Exercise Sheet 14 - Solution

Exercise 1

$$
\begin{aligned}
f_x =& \frac{\partial f}{\partial x} = \frac{2+3y}{2\sqrt{2x+3xy+4y}} \\
f_{xy} =& \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2+3y}{2\sqrt{2x+3xy+4y}}\right) = \frac{1}{2}\cdot\frac{\partial}{\partial y}\left(\frac{2+3y}{\sqrt{2x+3xy+4y}}\right) \\
=& \frac{1}{2}\cdot\frac{3\sqrt{2x+3xy+4y}-(2+3y)\frac{3x+4}{2\sqrt{2x+3xy+4y}}}{2x+3xy+4y} \\
=& \frac{1}{2}\cdot\frac{6\sqrt{2x+3xy+4y}-(2+3y)\frac{3x+4}{\sqrt{2x+3xy+4y}}}{2(2x+3xy+4y)} \\
=& \frac{1}{2}\cdot\frac{6(2x+3xy+4y)-(2+3y)(3x+4)}{2(2x+3xy+4y)\sqrt{2x+3xy+4y}} \\
=& \frac{1}{4}\cdot\frac{12x+18xy+24y-(6x+8+9xy+12y)}{\sqrt{(2x+3xy+4y)^3}} \\
=& \frac{1}{4}\cdot\frac{6x+9xy+12y-8}{\sqrt{(2x+3xy+4y)^3}}
\end{aligned}
$$

$$
\begin{aligned}
f_y =& \frac{\partial f}{\partial y} = \frac{3x+4}{2\sqrt{2x+3xy+4y}} \\
f_{yx} =& \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{3x+4}{2\sqrt{2x+3xy+4y}}\right) = \frac{1}{2}\cdot\frac{\partial}{\partial x}\left(\frac{3x+4}{\sqrt{2x+3xy+4y}}\right) \\
=& \frac{1}{2}\cdot\frac{3\sqrt{2x+3xy+4y}-(3x+4)\frac{2+3y}{2\sqrt{2x+3xy+4y}}}{2x+3xy+4y} \\
=& \frac{1}{2}\cdot\frac{6\sqrt{2x+3xy+4y}-(3x+4)\frac{2+3y}{\sqrt{2x+3xy+4y}}}{2(2x+3xy+4y)} \\
=& \frac{1}{2}\cdot\frac{6(2x+3xy+4y)-(3x+4)(2+3y)}{2(2x+3xy+4y)\sqrt{2x+3xy+4y}} \\
=& \frac{1}{4}\cdot\frac{12x+18xy+24y-(6x+9xy+8+12y)}{\sqrt{(2x+3xy+4y)^3}} \\
=& \frac{1}{4}\cdot\frac{6x+9xy+12y-8}{\sqrt{(2x+3xy+4y)^3}}
\end{aligned}
$$

So we have indeed $f_{xy} = f_{yx}$.

Exercise 2

a) In order to find the maximal domain of f we note that the root function is only defined for non-negative arguments and the logarithm is only defined for strictly positive arguments. So:

$$\begin{aligned} D(f) &= \{(x,y) \subset \mathbb{R}^2 |\ x^2+y^2-1 \geq 0 \text{ and } \sqrt{x^2+y^2-1} > 0\} \\ &= \{(x,y) \subset \mathbb{R}^2 |\ x^2+y^2 > 1\} = \mathbb{R}^2 \setminus B_1(0) \end{aligned}$$

where $B_1(0) = \{(x,y) \subset \mathbb{R}^2 |\ x^2+y^2 \leq 1\}$ is the closed unit circle. The function f is defined in all points of the two-dimensional plane outside the unit circle.

b) We compute derivatives:

$$\begin{aligned} f_x(x,y) &= 5(1-yx)e^{-xy} + \frac{x}{x^2+y^2-1} - \pi\sin(\pi x+y) + x, \\ f_y(x,y) &= -5x^2e^{-xy} + \frac{y}{x^2+y^2-1} - \sin(\pi x+y), \\ f_{yy}(x,y) &= 5x^3e^{-xy} + \frac{x^2-y^2-1}{(x^2+y^2-1)^2} - \cos(\pi x+y), \\ f_{xy}(x,y) &= 5x(xy-2)e^{-xy} - \frac{2xy}{(x^2+y^2-1)^2} - \pi\cos(\pi x+y), \end{aligned}$$

With this we get:

$$\begin{aligned} f_x(2;0) &= \frac{23}{3}, \qquad f_y(0;2) = \frac{2}{3} - \sin(2), \qquad f_{xy}(-2;0) = 20-\pi, \\ f_{yy}(5;0) &= 626 + \frac{1}{24}. \end{aligned}$$

Exercise 3

First we compute all partial derivatives. We find

$$\begin{aligned} h_x(x,y) &= \left(1+\frac{y}{x}\right)\mathrm{e}^{-\frac{y}{x}}, \\ h_y(x,y) &= -\,\mathrm{e}^{-\frac{y}{x}}, \\ h_{xy}(x,y) &= \partial_y h_x(x,y) = -\frac{y}{x^2}\,\mathrm{e}^{-\frac{y}{x}}, \\ h_{yy}(x,y) &= \frac{1}{x}\,\mathrm{e}^{-\frac{y}{x}}\,. \end{aligned}$$

Since

$$x \cdot h_{xy} + 2(h_x + h_y) - y \cdot h_{yy} = -\frac{y}{x}\,\mathrm{e}^{-\frac{y}{x}} + \left(2 + 2\frac{y}{x}\right)\mathrm{e}^{-\frac{y}{x}} - 2\,\mathrm{e}^{-\frac{y}{x}} - \frac{y}{x}\,\mathrm{e}^{-\frac{y}{x}}$$
$$= \left(-\frac{y}{x} + 2 + 2\frac{y}{x} - 2 - \frac{y}{x}\right)\mathrm{e}^{-\frac{y}{x}} = 0,$$

the function h satisfies the differential equation.

Exercise 4

a) From $f(x,y) = 3x^2 - 4x + 2xy + y^2$ we get:

$$f_x = 6x - 4 + 2y \qquad f_{xx} = 6 \qquad f_{xy} = 2$$
$$f_y = 2x + 2y \qquad f_{yy} = 2 \qquad f_{yx} = 2$$

$$\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix} \quad \Rightarrow \quad \left| \begin{matrix} 6x + 2y = 4 \\ 2x + 2y = 0 \end{matrix} \right| \quad \Rightarrow \quad \left| \begin{matrix} 3x + y = 2 \\ x + y = 0 \end{matrix} \right|$$

We find the point $x = 1$ and $y = -1$ and have to test further criteria. As $A = f_{xx}f_{yy} - f_{xy}^2 = 12 - 4 = 8 > 0$ this is an extremum and since $f_{xx} > 0$ the identified point is a minimum.

b) From $f(x,y) = (x^2 + y^2)^2 - 2(x^2 - y^2)$ we get:

$$f_x = 4x(x^2 + y^2) - 4x \qquad f_{xx} = 12x^2 + 4y^2 - 4 \qquad f_{xy} = 8xy$$
$$f_y = 4y(x^2 + y^2) + 4y \qquad f_{yy} = 4x^2 + 12y^2 + 4 \qquad f_{yx} = 8xy$$

$$\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix} \quad \Rightarrow \quad \left| \begin{matrix} 4x(x^2 + y^2 - 1) = 0 \\ 4y(x^2 + y^2 + 1) = 0 \end{matrix} \right| \quad \overset{(*)}{\Rightarrow} \quad \left| \begin{matrix} x^2 + y^2 - 1 = 0 \\ x^2 + y^2 + 1 = 0 \end{matrix} \right|$$

From (*) follows the solution $x = 0$ and $y = 0$. The last equation system has no solution since $x^2 + y^2$ cannot be equal to 1 and -1 at the same time. So the only point left is P(0/0).
We get: $A = f_{xx}f_{yy} - f_{xy}^2 = (-4) \cdot 4 - 0 = -16 < 0$ so this is not an extremum.

We need to look for other solutions for $f_x = 0$ and $f_y = 0$. From (*) we consider two cases

- $x = 0$ and $y \neq 0$

$$\begin{matrix} (1) \\ (2) \end{matrix} \left| \begin{matrix} 4x(x^2 + y^2 - 1) = 0 \\ 4y(x^2 + y^2 + 1) = 0 \end{matrix} \right| \quad \Rightarrow \quad \left| \begin{matrix} 0 = 0 \\ 4y(y^2 + 1) = 0 \end{matrix} \right|$$

  There are no additional solutions.

- $x \neq 0$ and $y = 0$

$$\begin{array}{l} (1) \\ (2) \end{array} \left| \begin{array}{r} 4x(x^2+y^2-1)=0 \\ 4y(x^2+y^2+1)=0 \end{array} \right| \quad \Rightarrow \quad \left| \begin{array}{r} 4x(x^2-1)=0 \\ 0=0 \end{array} \right|$$

We get two new solutions $x_1 = 1$ and $x_2 = -1$.
For $(\pm 1/0)$ we get $A = f_{xx}f_{yy} - f_{xy}^2 = (8) \cdot 8 - 0 = 64 > 0$. Since $f_{xx}(\pm 1, 0) = 8 > 0$ the points $(1/0)$ and $(-1/0)$ are minima.

c) From $f(x,y) = \frac{1}{3}x^3 - x^2 + y^3 - 12y$ we find

$$\begin{aligned} f_x &= x^2 - 2x & f_{xx} &= 2x - 2 & f_{xy} &= 0 \\ f_y &= 3y^2 - 12 & f_{yy} &= 6y & f_{yx} &= 0 \end{aligned}$$

$$\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array} \quad \Rightarrow \quad \left| \begin{array}{c} x^2 - 2x = 0 \\ 3y^2 - 12 = 0 \end{array} \right| \quad \Rightarrow \quad \left| \begin{array}{c} x(x-2) = 0 \\ y^2 - 4 = 0 \end{array} \right|$$

The equation system is easy to solve and we get the following solutions:

- $P(0/2)$ • $Q(0/-2)$ • $R(2/2)$ • $S(2/-2)$

We need to compute A and f_{xx} for all four points.

- P(0/2)

$$A = f_{xx}f_{yy} - f_{xy}^2 = (-2) \cdot 12 - 0 = -24 < 0 \quad \Rightarrow \text{no extremum}$$

- Q(0/-2)

$$\begin{aligned} A &= f_{xx}f_{yy} - f_{xy}^2 = (-2) \cdot (-12) - 0 = 24 > 0 \quad \text{and} \\ f_{xx} &= -2 < 0 \Rightarrow \text{maximum} \end{aligned}$$

- R(2/2)

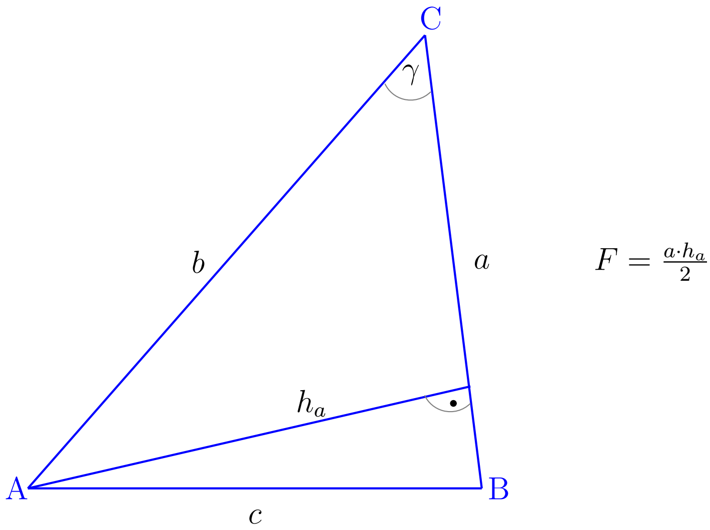
$$\begin{aligned} A &= f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 12 - 0 = 24 > 0 \quad \text{and} \\ f_{xx} &= 2 > 0 \Rightarrow \text{minimum} \end{aligned}$$

- S(2/-2)

$$A = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot (-12) - 0 = -24 < 0 \quad \Rightarrow \text{no extremum}$$

Exercise 5

We aim to express the are as a function of the perimeter U and two edges. So we get a function in two variables.
We start with the edges a and b and the angle γ in between. c is computed by using the law of cosines $c = \sqrt{a^2 + b^2 - 2ab\cos(\gamma)}$.



From the perimeter $U = a + b + c = a + b + \sqrt{a^2 + b^2 - 2ab\cos(\gamma)}$ we get:

$$\cos(\gamma) = \frac{(a^2 + b^2) - (U - a - b)^2}{2ab} = ... = \frac{2aU + 2bU - 2ab - U^2}{2ab}$$

So the altitude is $h_a = b \cdot \sin(\gamma) = b \cdot \sqrt{1 - \cos^2(\gamma)}$. The area can be written as

$$\begin{aligned} F =& \frac{1}{2}ab\sqrt{1 - \left(\frac{2aU + 2bU - 2ab - U^2}{2ab}\right)^2} \\ =& \frac{1}{2}\sqrt{(ab)^2 - \left(\frac{2aU + 2bU - 2ab - U^2}{2}\right)^2} \\ =& \frac{1}{4}\sqrt{(2ab)^2 - (2aU + 2bU - 2ab - U^2)^2} \end{aligned}$$

So the area can be expressed as a function of two variables and the given perimeter U. In order to maximise F it is enough to maximize the function inside the root.

So we need to solve the following exercise: Find the maximum of the function

$$f(a, b) = (2ab)^2 - (2aU + 2bU - 2ab - U^2)^2$$

We get:

$$\begin{aligned}
f_a =& \frac{\partial f}{\partial a} = 8ab^2 - 4(2aU + 2bU - 2ab - U^2)(U - b) \\
f_{aa} =& \frac{\partial^2 f}{\partial a^2} = 8b^2 - 8(U - b)^2 \\
f_{ab} =& \frac{\partial^2 f}{\partial a \partial b} = 16aU + 16bU - 12U^2 \\
f_b =& \frac{\partial f}{\partial b} = 8a^2b - 4(2aU + 2bU - 2ab - U^2)(U - a) \\
f_{bb} =& \frac{\partial^2 f}{\partial b^2} = 8a^2 - 8(U - a)^2 \\
f_{ba} =& \frac{\partial^2 f}{\partial b \partial a} = 16aU + 16bU - 12U^2
\end{aligned}$$

Find extrema:

$$\begin{aligned}
f_a =& 0 \qquad 8ab^2 - 4(2aU + 2bU - 2ab - U^2)(U - b) = 0 \quad (1) \\
f_b =& 0 \qquad 8a^2b - 4(2aU + 2bU - 2ab - U^2)(U - a) = 0 \quad (2)
\end{aligned}$$

From (1) we get: $2aU + 2bU - 2ab - U^2 = \frac{2ab^2}{U - b}$ plugging into (2)

$$\begin{aligned}
8a^2b - 4\frac{2ab^2}{U - b}(U - a) &= 0 && \Big| \div (8ab) \\
a - \frac{b}{U - b}(U - a) &= 0 && \Big| \cdot (U - b) \\
a(U - b) - b(U - a) &= 0 && \Big| \\
aU - ab - bU + ab &= 0 && \Big| \\
aU - bU &= 0 && \Big| \\
(a - b)U &= 0 && \Big|
\end{aligned}$$

As $U \neq 0$ we have $a = b$. Plugging $b = a$ into (1) we find the equation:

$$\begin{aligned} 2a^3 - (4aU - 2a^2 - U^2)(U - a) &= 0 \\ 2a^3 - (4aU^2 - 2a^2U - U^3 - 4a^2U + 2a^3 + aU^2) &= 0 \\ 2a^3 - (5aU^2 - 6a^2U - U^3 + 2a^3) &= 0 \\ 2a^3 - 5aU^2 + 6a^2U + U^3 - 2a^3 &= 0 \\ -5aU^2 + 6a^2U + U^3 &= 0 \\ 6Ua^2 - 5U^2a + U^3 &= 0 \end{aligned}$$

The solutions of this quadratic are:

$$a_{1,2} = \frac{-(-5U^2) \pm \sqrt{(-5U^2)^2 - 4 \cdot 6U \cdot U^3}}{2 \cdot 6U} = \frac{5U^2 \pm \sqrt{25U^4 - 24U^4}}{12U} = \frac{5U^2 \pm U^2}{12U}$$

We find the solutions $a_1 = \frac{U}{2}$ and $a_2 = \frac{U}{3}$.

a_1 is an extremum but a minimum since $b = a$ implies $c = 0$;, i.e. it corresponds to a degenerated triangle with area zero. Since we look for a maximum we consider a_2 as potential solution.

From this we get $a = U/3$, $b = U/3$ and $c = U/3$. We can stop the analysis here but we illustrate the remainder for the sake of completeness.

We need to compute $A = f_{aa}f_{bb} - f_{ab}^2$.

$$\begin{aligned} A =& \left(8\left(\frac{U}{3}\right)^2 - 8\left(U - \frac{U}{3}\right)^2\right)\left(8\left(\frac{U}{3}\right)^2 - 8\left(U - \frac{U}{3}\right)^2\right) - \\ & \left(16\left(\frac{U}{3}\right)U + 16\left(\frac{U}{3}\right)U - 12U^2\right)^2 \\ =& 8\left(\frac{U^2}{9} - 4\frac{U^2}{9}\right)8\left(\frac{U^2}{9} - 4\frac{U^2}{9}\right) - \left(\frac{32}{3}U^2 - 12U^2\right)^2 \\ =& 64\left(-\frac{U^2}{3}\right)\left(-\frac{U^2}{3}\right) - \left(-\frac{4}{3}U^2\right)^2 \\ =& \frac{64}{9}U^4 - \frac{16}{9}U^4 = \frac{48}{9}U^4 > 0 \\ f_{aa} =& \left(8\left(\frac{U}{3}\right)^2 - 8\left(U - \frac{U}{3}\right)^2\right) = -\frac{8}{3}U^2 < 0 \end{aligned}$$

With $A > 0$ and $f_{aa} < 0$ we have found a minimium which concludes the proof.