Dynamics of vortex filament interactions and Hamiltonian PDEs

Walter Craig

Department of Mathematics & Statistics

McMaster University

July 1 - 6, 2012
Nonlinear Hamiltonian PDEs
Monte Verità, Ascona

Work in collaboration with
Carlos Garcia – Azpeitia   McMaster University and UNAM

Acknowledgements:
NSERC, Canada Research Chairs Program,
Killam Research Fellows Program, Fields Institute

Outline

Vortex filaments

Natural questions in Hamiltonian dynamics

Hamiltonian PDEs

A variational formulation for invariant tori

Vortex filaments in $\mathbb{R}^3$

- **one vortex filament** with uniform vortex strength $\gamma = 1$ is stationary

  $$b(s) = (0, 0, s)$$

  It generates a flow field in $\mathbb{R}^3$ described by

  $$\mathbf{u} = (\partial_y \psi, -\partial_x \psi, 0)$$

  where

  $$\psi = \frac{1}{2} \log(x^2 + y^2) = \frac{1}{2} \log(|z|^2)$$

  is a stream function, and $z = x + iy$ complex horizontal coordinates.

Vortex filament pairs

Two exactly parallel (vertical) vortex filaments evolve as described by point vortices in $\mathbb{R}^2$

- Opposite vorticity $\gamma_1 = 1 = -\gamma_2$, initial configuration

$$b_1(s) = (\tfrac{1}{2}a, 0, s)\ , \qquad b_2(s) = (-\tfrac{1}{2}a, 0, s)$$

  then ballistic linear evolution

$$b_1(s,t) = (\tfrac{1}{2}a, \frac{t}{a^2}, s)\ , \qquad b_2(s,t) = (-\tfrac{1}{2}a, \frac{t}{a^2}, s)$$

- Same vorticity $\gamma_1 = 1 = \gamma_2$ with the above initial configuration have circular orbits with frequency $\omega = a^{-2}$

$$b_1(s,t) = (\tfrac{1}{2}ae^{it/a^2}, s)\ , \qquad b_2(s,t) = (\tfrac{1}{2}ae^{i(t/a^2+\pi)}, s)$$

- **Question:** Consider two near-vertical vortex filaments, slightly perturbed from exactly vertical. Do there persist similar orbital motions, whose time evolution is periodic or quasi-periodic.

  Fix the configuration to be 2π periodic in the vertical z variables.
- In 'center of vorticity' coordinates, describe the horizontal separation of the two vortex filaments by

$$w(s,t) = x(s,t) + iy(s,t)$$

  In a frame rotating with angular frequency ω

$$i\partial_t w + \partial_s^2 w - \omega w + \frac{w}{|w|^2} = 0 \tag{1}$$

- NB: For configurations which are greatly deformed from vertical, this is not an accurate approximation

Hamiltonian PDE

- Set $w = a + v(s,t)$ with $a \in \mathbb{R}$ and $v(s,t)$ a perturbation term,

$$i\partial_t v + \partial_s^2 v - \omega(a+v) + \frac{a+v}{|a+v|^2} = 0 \qquad (2)$$

  by the choice $\omega = a^{-2}$ then $v = 0$ is stationary

- This is a PDE in Hamiltonian form

$$H = \int_0^{2\pi} \tfrac{1}{2}|\partial_s v|^2 + \frac{1}{2a^2}|a+v|^2 - \tfrac{1}{2}\log|a+v|^2\, ds \qquad (3)$$

  Writing $v(s,t) = X(s,t) + iY(s,t)$ the dynamics are given by the Hamiltonian PDE

$$\begin{aligned} \partial_t X &= grad_Y H \\ \partial_t Y &= -grad_X H \end{aligned}$$

Linearized equations

- The linearized equations at equilibrium $(X, Y) = 0$ come from the quadratic Hamiltonian

$$H^{(2)} = \int_0^{2\pi} \tfrac{1}{2}\left[(\partial_s X)^2 + (\partial_s Y)^2 + \frac{2}{a^2} X^2\right] ds$$

- Linearized equations

$$\partial_t X = \mathrm{grad}_Y H^{(2)} = -\partial_s^2 Y$$

$$\partial_t Y = -\mathrm{grad}_X H^{(2)} = \partial_s^2 Y - \frac{2}{a^2} X$$

Linear flow

- Writing in a Fourier basis and using Plancherel
  $X(s) = (1/\sqrt{2\pi}) \sum_{k\in\mathbb{Z}} \hat{X}_k e^{iks}$
  $Y(s) = (1/\sqrt{2\pi}) \sum_{k\in\mathbb{Z}} \hat{Y}_k e^{iks}$

$$H^{(2)} = \sum_{k\in\mathbb{Z}} \tfrac{1}{2}\Big((k^2 + \frac{2}{a^2})|\hat{X}_k|^2 + k^2|\hat{Y}_k|^2\Big)$$

  An infinte series of uncoupled harmonic oscillators, with frequencies $\omega_k = k\sqrt{k^2 \pm (2/a^2)}$

- The solution operator, or the linear flow

$$\begin{aligned}\begin{pmatrix} X(s,t) \\ Y(s,t) \end{pmatrix} &= \Phi(t)\begin{pmatrix} X(s,0) \\ Y(s,0) \end{pmatrix} \\ &= \sum_{k\in\mathbb{Z}} e^{iks}\begin{pmatrix} \cos(\omega_k t) & k^2\sin(\omega_k t)/\omega_k \\ -\omega_k\sin(\omega_k t)/k^2 & \cos(\omega_k t) \end{pmatrix}\begin{pmatrix} \hat{X}_k \\ \hat{Y}_k \end{pmatrix}\end{aligned}$$

Elementary facts

1. All solutions are Periodic, or Quasi-Periodic, or in general Almost Periodic functions of time
2. More specifically, for initial data (X^0, Y^0) the active wavenumbers are $K := \{k \ : \ (\hat{X}_k^0, \hat{Y}_k^0) \neq 0\}$
   The dimension of the frequency basis is

$$m := \dim_{\mathbb{Q}}\big(\mathrm{span}_{\mathbb{Q}}\{\omega_k \ : \ k \in K\}\big)$$

3. Orbit space consists of tori

$$\overline{\mathrm{orbit}}(X^0, Y^0) = \overline{\{\Phi(t)(X^0, Y^0) \ : \ t \in \mathbb{R}\}} = \mathbb{T}^m$$

Periodic (P): $m = 1$
Quasi-Periodic (QP): $1 < m < +\infty$
Almost Periodic (AP): $m = +\infty$

NB: For generic a then $\omega_k(a)$ satisfy $1 \leq m \leq +\infty$

Elementary facts

4. Energy is conserved

$$H^{(2)}(\Phi(t)(X,Y)) = H^{(2)}(X,Y) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \overline{\begin{pmatrix}\hat{X}_k\\ \hat{Y}_k\end{pmatrix}} \begin{pmatrix}k^2+\frac{2}{a^2} & 0\\ 0 & k^2\end{pmatrix} \begin{pmatrix}\hat{X}_k\\ \hat{Y}_k\end{pmatrix}$$

5. Indeed each action variable is conserved

$$I_k = \frac{\sqrt{k^2+(2/a^2)}}{2|k|}|X_k|^2 + \frac{|k|}{2\sqrt{k^2+(2/a^2)}}|Y_k|^2$$

$$\frac{d}{dt}\left(\Phi_k(t)^T \begin{pmatrix}k^2+\frac{2}{a^2} & 0\\ 0 & k^2\end{pmatrix} \Phi_k(t)\right) = 0$$

Hence all Sobolev energy norms are preserved

$$H^{(2)} = \sum_{k\in\mathbb{Z}} \omega_k I$$

$$\|(X,Y)\|_r^2 := \sum_k |k|^{2r} I_k$$

Natural general questions

1. Whether any solutions of the nonlinear problem are Periodic, Quasi Periodic or Almost Periodic

   This refers to the KAM theory for PDEs

2. Whether the action variables $I_k(z)$ are approximately conserved (averaging theory), giving upper bounds on growth of action variables, or on higher Sobolev norms

   This is in the realm of averaging theory for PDEs, including Birkhoff normal forms and Nekhoroshev stability

3. Whether there exist some solutions which exhibit a growing lower bound on the growth of the action variables

   These would be cascade orbits, related to the question of Arnold diffusion

Results

Theorem (C. Garcia & WC (2012))

There exist Cantor families of periodic (i.e. $m = 1$*) solutions of the vortex filament equations* (2) *near the uniformly rotating solution* $v = 0$

Theorem (C. Garcia & WC (in progress))

Given wavenumbers $k_1, \dots k_m$ *there is* $\varepsilon_0 = \varepsilon_0(k_1, \dots k_m)$ *such that for a Cantor set of amplitudes* $(b_1, \dots b_m) \in B_{\varepsilon_0} \subseteq \mathbb{C}^m$ *there exist QP solutions of* (2) *with* m*-many* $\mathbb{Q}$ *independent frequencies* $\Omega_j(b)$*, of the form*

$$v(s,t) = \sum_{j=1}^{m} b_j e^{ik_j s} e^{i\Omega_j(b)t} + \mathcal{O}(\varepsilon^2)$$

Actually, these two theorems hold for any central configuration of vortices. The case of more complex configurations of near-vertical vortices is part of our future research program.

Small divisors

- This is a small divisor problem. The frequencies are $\omega_k = k\sqrt{k^2 \pm (2/a^2)}$.
  The eigenvalues associated with the linearized operator for a solution with temporal quasi-periods $\Omega = (\Omega_1, \ldots \Omega_m) \in \mathbb{R}^m$

$$\lambda_{jk}^{\pm} := k^2 + \frac{1}{a^2} \pm \sqrt{(\Omega \cdot j)^2 + \frac{1}{a^4}}$$

Proposition (small divisors)

For generic Ω the eigenvalues λ_{jk}^{-} accumulate at $\lambda = 0$.
For a set of full measure of Ω the eigenvalues satisfy a diophantine estimate

$$|\lambda_{jk}^{-}| \geq \frac{\gamma}{(|j| + |k|^2)^{m+1/2+}}$$

Hamiltonian PDEs

- Flow in *phase space*, where $z \in \mathcal{H}$ a Hilbert space with inner product $\langle X, Y \rangle_{\mathcal{H}}$,

$$\partial_t z = J \mathit{grad}_z H(z)\ , \quad z(x,0) = z^0(x)\ , \tag{4}$$

- Symplectic form

$$\omega(X,Y) = \langle X, J^{-1} Y \rangle_{\mathcal{H}}\ , \quad J^T = -J\ .$$

- The flow $z(x,t) = \varphi_t(z^0(x))$, defined for $z \in \mathcal{H}_0 \subseteq \mathcal{H}$

- Theorem
  The flow of (4) *preserves the Hamiltonian function:*

$$H(\varphi_t(z)) = H(z)\ , \qquad z \in \mathcal{H}_0$$

Proof: $\frac{d}{dt} H(\varphi_t(z)) = \langle \mathrm{grad}_z H, \dot{z} \rangle = \langle \mathrm{grad}_z H, J \mathit{grad}_z H(z) \rangle = 0\ .$

Invariant tori

- Equations for an invariant KAM torus

$$\Omega \cdot \partial_\theta S = J grad_z H(S)$$

- Linearize at S, set $\delta S = Z$ and place in self adjoint form

$$\Omega \cdot J^{-1} \partial_\theta Z - \partial_z^2 H(S) Z = F$$

  Eigenvalues of the RHS are the small divisors

- A symplectic version of eigenvalue perturbation theory given e-function/e-value pairs (Z, λ)

$$\Big(\Omega \cdot J\partial_\theta + \partial_z^2 H(S)\Big) Z = \lambda Z$$

- Proposition (eigenvalue perturbation theory (a version of the Feynman – Hellmann formula))

$$\partial_\Omega \lambda = -\langle Z | J \partial_\theta Z \rangle$$

- *Proof:* Normalize $\langle Z|Z\rangle = 1$ so that $\langle Z|\partial_\Omega Z\rangle = 0$

$$\big(\Omega \cdot J\partial_\theta + \partial_z^2 H(S)\big)\partial_\Omega Z + J\partial_\theta Z = (\partial_\Omega \lambda) Z + \lambda \partial_\Omega Z \qquad (5)$$

Taking inner products with Z

$$\langle Z|(\Omega \cdot J\partial_\theta + \partial_z^2 H(S))\partial_\Omega Z\rangle + \langle Z|J\partial_\theta Z\rangle = \partial_\Omega \lambda \langle Z|Z\rangle + \lambda \langle Z|\partial_\Omega Z\rangle$$

By the normalization this implies

$$\partial_\Omega \lambda = \langle Z|J\partial_\theta Z\rangle$$

When $\lambda = 0$ it is furthermore

$$\partial_\Omega \lambda = \langle Z|\partial_z^2 H(S) Z\rangle$$

which has a definite sign if the Hamiltonian H is definite on the eigenspace of (Z, λ).

Examples of Hamiltonian PDEs

- **Nonlinear Schrödinger equation**
  Domain $\mathbb{T}^d = \mathbb{R}^d/\Gamma$, for period lattice Γ

$$i\partial_t u - \frac{1}{2}\Delta_x u + Q(x, u, \overline{u}) = 0 \ , \quad x \in \mathbb{T}^d \tag{6}$$

  Hamiltonian

$$H_{NLS}(u) = \int_{\mathbb{T}^d} \tfrac{1}{2}|\nabla u|^2 + G(x, u, \overline{u})\, dx \ , \quad \partial_{\overline{u}} G = Q \ .$$

  Rewritten

$$\partial_t u = i \, \mathrm{grad}_{\overline{u}} H_{NLS}$$

- In many cases the Schrödinger equation admits a phase translational gauge symmetry, where $G = G(x, |u|^2)$

Nonlinear wave equation

- Domain $D = \mathbb{T}^d = \mathbb{R}^d/\Gamma$

$$\partial_t^2 u - \Delta u + g(x,u) = 0 \tag{7}$$

(Alternately, $u = 0$ on the boundary of a domain $D \subseteq \mathbb{R}^d$).

- Energy

$$H(u,p) = \int_{\mathbb{T}^d} \tfrac{1}{2}p^2 + \tfrac{1}{2}|\nabla u|^2 + G(x,u)\,dx \ ,$$

- Equation (7) can be rewritten

$$\begin{aligned} \partial_t u &= \mathrm{grad}_p H(u,p) &= p \\ \partial_t p &= -\mathrm{grad}_u H(u,p) &= \Delta u - \partial_u G(x,u) \ , \end{aligned}$$

in Darboux coordinates, where $g(x,\cdot) = \partial_u G(x,\cdot)$.

Generalized KdV

- **Korteweg – de Vries equation**

$$\partial_t r = \frac{1}{6}\partial_x^3 r - \partial_x(\partial_r G(x,r)) , \quad x \in \mathbb{T}^1 \tag{8}$$

Hamiltonian

$$H_{KdV}(r) = \int_{\mathbb{T}^1} \frac{1}{12}(\partial_x r)^2 + G(x,r)\, dx$$

Rewritten

$$\partial_t r = J \,\mathrm{grad}_r H_{KdV} , \quad \text{where} \quad J = -\partial_x$$

- Completely integrable cases are $G = r^3$ and $G = r^4$.

Euler's equations for free surface water waves

- Fluid domain $\{x \in \mathbb{R}^{d-1}, y \in (-h, \eta(x))\}$, $d = 2, 3$
- Incompressibility and irrotationality

$$\nabla \cdot \mathbf{u} = 0 \,, \qquad \nabla \wedge \mathbf{u} = 0$$

therefore $\mathbf{u} = \nabla\varphi$ where

$$\Delta\varphi = 0$$

- On the solid bottom boundary

$$N \cdot \mathbf{u} = 0$$

- **Euler's equations** for the free surface $\{y = \eta(x)\}$

$$\begin{aligned} \partial_t\eta &= \partial_y\varphi - \partial_x\eta \cdot \partial_x\varphi \\ \partial_t\varphi &= -g\eta - \tfrac{1}{2}|\nabla\varphi|^2 \,, \end{aligned}$$

kinetic and potential energy

- The energy H

$$\begin{aligned} H &= K + V := \int_{\mathbb{R}^{d-1}} \int_h^{\eta(x)} \tfrac{1}{2}|\mathbf{u}|^2 \, dydx + \int_{\mathbb{R}^{d-1}} \int_h^{\eta(x)} gy \, dydx \\ &= \int_{\mathbb{R}^{d-1}} \int_h^{\eta(x)} \tfrac{1}{2}|\nabla\varphi|^2 \, dydx + \int_{\mathbb{R}^{d-1}} \tfrac{g}{2}\eta^2 \, dx - C \, , \end{aligned}$$

  where the constant C can be neglected.
- Rewriting the kinetic energy

$$\begin{aligned} K &= \int_{\mathbb{R}^{d-1}} \int_h^{\eta(x)} \tfrac{1}{2}|\nabla\varphi|^2 \, dydx = - \int_{\mathbb{R}^{d-1}} \int_h^{\eta(x)} \tfrac{1}{2}\varphi\Delta\varphi \, dydx \\ &\quad + \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\varphi N \cdot \nabla\varphi \, dS_{\text{bottom}} + \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\varphi N \cdot \nabla\varphi \, dS_{\text{free surface}} \, . \end{aligned}$$

- The kinetic energy becomes

$$K = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\xi N \cdot \nabla\varphi \, dS_{\text{free surface}} = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\xi G(\eta)\xi \, dx$$

  where $G(\eta)$ is the Dirichlet – Neumann operator.

Dirichlet – Neumann operator

- The Dirichlet – Neumann operator For $\xi(x) = \varphi(x, \eta(x))$

  $$G(\eta)\xi(x) = (\partial_y - \partial_x \eta(x) \cdot \partial_x)\varphi(x, \eta(x)) = R(N \cdot \nabla\varphi)(x, \eta(x))$$

  with $R = \sqrt{1 + |\partial_x \eta|^2}$ a normalization factor so that $G(\eta)$ is self-adjoint on $L^2(dx)$.
- The Hamiltonian

  $$H = K + V = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\xi G(\eta)\xi + \tfrac{g}{2}\eta^2 \, dx$$

Theorem (Zakharov (1968))

Canonical variables for the water waves problem are $(\eta(x), \xi(x))$*, written in Darboux coordinates, with Hamiltonian* H.

Vortex sheets

- Hamiltonian $H = K + V$

$$
\begin{aligned}
H(y,p) &= \tfrac{1}{2}\int \partial_x p[\rho_1 G_1 + \rho G]^{-1}\partial_x p\, dx \\
&+ \tfrac{1}{2}\int y[\rho_1 G_1 - \rho G][\rho_1 G_1 + \rho G]^{-1}\partial_x p\, dx \\
&- \tfrac{1}{2}\int \frac{1}{4} y[\rho_1 G_1 + \rho G] y\, dx + \frac{g(\rho - \rho_1)}{2}\int y^2(x)\, dx
\end{aligned}
$$

- The canonical conjugate variables are

$$
(y, \partial_x p) = (y, \tfrac{1}{2}[\rho_1 G_1 + \rho G](\Psi + \Psi_1) - [\rho_1 G_1 - \rho G] y)
$$

The equations of evolution for a vortex sheet can be written as follows:

$$
\partial_t y = \mathrm{grad}_p H\,, \qquad \partial_t p = -\mathrm{grad}_y H\,. \tag{9}
$$

- Vortex sheet equations exhibit the Kelvin-Helmholtz instability.

A variational formulation for resonant invariant tori

- Mapping of a torus $S(\theta) : \mathbb{T}^m_\theta \mapsto \mathcal{H}$
- Flow invariance $S(\theta + t\Omega) = \varphi_t(S(\theta))$
  Frequency vector $\Omega \in \mathbb{R}^m$
- This implies that both

$$\partial_t S = \Omega \cdot \partial_\theta S \ , \quad \text{and} \quad \partial_t S = J \, \text{grad}_z H(S) \tag{10}$$

- Problem of KAM tori: Solve (10) for $(S(\theta), \Omega)$.
  This is generally a small divisor problem.

  Rewrite (10) in self-adjoint form

$$J^{-1}\Omega \cdot \partial_\theta S - \text{grad}_z H(S) = 0 \ . \tag{11}$$

Space of torus mappings

Consider the space of mappings $S \in X := \{S(\theta) : \mathbb{T}^m \mapsto \mathcal{H}\}$

- Define average action functionals

$$\begin{aligned} \bar{I}_j(S) &= \frac{1}{2}\int_{\mathbb{T}^m} \langle S, J^{-1}\partial_{\theta_j} S\rangle \, d\theta \\ \delta_S \bar{I}_j &= J^{-1}\partial_{\theta_j} S \end{aligned}$$

  The moment map for *mappings*

- The average Hamiltonian

$$\begin{aligned} \overline{H}(S) &= \int_{\mathbb{T}^m} H(S(\theta)) \, d\theta \\ \delta_S \overline{H} &= \mathrm{grad}_z H(S) \end{aligned}$$

A variational formulation

Consider the subvariety of X defined by fixed actions

$$\mathcal{M}_a = \{S \in X : \bar{I}_1(S) = a_1, \dots \bar{I}_m(S) = a_m\} \subseteq X$$

Variational principle: critical points of $\overline{H}(S)$ on $\mathcal{M}_a$ correspond to solutions of equation (11), with Lagrange multiplier Ω.

NB: All of $\overline{H}(S)$, $\bar{I}_j(S)$ and $\mathcal{M}_a$ are invariant under the action of the torus $\mathbb{T}^m$; that is $\tau_\alpha : S(\theta) \mapsto S(\theta + \alpha)\ ,\ \alpha \in \mathbb{T}^m$.

Two questions

- Two questions.
  1. Do critical points exist on $\mathcal{M}_a$?
     Note that the following operators are degenerate on the space of mappings X:
     $$\Omega \cdot J^{-1}\partial_\theta S\,, \qquad \Omega \cdot J^{-1}\partial_\theta S - \delta_S^2 \overline{H}(0)$$
  2. How to understand questions of multiplicity of solutions?
- Answers – proposal in some cases:
  1. Use infinite dimensional KAM theory or the Nash – Moser method, with parameters
     The latter relies on solutions of the linearized equations, via resolvant expansions (Fröhlich – Spencer estimates)
  2. Morse – Bott theory of critical $\mathbb{T}^m$ orbits.

The linearized vortex filament equations

Illustrate this with the linearized vortex filament equations

- The quadratic Hamiltonian

$$H^{(2)} = \int_0^{2\pi} \tfrac{1}{2}\left[(\partial_s X)^2 + (\partial_s Y)^2 + \frac{2}{a^2}X^2\right] ds$$

  with frequencies $\omega_k = k\sqrt{k^2 \pm (2/a^2)}$

- Linearized equations for an invariant torus

$$\Omega \cdot \partial_\theta X = grad_Y H^{(2)} = -\partial_s^2 Y$$

$$\Omega \cdot \partial_\theta Y = -grad_X H^{(2)} = \partial_s^2 Y - \frac{2}{a^2} X$$

- Fourier representation of torus mappings $S(\theta) : \mathbb{T}^m \mapsto M$

$$S(x,\theta) = \sum_{k\in\mathbb{R}} S_k(\theta) e^{iks} = \sum_{k\in\mathbb{R}, j\in\mathbb{Z}^m} S_{jk} e^{ij\cdot\theta} e^{iks}$$

  Eigenvalues $\lambda_{jk}^{\pm} = k^2 + \frac{1}{a^2} \pm \sqrt{(\Omega \cdot j)^2 + \frac{1}{a^4}}$

Null space

- Choose $(\omega_{k_1}, \dots \omega_{k_m})$ linear frequencies, and a frequency vector $\Omega^0 = (\Omega_1^0, \dots \Omega_m^0)$ solving the resonance relations

$$\lambda_{jk}^-(\Omega^0) = 0 \ .$$

- This identifies a null eigenspace in the space of mappings

$$X_1 \subseteq X \ .$$

Proposition

$X_1 \subseteq X$ *is even dimensional;* $\dim(X_1) = 2M \geq 2m$. *It is possibly infinite dimensional*

- Nonresonant case: $M = m$.
- Resonant case: $M > m$.

Lyapunov - Schmidt decomposition

- Decompose $X = \{S : \mathbb{T}^m \mapsto M\} = X_1 \oplus X_2 = QX \oplus PX$.
- Equation (11) is equivalent to

$$Q\big(J^{-1}\Omega \cdot \partial_\theta S \quad - \quad \mathrm{grad}_z H(S)\big) = 0 \; , \tag{12}$$

$$P\big(J^{-1}\Omega \cdot \partial_\theta S \quad - \quad \mathrm{grad}_z H(S)\big) = 0 \; . \tag{13}$$

- Decompose the mappings $S = S_1 + S_2$ as well.
- Small divisor problem for $S_2 = S_2(S_1, \Omega)$, which one solves for $(S_1, \Omega) \in \mathcal{E}$ a Cantor set.

Variational problem reduced to a link

It remains to solve the Q-equation (12). This can be posed variationally (with analogy to Weinstein - Moser theory).

- Define

$$
\begin{aligned}
\bar{I}_j^1(S_1) &= \bar{I}_j(S_1 + S_2(S_1, \Omega)) \\
\overline{H}^1(S_1) &= \overline{H}(S_1 + S_2(S_1, \Omega)) \\
\mathcal{M}_a^1 &= \{S_1 \in X_1 \ : \ \bar{I}_j^1(S_1) = a_j \ , \ j = 1 \ldots m\}
\end{aligned}
$$

- Critical points of $\overline{H}^1(S_1)$ on $\mathcal{M}_a^1$ are solutions of (12) with action vector a.

equivariant Morse – Bott theory

The group $\mathbb{T}^m$ acts on $\mathcal{M}^1_a$ leaving $\overline{H}^1(S_1)$ invariant.
One seeks critical $\mathbb{T}^m$ orbits.

Question: How many critical orbits of $\overline{H}^1$ on $\mathcal{M}^1_a$?

Depends upon its topology.

Conjecture (a reasonable guess)

For given a there exist integers $p_1, \ldots p_m$ such that $\sum_j p_j = M$ and

$$\mathcal{M}^1_a \simeq \otimes_{j=1}^m S^{2p_j - 1}$$

Morse – Bott theory

Check this fact, in endpoint cases.

- Periodic orbits $m = 1$, resonant case $M > 1$.

$$\mathcal{M}_a^1 \simeq S^{2M-1} \,, \qquad \mathcal{M}_a^1/\mathbb{T}^1 \simeq \mathbb{C}P_w(M-1)$$

This restates the estimate of Weinstein and Moser

$$\#\{\text{critical } \mathbb{T}^1 \text{ orbits}\} \geq M$$

Morse – Bott theory

- Nonresonant quasi-periodic orbits $m = M$.

$$\mathcal{M}_a^1 \simeq \otimes_{j=1}^{M} S^1 \ , \qquad \mathcal{M}_a^1/\mathbb{T}^m \simeq \text{a point}$$

  This corresponds to a KAM torus.

- The case $m = 2 \leq M$ occurs in the problem of doubly periodic traveling wave patterns on the surface of water.

$$\mathcal{M}_a^1 \simeq S^{2p-1} \otimes S^{2(M-p)-1}$$

topology of links

Theorem (Chaperon, Bosio & Meersmann (2006))

The topology of links $\mathcal{M}_a^1$ *can be complex. There are cases in which*

$$\mathcal{M}_a^1 \simeq \#_{\ell=1}^{q}(S^{2p_{\ell 1}-1} \otimes \cdots \otimes S^{p_{\ell k}-1}) \;, \quad \sum_j p_{\ell j} = M$$

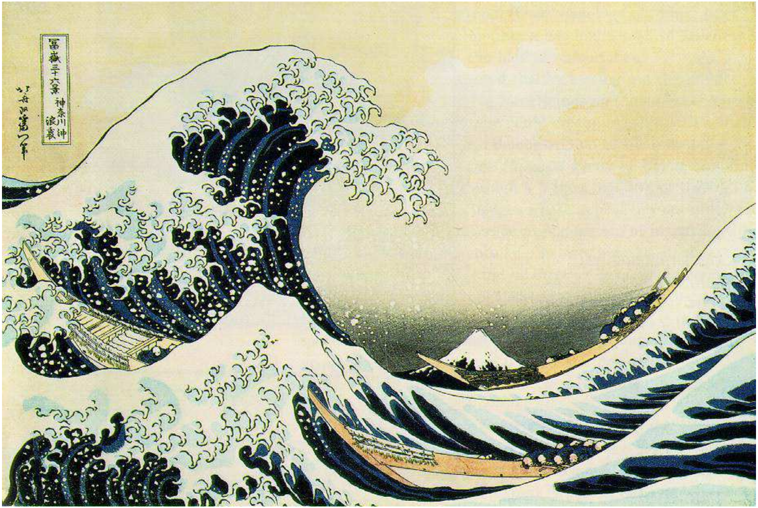
Furthermore, there are more complex quantities than this.

Proof: combinatorics and cohomolological calculations.

Conjecture (revised opinion)

The number of distinct critical $\mathbb{T}^m$ *orbits of* $\overline{H}^1$ *on* $\mathcal{M}_a^1$ *is bounded below:*

$$\#\{\text{critical orbits of } \overline{H}^1 \text{ on } \mathcal{M}_a^1\} \geq (M - m + 1) \;.$$



Thank you