# Exercise Sheet 11 - Solution

## Exercise 1

a) From $N'(t) = 10 \cdot e^{10 \cdot t} = 10 \cdot N(t) = \lambda \cdot N(t)$ we get $\lambda = 10$.

b) From $y' = x$ we get $y = \frac{1}{2}x^2 + C$. With $y(0) = C = 1$ the function is $y = \frac{1}{2}x^2 + 1$.

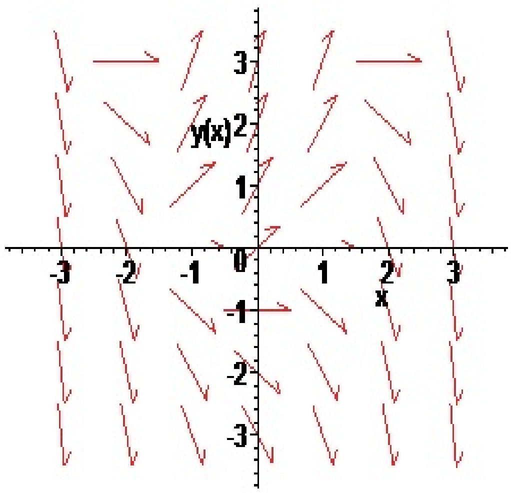
## Exercise 2 (5 points)

a) (2 points)

DE: $y' = y - x^2 + 1$ . Table for $y'$. (1 point)

| | -3 | -2 | -1 | 0 | 1 | 2 | 3 | ←x |
|---|---|---|---|---|---|---|---|---|
| -3 | -11 | -6 | -3 | -2 | -3 | -6 | -11 | |
| -2 | -10 | -5 | -2 | -1 | -2 | -5 | -10 | |
| -1 | -9 | -4 | -1 | 0 | -1 | -4 | -9 | |
| 0 | -8 | -3 | 0 | 1 | 0 | -3 | -8 | |
| 1 | -7 | -2 | 1 | 2 | 1 | -2 | -7 | |
| 2 | -6 | -1 | 2 | 3 | 2 | -1 | -6 | |
| 3 | -5 | 0 | 3 | 4 | 3 | 0 | -5 | |
| y↑ | | | | | | | | |

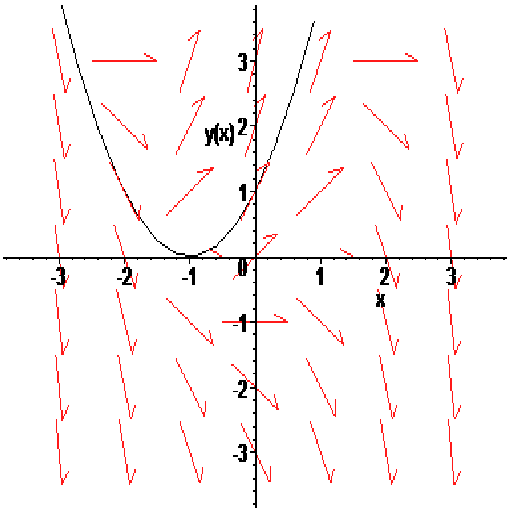
The direction field is shown below. (1 point)



b) (1 point) Plug $y$ and $y'$ into the DE:

$$\begin{aligned} y' &= y - x^2 + 1 \\ (x^2 + 2x + 1)' &\stackrel{!}{=} (x^2 + 2x + 1) - x^2 + 1 \\ 2x + 2 &\stackrel{!}{=} 2x + 2 \qquad \Rightarrow \qquad y' = y - x^2 + 1 \qquad \text{is satisfied} \end{aligned}$$

c) (2 points) Special solution $y = x^2 + 2x + 1$




You get (1 point) for the yellow marked entries in the table for $y'$. Anoter (1 point) for the correctly drawn parabola.

**Exercise 3** (6 points)

a) (2 points) We plug $i(t)$ and $\ddot{i}(t)$ into the DE $\ddot{i} + \omega_0^2 i = 0$.

$$\begin{aligned} i(t) &= \widehat{I} \cdot \sin(\omega_0 t + \varphi) \\ \dot{i}(t) &= \frac{di}{dt} = \omega_0 \widehat{I} \cdot \cos(\omega_0 t + \varphi) \\ \ddot{i}(t) &= \frac{d^2 i}{dt^2} = -\omega_0^2 \widehat{I} \cdot \sin(\omega_0 t + \varphi) \end{aligned}$$

(1 point)

$$\ddot{i} + \omega_0^2 i = -\omega_0^2 \widehat{I} \cdot \sin(\omega_0 t + \varphi) + \omega_0^2 (\widehat{I} \cdot \sin(\omega_0 t + \varphi)) = 0$$

(1 point)

b) (4 points)

$$\dot{i}(t) = \frac{di}{dt} = \widehat{I}\left[\omega\cos(\omega t+\varphi) - \sigma\sin(\omega t+\varphi)\right]e^{-\sigma t} \qquad \text{(1 point)}$$

$$\begin{aligned}\ddot{i}(t) = \frac{d^2 i}{dt^2} &= \widehat{I}\left[-\omega^2\sin(\omega t+\varphi) - \sigma\omega\cos(\omega t+\varphi)\right]e^{-\sigma t}\\ &+ \widehat{I}\left[-\omega\sigma\cos(\omega t+\varphi) + \sigma^2\sin(\omega t+\varphi)\right]e^{-\sigma t}\\ &= \widehat{I}\left[-\omega^2\sin(\omega t+\varphi) - 2\omega\sigma\cos(\omega t+\varphi) + \sigma^2\sin(\omega t+\varphi)\right]e^{-\sigma t} \qquad \text{(1 point)}\end{aligned}$$

Plugging into the DE:

$$\begin{aligned}\ddot{i} + 2\sigma\dot{i} + \omega_0^2 i &= 0\\ \widehat{I}\left[-\omega^2\sin(\omega t+\varphi) - 2\omega\sigma\cos(\omega t+\varphi) + \sigma^2\sin(\omega t+\varphi)\right]e^{-\sigma t}&\\ + 2\sigma\widehat{I}\left[\omega\cos(\omega t+\varphi) - \sigma\sin(\omega t+\varphi)\right]e^{-\sigma t}&\\ + \omega_0^2\widehat{I}\sin(\omega t+\varphi)e^{-\sigma t} &\stackrel{!}{=} 0\\ \widehat{I}\Big[-\omega^2\sin(\omega t+\varphi) - 2\omega\sigma\cos(\omega t+\varphi) + \sigma^2\sin(\omega t+\varphi)&\\ + 2\sigma\omega\cos(\omega t+\varphi) - 2\sigma^2\sin(\omega t+\varphi)&\\ + \omega_0^2\sin(\omega t+\varphi)\Big]e^{-\sigma t} &\stackrel{!}{=} 0\\ \widehat{I}\Big[-\omega^2\sin(\omega t+\varphi) - \cancel{2\omega\sigma\cos(\omega t+\varphi)} + \sigma^2\sin(\omega t+\varphi)&\\ + \cancel{2\sigma\omega\cos(\omega t+\varphi)} - 2\sigma^2\sin(\omega t+\varphi)&\\ + \omega_0^2\sin(\omega t+\varphi)\Big]e^{-\sigma t} &\stackrel{!}{=} 0\\ \widehat{I}\Big[-\omega^2\sin(\omega t+\varphi) - \sigma^2\sin(\omega t+\varphi) + \omega_0^2\sin(\omega t+\varphi)\Big]e^{-\sigma t} &\stackrel{!}{=} 0\\ \widehat{I}\Big[-\omega^2 - \sigma^2 + \omega_0^2\Big]\sin(\omega t+\varphi)e^{-\sigma t} &\stackrel{!}{=} 0\\ \widehat{I}\Big[-(\omega_0^2 - \sigma^2) - \sigma^2 + \omega_0^2\Big]\sin(\omega t+\varphi)e^{-\sigma t} &\stackrel{!}{=} 0\\ \widehat{I}\Big[-\omega_0^2 + \sigma^2 - \sigma^2 + \omega_0^2\Big]\sin(\omega t+\varphi)e^{-\sigma t} &\stackrel{!}{=} 0\\ \widehat{I}\Big[0\Big]\sin(\omega t+\varphi)e^{-\sigma t} &= 0 \qquad \text{(2 points)}\end{aligned}$$

**Exercise 4** (8 points)

a) • (3 points)

$$y = xe^x$$
$$y' = e^x + xe^x \quad \text{(1 point)}$$
$$y'' = e^x + e^x + xe^x = 2e^x + xe^x \quad \text{(1 point)}$$

Plugging into the DE: (1 point)

$$e^{-x}y'' + e^{-x}y' = e^{-x}(2e^x + xe^x) + e^{-x}(e^x + xe^x) = 2 + x + 1 + x = 2x + 3$$

• (3 points)

$$y = e^x \sin(x)$$
$$y' = e^x \sin(x) + e^x \cos(x) \quad \text{(1 point)}$$
$$y'' = e^x \sin(x) + e^x \cos(x) + e^x \cos(x) - e^x \sin(x) = 2e^x \cos(x) \quad \text{(1 point)}$$

Plugging into the DE: (1 point)

$$\begin{aligned} xy'' - 2xy' + xy &= 2xe^x \cos(x) - 2x(e^x \sin(x) + e^x \cos(x)) + xe^x \sin(x) \\ &= 2xe^x \cos(x) - 2xe^x \sin(x) - 2xe^x \cos(x) + xe^x \sin(x) \\ &= -xe^x \sin(x) \end{aligned}$$

b) • (1 point) From $y^2 + y - 2 = 0$ we get: $(y-1)(y+2) = 0$. This yields the solutions $y_1 = 1$ and $y_2 = -2$.

• (1 point) Guessing yields $y_1 = 1$ as a first solution of $y^3 - 2y^2 - y + 2 = 0$. Polynomial division $(y^3 - 2y^2 - y + 2) : (y-1)$ yields $y^2 - y - 2 = (y+1)(y-2)$ which implies the solutions $y_2 = -1$ and $y_3 = 2$.

**Exercise 5** (2 points)
We plug $y = f(x)\sin(x)$ into the DE and check if equality holds.

$$\begin{aligned} y' &\stackrel{!}{=} \left[\frac{f'(x)}{f(x)} + \cot(x)\right] \cdot y \\ \frac{d}{dx}\left[f(x)\sin(x)\right] &\stackrel{!}{=} \left[\frac{f'(x)}{f(x)} + \cot(x)\right] \cdot \Big(f(x)\sin(x)\Big) \\ f'(x)\sin(x) + f(x)\cos(x) &\stackrel{!}{=} \left[\frac{f'(x)}{f(x)} + \frac{\cos(x)}{\sin(x)}\right] \cdot f(x)\sin(x) \\ f'(x)\sin(x) + f(x)\cos(x) &\stackrel{!}{=} \frac{f'(x)}{\cancel{f(x)}} \cdot \cancel{f(x)}\sin(x) + \frac{\cos(x)}{\cancel{\sin(x)}} \cdot f(x)\cancel{\sin(x)} \\ f'(x)\sin(x) + f(x)\cos(x) &\stackrel{!}{=} f'(x)\sin(x) + f(x)\cos(x) \qquad \Rightarrow \qquad \text{QED} \end{aligned}$$