Exercise Sheet 12 - Solution

Exercise 1

a) A separable differential equation is of the form $y' = r(x) \cdot s(y)$.
Two arbitary examples are:

- $y' = \sin(x) \cdot \tan(y)$
- $y' = (x^2 + 2x + \pi) \cdot \log(y)$.

b) A linear DE of first order is of the form $y' = p(x) \cdot y + q(x)$.
The DE is called homogeneous if $q(x) = 0$, otherwise inhomogeneous.
A homogeneous DE has the form $y' = p(x) \cdot y$. This is also a seperable DE.

c) If a DE is seperable it can be transformed to $\dfrac{dy}{s(y)} = r(x)dx$.

(1) $y' = xy \quad \Rightarrow \quad$ separable $\quad \Rightarrow \quad \dfrac{dy}{y} = x\,dx$

(2) $y' = x - y \quad \Rightarrow \quad$ not separable

(3) $y' = x^2\sqrt{y} \quad \Rightarrow \quad$ separable $\quad \Rightarrow \quad \dfrac{dy}{\sqrt{y}} = x^2\,dx$

(4) $\dfrac{1}{y'} = xy \quad \Rightarrow \quad$ separable $\quad \Rightarrow \quad y' = \dfrac{1}{xy} \quad \Rightarrow \quad y\,dy = \dfrac{dx}{x}$

(5) $\dfrac{1}{y'+1} = xy \quad \Rightarrow \quad$ not separable $\quad \Rightarrow \quad y' = \dfrac{1}{xy} - 1$

(6) $\dfrac{1}{y'+1} = \dfrac{1}{xy+1} \quad \Rightarrow \quad$ separable $\quad \Rightarrow \quad y' = xy,$ see (1)

Exercise 2 (6 points)

a) (1 point) The solution of the homogeneous DE is

$$y = Ce^{\frac{x^{\frac{5}{2}}}{\frac{5}{2}}} = Ce^{\frac{2}{5}\cdot x^{\frac{5}{2}}}$$

1) (1/2 point) $\quad y(0) = 0 \quad \Rightarrow \quad C = 0 \quad$ and $\quad y = 0$

2) (1/2 point) $\quad y(0) = 1 \quad \Rightarrow \quad C = 1 \quad$ and $\quad y = e^{\frac{2}{5}\cdot x^{\frac{5}{2}}}$

b) (5 points)

1) (3 points) The solution of the homogeneous DE $y' = y$ is $y = Ke^x$.

Variation of parameters: $y = K(x)e^x \quad \Rightarrow \quad y' = K'(x)e^x + K(x)e^x$

$$
\begin{aligned}
y' &= y + x^2 \\
K'(x)e^x + K(x)e^x &= K(x)e^x + x^2 && | - K(x)e^x \\
K'(x)e^x &= x^2 && | \div e^x \\
K'(x) &= e^{-x}x^2 && | \qquad \text{(1 point)}
\end{aligned}
$$

$K(x)$ can be determined by integration by parts. We add the integration constant C at the very end:

$$
\begin{aligned}
K(x) = \int e^{-x}x^2 dx &= -e^{-x}x^2 - \int -e^{-x}2x dx = -e^{-x}x^2 + 2\int e^{-x}x dx \\
&= -e^{-x}x^2 + 2\Big[-e^{-x}x - \int -e^{-x}dx\Big] = -e^{-x}x^2 - 2e^{-x}x + 2\int e^{-x}dx \\
&= -e^{-x}x^2 - 2e^{-x}x - 2e^{-x} + C = -(x^2 + 2x + 2)e^{-x} + C \qquad \text{(1 point)}
\end{aligned}
$$

So the general solution of the differential equaiton is:

$$y = K(x)e^x = Ce^x - x^2 - 2x - 2$$

From $y(0) = C - 2 = -1$ we get $C = 1$ and thus

$$y = e^x - x^2 - 2x - 2 \qquad \text{(1 point)}$$

2) (2 points) The solution of the homogeneous DE is $y = Ke^{-2x}$.

Variation of parameters: $y = K(x)e^{-2x} \quad \Rightarrow \quad y' = K'(x)e^{-2x} - 2K(x)e^{-2x}$

$$
\begin{aligned}
y' &= -2y + e^{-2x} \\
K'(x)e^{-2x} - 2K(x)e^{-2x} &= -2K(x)e^{-2x} + e^{-2x} && | + 2K(x)e^{-2x} \\
K'(x)e^{-2x} &= e^{-2x} && | \div e^{-2x} \\
K'(x) &= 1
\end{aligned}
$$

Determine K(x) by integration:

$$K(x) = \int 1dx = x + C \qquad \text{(1 point)}$$

So the general solution of the differential equation is:

$$y = K(x)e^{x} = (x + C)e^{-2x} = Ce^{-2x} + xe^{-2x}$$

From $y(0) = C = 1$ we get $C = 1$ and thus

$$y = e^{-2x} + xe^{-2x} \qquad \text{(1 point)}$$

Exercise 3 (6 points)

1) (2 points) Rearranging terms gives: $y' = 3x^2y + 2xe^{x^3}$. The solution of the homogeneous DE is $y = Ke^{x^3}$. Variation of parameters:

$$y = K(x)e^{x^3} \qquad \Rightarrow \qquad y' = K'(x)e^{x^3} + 3x^2K(x)e^{x^3} \qquad \text{(1 point)}$$

Plugging into the DE:

$$\begin{aligned} y' &= 3x^2y + 2xe^{x^3} \\ K'(x)e^{x^3} + 3x^2K(x)e^{x^3} &= 3x^2K(x)e^{x^3} + 2xe^{x^3} \\ K'(x)e^{x^3} &= 2xe^{x^3} \\ K'(x) &= 2x \\ K(x) &= x^2 + C \end{aligned}$$

The general solution is: $y = K(x)e^{x^3} = Ce^{x^3} + x^2e^{x^3}$ (1 point)

2) (2 points) Rearranging terms gives:

$$\begin{aligned} y' &= -\frac{x^2}{1+x^3} \cdot y && \text{(1 point)} \\ \frac{dy}{dx} &= -\frac{x^2}{1+x^3} \cdot y \\ \frac{dy}{y} &= -\frac{x^2}{1+x^3}dx \\ \frac{dy}{y} &= -\frac{1}{3} \cdot \frac{3x^2}{1+x^3}dx \\ \ln|y| &= -\frac{1}{3} \cdot \ln(x^3+1) + \tilde{C} \\ y &= e^{\tilde{C}}(x^3+1)^{-\frac{1}{3}} = C\frac{1}{\sqrt[3]{x^3+1}} && \text{(1 point)} \end{aligned}$$

The domain is: $x \in \mathbb{R} \setminus \{-1\}$

3) (2 points) Rearranging terms gives: $y' = y \cdot \tan(x) + \frac{1}{\cos(x)}$. The solution of the homogeneous DE is $y = K\frac{1}{\cos(x)}$. Variation of parameters:

$$y = K(x)\frac{1}{\cos(x)} \qquad \Rightarrow \qquad y' = K'(x)\frac{1}{\cos(x)} + K(x)\frac{\sin(x)}{\cos^2(x)} \qquad \text{(1 point)}$$

Plugging into the DE:

$$\begin{aligned}
y' &= y\tan(x) + \frac{1}{\cos(x)} \\
K'(x)\frac{1}{\cos(x)} + K(x)\frac{\sin(x)}{\cos^2(x)} &= K(x)\frac{1}{\cos(x)}\tan(x) + \frac{1}{\cos(x)} \\
K'(x)\frac{1}{\cos(x)} + K(x)\frac{\sin(x)}{\cos^2(x)} &= K(x)\frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} + \frac{1}{\cos(x)} \\
K'(x)\frac{1}{\cos(x)} + \cancel{K(x)\frac{\sin(x)}{\cos^2(x)}} &= \cancel{K(x)\frac{\sin(x)}{\cos^2(x)}} + \frac{1}{\cos(x)} \\
K'(x)\frac{1}{\cancel{\cos(x)}} &= \frac{1}{\cancel{\cos(x)}} \\
K'(x) &= 1 \\
K(x) &= x + C
\end{aligned}$$

The general solution is:

$$y = K(x)\frac{1}{\cos(x)} = (x+C)\frac{1}{\cos(x)} = C\frac{1}{\cos(x)} + \frac{x}{\cos(x)} \qquad \text{(1 point)}$$

The domain is: $x \in \mathbb{R} \setminus \{\frac{\pi}{2} \pm n \cdot \pi\}$, where $n \in \mathbb{N}$

Exercise 4 (4 points)

a) (3 points) The DE $y' = 3 \cdot (x(y-2))^2 = 3 \cdot x^2 \cdot (y-2)^2$ is separable.

$$
\begin{aligned}
y' = \frac{dy}{dx} &= 3 \cdot x^2 \cdot (y-2)^2 && \text{(1 point)} \\
\frac{dy}{(y-2)^2} &= 3 \cdot x^2 \, dx \\
\int \frac{dy}{(y-2)^2} &= \int 3 \cdot x^2 \, dx \\
\int (y-2)^{-2} \, dy &= \int 3 \cdot x^2 \, dx \\
\frac{(y-2)^{-1}}{-1} &= x^3 + \tilde{C} && \text{(1 point)} \\
-\frac{1}{y-2} &= x^3 + \tilde{C} \\
\frac{1}{y-2} &= -x^3 + C \\
y - 2 &= \frac{1}{-x^3 + C} \\
y &= \frac{1}{-x^3 + C} + 2 && \text{(1 point)}
\end{aligned}
$$

b) (1 point) Yes, for $y' = 0$ we find the solution $y = 2$. This solution satisfies the DE and is not contained in the geneal solution computed in subtask b).

Exercise 5 (4 points)

a) (3 points) The DE can be rewritten as

$$E'(x) = -\alpha E(x) + \alpha E_m$$

which has the form of a linear DE. However, we solve this equation directly with separation. (1 point)

$$
\begin{aligned}
E'(x) &= \alpha(E_m - E(x)) \\
\frac{dE}{dx} &= \alpha(E_m - E) \\
\frac{dE}{E_m - E} &= \alpha\, dx \\
\ln|E_m - E| &= -\alpha x + C_1 \\
|E_m - E| &= e^{-\alpha x + C_1} \\
|E_m - E| &= C_2 e^{-\alpha x} \qquad \text{, with} \quad C_2 > 0 \\
E_m - E &= C e^{-\alpha x} \qquad \text{, with} \quad C \in R \quad \text{(we allow C=0 so that the singular solution is included)} \\
E &= E_m - C e^{-\alpha x} \qquad \text{(1 Punkt)}
\end{aligned}
$$

With the initial condition $E(0) = E_0 < E_m$ we can compute C.

$$E(0) = E_m - C = E_0 \quad \Rightarrow \quad C = E_m - E_0 \qquad \text{(1 point)}$$

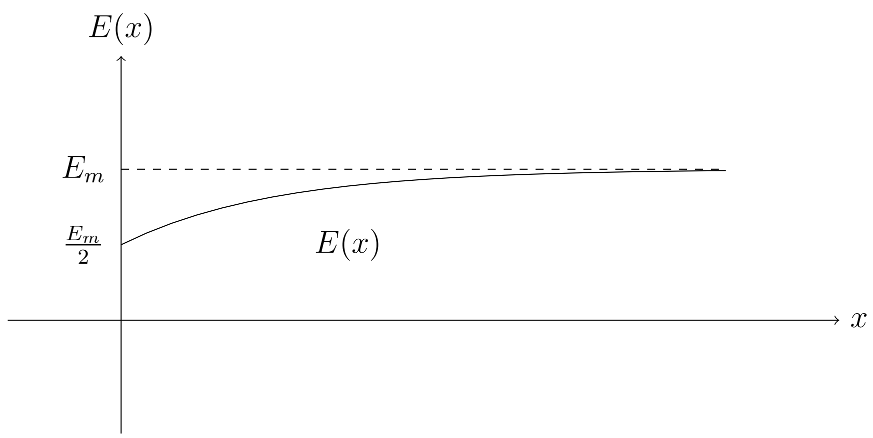
So we find the function $E(x)$:

$$E(x) = E_m - (E_m - E_0)e^{-\alpha x}$$

b) (1 point) Plugging in $E_0 = \frac{E_m}{2}$ we get $E(x)$:

$$E(x) = E_m - (E_m - \frac{E_m}{2})e^{-\alpha x} = E_m - \frac{E_m}{2}e^{-\alpha x} = E_m\left(1 - \frac{1}{2}e^{-\alpha x}\right)$$

x only takes positive values!



Remark: This is an example of a saturation process and a special case of the economic law of diminishing returns: increased use of a variable factor of production (fertilizer) on a fixed factor of production (soil) yields decreasing increments of return.