lambda_m V) \, d\mu.$$

Consider the Hamiltonian equation $\dot{V} = X_{G_0}(V)$ with $X_{G_0}(V) = J\nabla G_0(V)$. In terms of a scalar function v, setting $V = \left[\begin{smallmatrix} \Lambda_m^{-1/2}\partial_t v \\ \Lambda_m^{1/2} v \end{smallmatrix}\right]$, this equation is equivalent to

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$$(\partial_t^2 - \Delta + m^2) v = 0.$$

Problem: Study the stability of the zero solution under non-linear perturbations of the Hamiltonian: For Cauchy data of size $\epsilon \to 0+$ in a convenient space included in $H^s, s \gg 1$, show that the solution exists and remains of size $O(\epsilon)$ in H^s over a "long" time interval $]-T_\epsilon, T_\epsilon[$.

Set $G = \sum_{p=0}^{P_0} G_p$ with, for $p \geq 1$,

$$G_0(V) = \frac{1}{2}\int_X (\Lambda_m^{1/2} V)\cdot(\Lambda_m^{1/2} V)\, d\mu,\ G_p(V) = \int_X \underbrace{P_p(\Lambda_m^{1/2} V)}_{\text{pol. hom. of order } p+2} d\mu.$$

Then the perturbed Hamiltonian equation $\dot{V} = X_G(V)$ is equivalent to a quasi-linear scalar equation

$$(\partial_t^2 - \Delta + m^2)v = N(v, \partial v)\partial^2 v.$$

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Dispersive case: $X = \mathbb{R}^d$, $v|_{t=0}, \partial_t v|_{t=0} = O(\epsilon)$ in $C_0^\infty(\mathbb{R}^d)$, with a non-linearity that is not necessarily Hamiltonian.

• $d \geq 3, \epsilon \ll 1$: Global solutions of size ϵ (Klainerman – Vector fields methods; Shatah – Normal forms).

• $d = 2, \epsilon \ll 1$: Same result (Ozawa, Tsutaya and Tsutsumi – Vector fields and normal forms).

• $d = 1, \epsilon \ll 1$: Stability in H^s for $|t| \leq ce^{c/\epsilon^2}$ (Tsutsumi) and for $t \in \mathbb{R}$ under a null condition (D.).

Relation with Germain-Masmoudi-Shatah method of space-time resonances:

- Space resonances $\leftrightarrow$ dispersion $\leftrightarrow$ time decay of linear solutions.
- Time resonances $\leftrightarrow$ "Usual" normal forms.

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Theorem

Assume $X = \mathbb{S}^d$. There is $\mathcal{N} \subset]0, +\infty[$ of zero measure and for any $m \in]0, +\infty[- \mathcal{N}$, any $P \in \mathbb{N}$, there is $s_0 > 0$ and for any $s \geq s_0$, there are $\epsilon_0 > 0, c > 0, C > 0$ so that $\dot{V} = X_G(V)$, $V|_{t=0} = \epsilon V_0$, with $\epsilon \in]0, \epsilon_0[$, $V_0 \in H^s(X, \mathbb{R}^2)$, $\|V_0\|_{H^s} \leq 1$, has a unique solution $V \in C^0(] - T_\epsilon, T_\epsilon[, H^s(X, \mathbb{R}^2))$ with $T_\epsilon \geq c\epsilon^{-P}$. Moreover, $\sup_{|t| \leq T_\epsilon} \|V(t, \cdot)\|_{H^s} \leq C\epsilon$.

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Dimension $d = 1$ ($X = \mathbb{S}^1$): Bourgain, Bambusi (see also Bambusi-Grébert, Grébert).
Dimension $d \geq 2$ ($X = \mathbb{S}^d$): Bambusi-D.-Grébert-Szeftel.
Remarks on the proof: Use a Birkhoff normal forms method to eliminate higher and higher degree terms in Sobolev energy.
Main problem: not lo lose derivatives in the process.
Two sources of potential losses of derivatives:

- Small divisors (see below)
- Quasi-linear character of the equation.

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2. Birkhoff normal forms

Let G be a Hamiltonian, P an integer. Look for $\Theta_s : H^s \to \mathbb{R}$ so that $\Theta_s(V) \sim \|V\|^2_{H^s}$, $V \to 0$ and

$$\frac{d}{dt}\Theta_s(V(t,\cdot)) = O(\|V(t,\cdot)\|^{P+2}_{H^s}), \ V \to 0$$

for any V solution of $\dot{V} = X_G(V)$, $V|_{t=0} = O_{H^s}(\epsilon)$ (This implies the wanted result of almost global existence).

Take $\Theta_s \stackrel{\text{def}}{=} \Theta_s^0 \circ \chi$ where $\Theta_s^0(V) = \frac{1}{2}\langle \Lambda_m^{2s} V, V\rangle$, $\chi : H^s \to H^s$ local symplectomorphism at zero. Then

$$\begin{aligned}\frac{d}{dt}\Theta_s(V(t,\cdot)) &= D\Theta_s \cdot X_G = \{\Theta_s, G\} \\ &= \{\Theta_s^0 \circ \chi, G\} = \underbrace{\{\Theta_s^0, G \circ \chi^{-1}\}}_{=O(\|V\|^{P+2}_{H^s}),\ V \to 0?} \circ \chi.\end{aligned}$$

If $F : H^s \to \mathbb{R}$ is an auxiliary function, if χ is defined as $\chi(V) = \Phi(-1, V)$ where $\dot{\Phi} = X_F(\Phi)$, $\Phi(0, V) = V$,

$$\{\Theta_s^0, G \circ \chi^{-1}\} \sim \left\{\Theta_s^0, \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \mathrm{Ad}^k F \cdot G\right\} = O(\|V\|_{H^s}^{P+2}),\ V \to 0,$$

(with $\mathrm{Ad} F \cdot G = \{F, G\}$).

If $G = \sum_{p=0}^{P_0} G_p$, $F = \sum_{p\geq 1} F_p$ with F_p, G_p homogeneous of degree $p+2$, sorting by homogeneity, one is reduced to solving the homological equation

$$\{\Theta_s^0, -\{F_p, G_0\} + H_p\} = 0 \text{ if } p < P,$$

where H_p depends only on $F_{p'}$ for $p' < p$ and is computed from expressions $\{F_{p'}, G_{p''}\}$ with $p' + p'' = p$.

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3. Solving the homological equation

Model problem: $\dot{u} = i\nabla_{\bar{u}} G(u, \bar{u})$ with $u|_{t=0} = \epsilon u_0$, $u_0 \in H^s(X, \mathbb{C})$ and $G(u, \bar{u}) = \sum_{p=0}^{P_0} G_p(u, \bar{u})$, with

$$G_0(u, \bar{u}) = \int_X (\Lambda_m u)\bar{u}\, d\mu, \;\; G_p(u, \bar{u}) = \int_X (A_p(U)u)\bar{u}\, d\mu,$$

where $\Lambda_m = \sqrt{-\Delta + m^2}$ and $U \to A_p(U)$ is a function homogeneous of degree p in $U = (u, \bar{u})$, defined on H^s, with values in the space of self-adjoint para-differential operators of order one.

One looks for $F_p(U) = \int_X (B_p(U)u)\bar{u}\, d\mu$, with B_p of order one, to be determined. Then

$$\{F_{p'}, G_{p''}\} = i \int_X (\underbrace{[B_{p'}(U), A_{p''}(U)]}_{\text{para-diff. op. of order 1}} u)\bar{u}\, d\mu + \cdots$$

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Case p odd: In this case, one can solve the stronger equation $\{F_p, G_0\} = H_p$, with H_p given of the form $H_p(U) = \int_X (A_p(U)u)\bar{u}\,d\mu$ and F_p looked for as $F_p(U) = \int_X (B_p(U)u)\bar{u}\,d\mu$. This may be reduced to the following problem:

Set $\widetilde{\Lambda} = \sqrt{-\Delta + \left(\frac{d-1}{2}\right)^2}$. Let $\mathcal{U} = (u_1, \dots, u_p) \to A_\ell(\mathcal{U})$ be p-linear on $C^\infty(X)^p$ with values in para-differential operators of order one be given. Look for $\mathcal{U} \to B_\ell(\mathcal{U})$ of the same type so that

$$[B_\ell(\mathcal{U}), \widetilde{\Lambda}] + \sum_{j=1}^{\ell} B_\ell(u_1, \dots, \Lambda_m u_j, \dots, u_p)$$
$$- \sum_{j=\ell+1}^{p} B_\ell(u_1, \dots, \Lambda_m u_j, \dots, u_p) = -iA_\ell(\mathcal{U}) \text{ mod. } O(\epsilon^P).$$

Denote $\mathrm{Spec}\left(\sqrt{-\Delta_{\mathbb{S}^d}}\right) = \{\lambda_n; n \in \mathbb{N}^*\}$, $\lambda_n^2 = (n-1)(n+d-2)$, Π_n the spectral projector associated to λ_n. Substitute $\Pi_{n_j} u_j$ to u_j.

Then $\Lambda_m \Pi_{n_j} u_j = \sqrt{-\Delta + m^2}\,\Pi_{n_j} u_j = \sqrt{m^2 + \lambda_{n_j}^2}\,\Pi_{n_j} u_j$.

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Equivalent equation:

$$[B_\ell(\Pi_{n'}\mathcal{U}), \widetilde{\Lambda}] + G_m(n')B_\ell(\Pi_{n'}\mathcal{U}) = -iA_\ell(\Pi_{n'}\mathcal{U}),$$

where $n' = (n_1, \ldots, n_p)$, $\Pi_{n'}\mathcal{U} = (\Pi_{n_1}u_1, \ldots, \Pi_{n_p}u_p)$,
$G_m(n') = \sum_{j=1}^{\ell} \sqrt{m^2 + \lambda_{n_j}^2} - \sum_{j=\ell+1}^{p} \sqrt{m^2 + \lambda_{n_j}^2}$.

One may choose $\mathcal{N}$ of zero measure such that, if $m \notin \mathcal{N}$, for any $n' \in (\mathbb{N}^*)^p$, one has $d(G_m(n'), \frac{1}{2}\mathbb{Z}) \geq c|n'|^{-L_0}$ (for some $L_0 > 0$).
(Does not use that $X = \mathbb{S}^d$, but only weak separation properties of eigenvalues).

Take $\theta \in C_0^\infty(\mathbb{R})$ with $\theta \equiv 1$ close to zero and define

$$B_\ell(\Pi_{n'}\mathcal{U}) = -\int_0^{+\infty} e^{-it\widetilde{\Lambda}} A_\ell(\Pi_{n'}\mathcal{U}) e^{it\widetilde{\Lambda}} e^{itG_m(n')} \theta(\epsilon t)\, dt.$$

Since $\mathrm{Spec}(\widetilde{\Lambda}) \subset \frac{1}{2}\mathbb{N}$, $t \to e^{-it\widetilde{\Lambda}}$ is 4π-periodic, so

$$e^{-it\widetilde{\Lambda}} A_\ell(\Pi_{n'}\mathcal{U}) e^{it\widetilde{\Lambda}} = \sum_\alpha C_\alpha(\Pi_{n'}\mathcal{U}) e^{it\alpha/2}.$$

Consequently,

$$B_\ell(\Pi_{n'}\mathcal{U}) = -\sum_\alpha C_\alpha(\Pi_{n'}\mathcal{U}) \underbrace{\int_0^{+\infty} e^{it[\frac{\alpha}{2}+G_m(n')]} \theta(\epsilon t)\, dt}_{=O\left(d\left(G_m(n'),\frac{1}{2}\mathbb{Z}\right)^{-1}\right)=O(|n'|^{L_0})} .$$

One shows that $C_\alpha(\mathcal{U})$ are para-differential operators of order one. The loss $|n'|^{L_0}$ represents a loss of derivatives on the coefficients, so that B_ℓ is in the same class. Moreover, B_ℓ is a solution to the equation.

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