Quasi-periodic solutions of PDEs

Massimiliano Berti
Università Federico II, Napoli

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KAM for PDEs

Goal: quasi-periodic solutions of PDEs like

- **Nonlinear wave equation** (NLW), $d \geq 1$
- **Nonlinear Schrödinger equation** (NLS), $d \geq 1$

- **1d-Derivative-NLW**
- **Fully nonlinear perturbations of KdV**

KAM for PDEs

- **Nonlinear wave equation** (NLW), $d \geq 1$
- **Nonlinear Schrödinger equation** (NLS), $d \geq 1$
  1. any space dimension $x \in \mathbb{T}^d$, $d \geq 1$
  2. Hamiltonian PDEs, semi-linear nonlinearities $f(x, u)$
  3. existence of quasi-periodic solutions,
  4. no-reducibility results, no informations on Lyapunov exponents/stability

- **1d-Derivative-NLW**
- **Fully nonlinear perturbations of KdV**
  1. 1-space dimension $x \in \mathbb{T}^1$
  2. not Hamiltonian, other algebraic structures: reversibililty, ...
  3. quasi-linear/ fully-nonlinear
  4. reducibility results, informations on Lyapunov exponents/stability, ...

Techniques:

- Nash-Moser implicit function theorems
- KAM (Kolmogorov-Arnold-Moser) theory

- Key: new perturbative spectral analysis for the linearized PDE on approximate solutions

A model case: NLW on $\mathbb{T}^d$

$$\textbf{(NLW)} \quad u_{tt} - \Delta u + V(x)u = \varepsilon f(\omega t, x, u)$$

- $x \in \mathbb{T}^d$, $d \geq 1$, periodic boundary conditions
- $\varepsilon > 0$ is small
- $V(x) \in C^k(\mathbb{T}^d; \mathbb{R})$ MULTIPLICATIVE POTENTIAL
- $f \in C^k(\mathbb{T}^\nu \times \mathbb{T}^d \times \mathbb{R}; \mathbb{R})$ FINITELY DIFFERENTIABLE NONLINEARITIES
- $\omega \in \mathbb{R}^\nu$ diophantine, forcing frequencies

INFINITE DIMENSIONAL HAMILTONIAN (LAGRANGIAN) SYSTEM

The problem:

- QUESTION: $\exists$ **quasi-periodic solutions of NLW for** $\varepsilon \neq 0$?

- DEFINITION OF QUASI-PERIODIC SOLUTION:
  $u(\omega t, x)$ where $u(\varphi, x) : \mathbb{T}^\nu \times \mathbb{T}^d \to \mathbb{R}$

The torus-manifold

$$\mathbb{T}^\nu \ni \varphi \mapsto u(\varphi, x) \in \text{Infinite Dimensional Phase Space}$$

is invariant under the flow evolution of NLW

Linear wave equation

$$u_{tt} - \Delta u + V(x)u = 0$$

Solutions: (superposition principle)

$$u(t,x) = \sum_j a_j e^{\mathrm{i}\sqrt{\lambda_j}t} \psi_j(x) \quad \text{where} \quad (-\Delta + V(x))\psi_j = \lambda_j \psi_j$$

Eigenfunctions $\psi_j(x)$ orthonormal in L^2: "NORMAL MODES"
Eigenvalues $\lambda_j \to +\infty$: $\sqrt{\lambda_j}$ = "NORMAL FREQUENCIES"
Periodic, Quasi-Periodic, Almost Periodic solutions

QUESTION: what happens for the nonlinear PDE for $\varepsilon \neq 0$ small?

Functional setting

Look for quasi-periodic solutions

$$u(\omega t, x) \text{ of NLW}$$

$\Longrightarrow$ the embedding $\mathbb{T}^\nu \ni \varphi \mapsto u(\varphi, x)$ solves

$$(NLW) \qquad (\omega \cdot \partial_\varphi)^2 u - \Delta u + V(x) u = \varepsilon f(\varphi, x, u)$$

in the Sobolev space $H^s(\mathbb{T}^\nu \times \mathbb{T}^d; \mathbb{R})$ for some $s \leq k$

$$H^s := \Big\{ u(\varphi, x) := \sum_{(\ell, j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d} u_{\ell, j} e^{\mathrm{i}(\ell \cdot \varphi + j \cdot x)} \ \text{ with}$$

$$\|u\|_s^2 := \sum_{(\ell, j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d} |u_{\ell, j}|^2 (1 + |\ell|^{2s} + |j|^{2s}) < +\infty \Big\}$$

Bifurcation problem: Let $F : [0, \varepsilon_0) \times H^s \to H^{s-2}$ be

$$F(\varepsilon, u) := (\omega \cdot \partial_\varphi)^2 u - \Delta u + V(x)u - \varepsilon f(\varphi, x, u)$$

Look for zeros $F(\varepsilon, u) = 0$.
Small amplitude solutions:

$$F(0, 0) = 0$$

Compute the partial derivative with respect to u at $(\varepsilon, u) = (0, 0)$,

$$D_u F(0, 0) = (\omega \cdot \partial_\varphi)^2 - \Delta + V(x)$$

$$D_u F(0,0) = (\omega \cdot \partial_\varphi)^2 - \Delta + V(x)$$

eigenvectors: $e^{\mathrm{i}\ell\cdot\varphi}\psi_j(x)$, normal modes: $(-\Delta + V(x))\psi_j = \lambda_j \psi_j$

eigenvalues: $-(\omega \cdot \ell)^2 + \lambda_j$

Assumption: NON-RESONANT case: SMALL DIVISORS

$$\left|(\omega \cdot \ell)^2 - \lambda_j\right| \geq \frac{\gamma}{1 + |\ell|^\tau}, \quad \forall (\ell, j), \ \tau > 0$$

$\Longrightarrow D_u F(0,0)$ is invertible, but the inverse is **unbounded**:

$$((\omega \cdot \partial_\varphi)^2 - \Delta + V(x))^{-1} : H^s \to H^{s-\tau}$$

$\tau :=$ "*LOSS OF DERIVATIVES*"

$\Longrightarrow$ classical Implicit function theorem fails

Nash-Moser IFT: Newton method + "Smoothing"

Newton tangent method for zeros of $F(u) = 0$ + "smoothing":

$$u_{n+1} := u_n - S_n (D_u F)^{-1}(u_n) F(u_n)$$

where S_n are regularizing operators

- Advantage: QUADRATIC scheme

$$\|u_{n+1} - u_n\|_s \leq C(n) \|u_n - u_{n-1}\|_s^2$$

$\Longrightarrow$ convergent also if $C(n) \to +\infty$

- Difficulty: invert $(D_u F)(u)$ in a whole neighborhood of the expected solution

Example for NLW: linearized equation on an **approximate** solution

$$h \to (D_u F)(u,\varepsilon)[h] := ((\omega \cdot \partial_\varphi)^2 - \Delta + V(x))h + \varepsilon p(\varphi, x)h$$

$$p(\varphi, x) := (\partial_u f)(\varphi, x, u(\varphi, x))$$

- Linear differential operator with non-constant coefficients
- not diagonal in Fourier basis
- "singular" perturbation problem: $L_\omega^{-1} T$ is unbounded

$$L_\omega := (\omega \cdot \partial_\varphi)^2 - \Delta + V(x)\,, \quad Th := p(\varphi, x)h$$

$$L_\omega^{-1} = \text{order } \tau\,, \quad T = \text{order } 0$$

Literature $d = 1$

- Kuksin '89, Wayne '90. KAM theory, analytic NLS, NLW with DIRICHLET boundary conditions:
  - Eigenvalues of $-\partial_{xx} + V(x)$ are simple $\Longrightarrow$
    2^{th} **order Melnikov of non-resonance conditions OK**
  - $V(x)$ are "parameters"

- $x \in \mathbb{T}$
  - Craig-Wayne '93: periodic solutions
  - Bourgain '94: quasi-periodic solutions

- Lyapunov-Schmidt, Newton method, f analytic,
  1^{th} **order Melnikov non-resonance conditions OK**

dimension $d \geq 2$

Main difficulties:

- 1) the eigenvalues of $-\Delta + V(x)$ appear in clusters of increasing size

For example $-\Delta e^{\mathrm{i} j \cdot x} = |j|^2 e^{\mathrm{i} j \cdot x}$ then $|j|^2 = |j_0|^2$, $j \in \mathbb{Z}^d$

- 2) The eigenfunctions of $-\Delta + V(x)$ may be "NOT localized with respect to exponentials"! (Feldman- Knörrer-Trubowitz)

$\Longrightarrow$ often used pseudo-PDE with Fourier multipliers

$$\mathrm{i} u_t - \Delta u + M_\sigma u = \varepsilon f \ , \quad M_\sigma e^{\mathrm{i} j \cdot x} = m_\sigma e^{\mathrm{i} j \cdot x}$$

and m_σ are used as parameters

Literature: $d \geq 2$: quasi-periodic solutions

- Newton method, 1^{th} order Melnikov
  - Bourgain, Annals '98 ($d = 2$), Annals '05
    NLS and NLW with Fourier multipliers
    - Anderson localization theory: Bourgain, Goldstein, Schlag
    - Polynomial nonlinearities (analytic): semialgebraic and subharmonicity theory for "measure and complexity" estimates
  - Wang, '10- '11 completely resonant NLS-NLW, no external parameters,
  - Berti-Bolle, '10-'12, NLS-NLW, finite regularity, $V(x)$ multiplicative potential
- KAM theory: 2^{th} order Melnikov
  - Kuksin-Eliasson, Annals '10, NLS with Fourier multipliers
  - $d = 2$, Geng-You-Xu, cubic NLS, no external parameters, '10
  - Procesi-Xu '11, Procesi-Procesi '11, any dimension, reducible Birkhoff normal form for completely resonant NLS

New results of quasi-periodic solutions in $d \geq 2$

We look for quasi-periodic solutions of

$$(NLW) \quad (\omega \cdot \partial_\varphi)^2 u - \Delta u + V(x) u = \varepsilon f(\varphi, x, u)$$

with $\mathrm{Ker}(-\Delta + V(x)) = 0$, and

$$\omega = \lambda \bar{\omega} \,, \quad \lambda \approx 1$$

in a FIXED diophantine direction

$$|\bar{\omega} \cdot \ell| \geq \frac{\gamma_0}{|\ell|^{\tau_0}} \,, \ \forall \ell \in \mathbb{Z}^\nu \setminus \{0\} \,, \quad \Big| \sum_{1 \leq i \leq j \leq \nu} \bar{\omega}_i \bar{\omega}_j p_{ij} \Big| \geq \frac{\gamma_0}{1 + |p|^{\tau_0}}$$

for all $p_{ij} \in \mathbb{Z}$ which are not all naught.
In FINITE DIMENSION Eliasson '89 and Bourgain '94

Theorem

(M.Berti, P.Bolle, '11)

Existence: $\exists s := s(d,\nu)$, $k := k(d,\nu) \in \mathbb{N}$, *such that:*

$\forall V, f \in C^k$, *there exist* $\varepsilon_0 > 0$, *such that* $\forall 0 < \varepsilon < \varepsilon_0$, *there exists a map*

$$u(\varepsilon,\cdot) \in C^1([1/2,3/2]; H^s) \quad \text{with} \quad \sup_{\lambda\in[1/2,3/2]} \|u(\varepsilon,\lambda)\|_s \overset{\varepsilon\to 0}{\longrightarrow} 0\,,$$

and a Cantor like set $\mathcal{C}_\varepsilon \subset [1/2,3/2]$ *of asymptotically full Lebesgue measure, i.e.*

$$|\mathcal{C}_\varepsilon| \to 1 \quad \text{as} \quad \varepsilon \to 0,$$

such that, $\forall \lambda \in \mathcal{C}_\varepsilon$, $u(\varepsilon,\lambda)$ *is a solution of NLW with* $\omega = \lambda\bar{\omega}$.

Regularity: *If* $V, f \in C^\infty$ *then* $u \in C^\infty$ *in space and time.*

Smoothness: for $\nu = 1$ (periodic sol.), $d = 1$, we got $k = 6$

Remarks

- A similar result holds for the HAMILTONIAN NLS

$$\mathrm{i}u_t - \Delta u + V(x)u = \varepsilon f(\omega t, x, u, \bar{u})$$

M. Berti, P. Bolle, to appear on **Journal Eur. Math Soc.**

- The restriction of $\mathcal{C}_\varepsilon$ is not technical! Outside: "Chaos", "homoclinc/heteroclinics solutions", "Arnold Diffusion", ... **"Growth of Sobolev norms in 2-d cubic NLS"**

$$\mathrm{i}u_t - \Delta u = |u|^2 u\,, \quad x \in \mathbb{T}^2$$

Colliander-Keel-Staffilani-Takaoka-Tao, Invent. Math. 2010.

- For Differentiable nonlinearities $f \in C^k$ the "chaotic effects" are stronger... and KAM theory more difficult
  - Delort '10, $f \in C^\infty$, periodic sol. of NLS, paradiff. calculus,

- Pre-assigned direction of tangential frequencies
  - for NLW in Geng-Ren, '10,
  - Berti-Biasco, CMP '11.

  Use JUST 1 PARAMETER. See "Degenerate KAM theory"
  -Bambusi-Berti-Magistrelli, JDE '11,
- For "measure and complexity" estimates we use simple eigenvalue variation arguments not sub-harmonicity theory (not available in C^k)
- Many of these results should carry over spheres, Zoll manifolds, Lie groups, homogeneous spaces
  - symmetries and properties of eigenfunctions and eigenvalues
  - For periodic solutions proved in Berti-Procesi, DUKE '11
  - related to Birkhoff normal form results by Bambusi, Delort, Grebert, Szeftel for spheres and Zoll manifolds

About the Proof

KEY STEP: For "most" parameters $\lambda \in [1/2, 3/2]$ the linearized operator

$$\mathcal{L}_\varepsilon(\lambda) := (\lambda \bar{\omega} \cdot \partial_\varphi)^2 - \Delta + V(x) + \varepsilon (\partial_u f)(\varphi, x, u(\varphi, x))$$

is invertible and TAME estimate in HIGHER Sobolev norms, i.e.

$$\|\mathcal{L}_\varepsilon^{-1}(\lambda) h\|_s \leq \|h\|_{s+\tau} \|u\|_{s_0} + \|h\|_{s_0} \|u\|_s \,, \ \forall s_0 \leq s \leq k$$

- **Step 1)** L^2-estimates: lower bounds for the eigenvalues of the **self adjoint** operator $\mathcal{L}_\varepsilon(\lambda)$: eigenvalues are smooth in $\lambda \in [1/2, 3/2]$
- **Step 2)** Tame-estimates in high norm
  KEY OBSERVATION: many eigenvalues are NOT small !

Separation properties of singular sites

Singular sites : $(\ell, j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d$ such that

$$|-(\omega \cdot \ell)^2 + |j|^2 + m| \le \rho$$

must be more and more "rare" as $\rho \to 0$. Integer points near a "cone": the slope ω must be "irrational"

$$\Big| \sum_{1 \le i \le j \le \nu} \bar{\omega}_i \bar{\omega}_j p_{ij} \Big| \ge \frac{\gamma_0}{1 + |p|^{\tau_0}}$$

(this should be the **optimal** -minimal- condition)

NLS $\quad |-\omega \cdot \ell + |j|^2 + m| << 1$

near a "paraboloid": more torsion $\Longrightarrow$ less integers nearby

Two different approaches to KAM theory for PDEs

Solve the linearized equations on approximate solutions using:
"1^{th}-Melnikov conditions" (Nash-Moser)

- $|\omega \cdot \ell - \mu_j(\varepsilon)| \geq \frac{\gamma}{1+|\ell|^\tau}$ minimal assumption ($\mu_j(\varepsilon)$ are the perturbed frequencies of the linearized equations at each iterative step),
- $\Longrightarrow$ works well in case of multiple eigenvalues
- DRAWBACK: linearized eq. with non-constant coefficients

"2^{th}-Melnikov conditions"

- $|\omega \cdot \ell - \mu_j(\varepsilon) + \mu_i(\varepsilon)| \geq \frac{\gamma}{1+|\ell|^\tau}$
- ADVANTAGE: linearized equation with constant coefficients
- ADVANTAGE: $\exists$ torus + REDUCIBLE normal form $\Longrightarrow$ stability results
- the linearized eq. with non-constant coefficients of case 1, can be conjugated to a constant coefficient eq.

KAM for 1-d unbounded perturbations

Kuksin '98 for KdV, Kappeler-Pöschel '03

$$u_t + u_{xxx} + uu_x + \varepsilon \partial_x f(x, u) = 0$$

Liu-Yuan '10 for Hamiltonian DNLS (and Benjamin-Ono)

$$\mathrm{i} u_t - u_{xx} + M_\sigma u + \mathrm{i}\varepsilon f(u, \bar{u}) u_x = 0$$

1. Main difficulty: the vector field whose flow defines usual KAM changes of variables is UNBOUNDED
2. no stability result, non constant coefficients KAM normal form

2^{th} order Melnikov-non-resonance conditions

KdV $$|\omega \cdot \ell + j^3 - i^3| \geq \gamma \frac{j^2 + i^2}{1 + |\ell|^\tau}, \; j \neq i,$$

$\Longrightarrow$ gains 2 derivatives

DNLS $$|\omega \cdot \ell + j^2 - i^2| \geq \gamma \frac{|j| + |i|}{1 + |\ell|^\tau}, \; j \neq i,$$

$\Longrightarrow$ gains 1 derivative

DNLS is less dispersive than KdV

(solutions in Liu-Yuan are C^∞)

The derivative wave equation is not dispersive $\Longrightarrow$ is excluded

Derivative NLW

$$y_{tt} - y_{xx} + my + f(x, y, y_x, y_t) = 0, \quad x \in \mathbb{T}$$

NOT Hamiltonian. For example: there are **no** non-trivial periodic/quasi-periodic solutions of

$$y_{tt} - y_{xx} + my + y_t^3 = 0\,, \quad y_{tt} - y_{xx} + my + y_x^3 + f(y) = 0\,,$$

$$y_{tt} - y_{xx} + my + \partial_x(y^3) + f(y) = 0$$

But: all solutions of

$$y_{tt} - y_{xx} = y_t^2 - y_x^2 \quad \text{are} \quad y = -\ln(p(t+x) + q(t-x))$$

2π-periodic ("null-condition" of Klainerman)

$\exists$ periodic solutions of

$$y_{tt} - y_{xx} + my + y_t^2 = 0\,, \;\; m \neq 0\,, \quad \text{Bourgain '96}\,,$$

The above equations are NOT Hamiltonian but **Reversible** PDE

$$y(t,x) \mapsto y(-t,x)$$

Reversible KAM theory:

- Finite dimension: Moser '67, Arnold, Sevryuk, ...
- Infinite dimension:
  1-d-NLS reversible, Zhang-Gao-Yuan '11

$$\mathrm{i}u_t + u_{xx} = |u_x|^2 u$$

Derivative NLW

$$y_{tt} - y_{xx} + my + f(x, y, y_x, y_t) = 0$$

"reversibility condition"

$$f(x, y, y_x, -v) = f(x, y, y_x, v)$$

it rules out nonlinearities like y_t^3

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ y_{xx} - my - f(x, y, y_x, v) \end{pmatrix} =: F(y, v)$$

$$SF = -FS \quad S(y, v) := (y, -v) \,, \quad S^2 = I \,,$$

"parity condition"

$$f(-x, y, -y_x, v) = f(x, y, y_x, v)$$

it rules out nonlinearities like y_x^3

Theorem

(M.Berti, L. Biasco, M. Procesi '12)
For all $m > 0$*, for every choice of the* "TANGENTIAL SITES"

$$\mathcal{I} := \{j_1, \ldots, j_n\} \subset \mathbb{N} \setminus \{0\},$$

the DNLW eq. satisfying "reversibility"+"parity" conditions, ex.

$$y_{tt} - y_{xx} + my + yy_x^2 = 0,$$

possesses small amplitude, analytic, quasi-periodic solutions, with **zero Lyapunov exponents***, of the form*

$$y = \sum_{j \in \mathcal{I}} \sqrt{\xi_j} \cos(\omega_j^\infty(\xi)\, t) \cos(jx) + o(\sqrt{\xi}), \ \omega_j^\infty(\xi) \overset{\xi \to 0}{\approx} \sqrt{j^2 + m}$$

for a "Cantor-like" set of parameters $\xi \in \mathbb{R}^n$ *with asymptotically density* 1 *at* $\xi = 0$*. The linearized equations on these quasi-periodic solutions are* **reduced to constant coefficients***.*

SOME IDEAS OF PROOFS

1. KAM theory is NOT an Hamiltonian theory: work at level of VECTOR FIELDS (not Hamiltonian), COMMUTATORS (not Poisson brackets),
2. "**reversibility**" and "**parity**" give PURELY REAL corrections to the eigenvalues, i.e. frequencies, which avoids "friction terms" and "secular terms"
3. KEY: Verify the 2^{th}-Melnikov non resonance conditions

$$|\omega \cdot \ell + \mu_j(\varepsilon) - \mu_i(\varepsilon)| \geq \gamma / |\ell|^\tau \, , \ \forall i, j, \ell$$

Usual perturbation theory implies the estimate

$$\mu_j(\varepsilon) = \sqrt{j^2 + m} + O(\varepsilon) \, , \qquad j \to +\infty$$

which is not sufficient... semi-linear nonlinearities:

$$\mu_j(\xi) = \sqrt{j^2 + m} + O(\varepsilon / j)$$

KEY: First order asymptotic expansion

$$\mu_j(\varepsilon) = \sqrt{j^2 + m} + c_\varepsilon + O(\varepsilon/j) = j + c_\varepsilon + O(\tfrac{m}{j})$$

where $c_\varepsilon = O(\varepsilon)$ **is independent of** j
$\Longrightarrow$ in the 2^{th}-Melnikov conditions

$$|\omega \cdot \ell + \mu_j(\varepsilon) - \mu_i(\varepsilon)| \geq \gamma/|\ell|^\tau \,, \ \forall i, j, \ell$$

the difference of c_ε cancels out $\Longrightarrow$ OK
Proof by QUASI-TÖPLITZ VECTOR FIELD (related to Procesi-Xu '11, Eliasson-Kuksin for NLS, see also Grebert-Thomann '11 for harmonic oscillators) Stable under KAM operations:

1. POISSON BRACKETS
2. LIE TRANSFORM
3. SOLUTION OF HOMOLOGICAL EQUATION

Open problem: quasi-linear NLW (Klein-Gordon)?

$$y_{tt} - y_{xx} + my + \varepsilon f(x, y, y_x, y_{xx}) = 0$$

- Difficulty: 2-derivatives in the nonlinearity!

Lax, Zabusky '64, Klainemann-Majda '82

$$y_{tt} - (1 + \varepsilon\sigma(y_x))y_{xx} = 0\,, \quad \sigma(y_x) = y_x^p + \ldots.$$

have NO smooth solutions for all times:

$$\exists T_{crit} > 0 \quad \text{such that } y_{xx} \text{ becomes discontinuous}$$

Rabinowitz '71: periodic solutions of

$$y_{tt} - y_{xx} + \alpha y_t = \varepsilon F(x, t, y, y_t, y_x, y_{tx}, y_{xx}, y_{tt})$$

The small dissipation y_t allows the existence of periodic solutions!

Quasi-linear perturbations of g-Kdv

$$u_t + u_{xxx} + \varepsilon \partial_{xx}(f(\omega t, x, u_x)) = 0\,, \quad x \in \mathbb{T}\,, \ \omega \in \mathbb{R}^\nu$$

Hamiltonian:

$$u_t = \partial_x \nabla_{L^2} H(u)\,, \quad H(u) = \int_{\mathbb{T}} \frac{u_x^2}{2} + \varepsilon F(\omega t, x, u_x) dx$$

Physically important for perturbative derivation from water-waves, ex. Craig

Reversible $f : \mathbb{T}^\nu \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$:

$$-f(\varphi, x, u_x) = f(-\varphi, -x, -u_x)$$

Involution

$$(Su)(x) := u(-x)\,, \quad S^2 = I\,,$$

Theorem

(Baldi, Berti, Montalto , '12) Let $\bar{\omega} \in \mathbb{R}^\nu$ diophantine.
$\exists k := k(\nu) \in \mathbb{N}$ such that:
$\forall f \in C^k$, f reversible, $\forall 0 < \varepsilon < \varepsilon_0$ (small enough), for all λ in a Cantor like set $\mathcal{C}_\varepsilon \subset [1, 2/3/2]$ of asymptotically full Lebesgue measure, i.e.

$$|\mathcal{C}_\varepsilon| \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

there is a **quasi-periodic** *solution $u(\varepsilon, \lambda) \in H^s$, $s \leq k$, even in (t, x), with frequency $\omega = \lambda\bar{\omega}$, of the gKdV equation*

$$u_t + u_{xxx} + \varepsilon\partial_{xx}(f(\omega t, x, u_x)) = 0\,, \quad x \in \mathbb{T}\,.$$

The solution $\|u(\varepsilon, \lambda)\|_s \to 0$ as $\varepsilon \to 0$. The linearized equations on these quasi-periodic solutions are **reduced to constant coefficients** *and they have* **zero Lyapunov exponents**.

Linearized operator for quasi-linear KdV

$$
\begin{aligned}
\mathcal{L} &:= \omega \cdot \partial_\varphi + \partial_{xxx} + \varepsilon \partial_{xx}(p(x,\varphi)\partial_x \,) \\
&= \omega \cdot \partial_\varphi + (1 + \varepsilon p(\varphi, x))\partial_{xxx} + 2\varepsilon p_x \partial_{xx} + \varepsilon p_{xx} \partial_x
\end{aligned}
$$

Main difficulty: the non constant coefficients term $\varepsilon p(\varphi, x)\partial_{xxx}$!

Usual perturbation theory implies the estimate for eigenvalues

$$\mu_j(\varepsilon) = \omega \cdot \ell + j^3 + O(\varepsilon j^3)$$

Not sufficient!

Theorem

Conjugate $\mathcal{L}$ to a diagonal (constant coefficients) linear operator:

$$\Phi^{-1} \circ \mathcal{L} \circ \Phi = \mathrm{diag}\{\mathrm{i}\mu_j(\varepsilon, \omega)\}$$

where

$$\mu_j(\varepsilon, \omega) = \omega \cdot \ell - (1 + \varepsilon c_0(\varepsilon, \omega))j^3 + \varepsilon c_1(\varepsilon, \omega)j + r_j(\varepsilon),$$

$$\sup_{j \in \mathbb{Z}} |r_j(\varepsilon)| = O(\varepsilon)$$

The functions $c_0(\varepsilon, \omega)$, $c_1(\varepsilon, \omega)$ are independent of j

$\Longrightarrow$ we may verify II Melnikov conditions

Higher order operator: $\mathcal{L} := \omega \cdot \partial_\varphi - \partial_{xxx} + \varepsilon p(\varphi, x)\partial_{xxx}$

STEP 1) Under a change of variables

$$(Au) := u(\varphi, x + \beta(\varphi, x))$$

we get

$$\mathcal{L}_1 := A^{-1}\mathcal{L}A = \omega \cdot \partial_\varphi + c_\varepsilon(\varphi)\partial_{xxx} + O(\partial_{xx})$$

STEP 2) Rescaling time

$$(Bu)(\varphi, x) = u(\varphi + \omega q(\varphi), x)\,,$$

we get

$$\mathcal{L}_2 = \omega \cdot \partial_\varphi + \lambda(\varepsilon)\partial_{xxx} + O(\partial_{xx}), \;\; \lambda(\varepsilon) = 1 + O(\varepsilon)$$

which has the leading order with CONSTANT COEFFICIENTS

STEP 3) Descent method. Goal: Conjugate

$$\mathcal{L}_2 := \omega \cdot \partial_\varphi + \lambda(\varepsilon, \omega)\partial_{xxx} + b_2(\varphi, x)\partial_{xx} + b_1(\varphi, x)\partial_x$$

with $b_1, b_2 = O(\varepsilon)$, to

$$\mathcal{L}_3 := \Phi^{-1}\mathcal{L}_2\Phi = \mathcal{D}_3 + R_0\,, \qquad R_0 = \text{order } 0$$

$$\mathcal{D}_3 := \omega \cdot \partial_\varphi + \lambda(\varepsilon, \omega)\partial_{xxx} + m(\varepsilon, \omega)\partial_x$$

via

$$\Phi(h) := (1 + d(\varphi, x))h + f(\varphi, x)\partial_x^{-1} h$$

STEP 4) Super-quadratic reducibility scheme...

Remarks

The transformation

$$(Au) := u(x + \beta_\varepsilon(x))$$

1. - **NOT symplectic for** ∂_x
     $\Longrightarrow$ *does not preserves Hamiltonian structure*
   - **Anti-reversible**
     $\Longrightarrow$ *preserves reversible structure*
   - Hamiltonian structure used to eliminate $b(t)\partial_{xx}$
2. "*A not very close to identity*":
   A tends to 0 as $\varepsilon \to 0$ **pointwise**, $\forall u(x)$, not in operatorial norm.

Free vibrations

In preparation

Autonomous g-KdV: free quasi-periodic vibrations

$$u_t + u_{xxx} + \partial_x u^3 + \partial_{xx} f(u_x) = 0\,, \quad x \in \mathbb{T}\,,$$

$$f(u_x) = u_x^5 + h.o.t.$$

FURTHER DIFFICULTIES:

- add-reversibility
- no external parameters
- Birkhoff normal form
- amplitude-frequency relation

Open problem: quasi-periodic solutions of water wawes

Euler equations of hydrodynamics: water waves

(WW)
$$\begin{cases} \partial_t \eta = G(\eta)\xi \\ \partial_t \xi = -g\eta - \frac{\xi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\Big(G(\eta)\xi + \eta_x \xi_x\Big)^2 \end{cases}$$

$G(\eta)$ = Dirichlet-Neumann operator: pseudo-diff. operator

- Even less dispersive + derivatives in the dominant operator, ...
- Periodic solutions: Iooss, Plotnikov, Toland, '02-'10