Exercise Sheet 10 - Solution

Exercise 1

a)

$$\int_{-\infty}^{0} e^x dx = \lim_{x_u \to -\infty} \left(\int_{x_u}^{0} e^x dx \right) = \lim_{x_u \to -\infty} \left[e^x \right]_{x_u}^{0} = e^0 - \lim_{x_u \to -\infty} e^{x_u} = 1 - 0 = 1$$

b)

$$\int_0^9 \frac{1}{\sqrt{x}}\, dx = \int_0^9 \frac{1}{x^{\frac{1}{2}}}\, dx = \int_0^9 x^{-\frac{1}{2}}\, dx = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^9 = 2 \cdot \left[\sqrt{x} \right]_0^9 = 2 \cdot 3 = 6$$

Exercise 2 (4 points)
We denote the lower limit by x_u and the upper limit by x_o.

a) (1 point)

$$\int_1^{\infty} \frac{1}{\sqrt[3]{x^4}} dx = \int_1^{\infty} \frac{1}{x^{\frac{4}{3}}} dx = \int_1^{\infty} x^{-\frac{4}{3}} dx = \lim_{x_o \to \infty} \int_1^{x_o} x^{-\frac{4}{3}} dx$$
$$= \lim_{x_o \to \infty} \left(\frac{x^{-\frac{1}{3}}}{-\frac{1}{3}} \right)_1^{x_o} = (-3) \cdot \lim_{x_o \to \infty} \left(x^{-\frac{1}{3}} \right)_1^{x_o} = (-3) \cdot \lim_{x_o \to \infty} \left(\frac{1}{\sqrt[3]{x}} \right)_1^{x_o}$$
$$= (-3) \cdot \lim_{x_o \to \infty} \left(\frac{1}{\sqrt[3]{x_o}} - 1 \right) = (-3) \cdot (-1) = 3$$

b) (1 point) Similar as in Storrer on page 170 below we set $g(t) = \frac{1}{\sqrt{\sin(t)}} \cdot \cos(t)$. We choose $f(u) = \frac{1}{\sqrt{u}}$ and $u(t) = \sin(t)$ so that $u'(t) = \cos(t)$.

$$\int_0^{\pi/2} \frac{\cos(t)}{\sqrt{\sin t}} dt = \int_0^{\pi/2} \frac{1}{\sqrt{\sin t}} \cdot \cos(t) dt = \int_{u=0}^{u=1} \frac{1}{\sqrt{u}} du = \int_0^1 \frac{1}{\sqrt{u}} du = \int_0^1 u^{-\frac{1}{2}} du$$
$$= \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^1 = 2 \cdot \left[\sqrt{u} \right]_0^1 = 2 \cdot (1 - 0) = 2$$

c) (1 point) Similarly with $g(x) = e^{-\sqrt{x}} \cdot \frac{1}{\sqrt{x}} = 2e^{-\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$ we set $f(u) = e^{-u}$ and

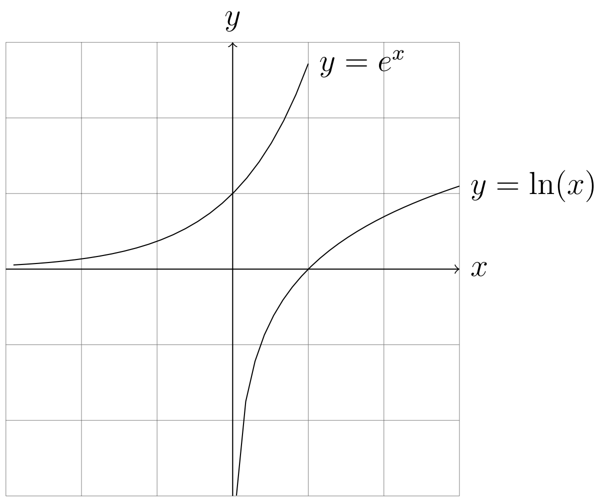
$u(x) = \sqrt{x}$ so that $u'(t) = \frac{1}{2\sqrt{x}}$.

$$\begin{aligned}\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx &= \int_0^\infty 2e^{-\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}dx = \int_0^\infty 2e^{-u}du = 2 \cdot \int_0^\infty e^{-u}du \\ &= 2 \cdot \lim_{u_o\to\infty} \int_0^{u_o} e^{-u}du = 2 \cdot \lim_{u_o\to\infty} \left[\frac{e^{-u}}{-1}\right]_0^{u_o} = -2 \cdot \lim_{u_o\to\infty} \left[e^{-u}\right]_0^{u_o} \\ &= -2 \cdot \left[\lim_{u_o\to\infty} \left(e^{-u_o}\right) - 1\right] = (-2)\cdot(-1) = 2\end{aligned}$$

d) (1 point)

$$\begin{aligned}\int_0^1 \ln(x)dx &= \int_0^1 1 \cdot \ln(x)dx = \left[x \cdot \ln(x)\right]_0^1 - \int_0^1 x \cdot \frac{1}{x}dx \\ &= \lim_{x_u\to 0} \left[x \cdot \ln(x)\right]_{x_u}^1 - \int_0^1 1dx = \left[\lim_{x_u\to 0} \Big(0 - x_u \cdot \ln(x_u)\Big)\right] - \left[x\right]_0^1 \\ &= \left[0 - 0\right] - \left[1 - 0\right] = -1\end{aligned}$$

Hint: There is no need to compute the integral explicitly. The function $\ln(x)$ is the inverse function of e^x.



It holds:

$$\int_0^1 \ln(x)dx = -\int_{-\infty}^0 e^x dx = -1 \qquad \text{(see 1a))}$$

Exercise 3 (6 points)

a) (2 points) The path $\vec{x}(t)$ and $\dot{\vec{x}}(t)$ can be represented by:

$$\vec{x}(t) = r\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \quad \text{and} \quad \dot{\vec{x}}(t) = \frac{d\vec{x}}{dt} = r\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}.$$

b) (2 points) Integrating counterclockwise from $E(r/0)$ to $W(-r/0)$

$$\begin{aligned}\int_C \vec{F}(\vec{x})d\vec{x} &= \int_0^\pi \frac{1}{r^2\cos^2(t) + r^2\sin^2(t)}\begin{pmatrix} 2 \\ 4 \end{pmatrix} \cdot r\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} dt \\ &= \int_0^\pi \frac{r}{r^2(\cos^2(t) + \sin^2(t))}\Big(-2\sin(t) + 4\cos(t)\Big) dt \\ &= \int_0^\pi \frac{1}{r}\Big(-2\sin(t) + 4\cos(t)\Big) dt = \frac{1}{r}\Big[2\cos(t) + 4\sin(t)\Big]_0^\pi \\ &= \frac{1}{r}\Big[2(-1) - 2\Big] = -\frac{4}{r}\end{aligned}$$

c) (2 points) Integrating clockwise from $N(0/r)$ to $S(0/-r)$

$$\begin{aligned}\int_C \vec{F}(\vec{x})d\vec{x} &= \int_{\pi/2}^{-\pi/2} \frac{1}{r^2\cos^2(t) + r^2\sin^2(t)}\begin{pmatrix} 2 \\ 4 \end{pmatrix} \cdot r\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} dt \\ &= \int_{\pi/2}^{-\pi/2} \frac{r}{r^2(\cos^2(t) + \sin^2(t))}(-2\sin(t) + 4\cos(t))dt \\ &= \int_{\pi/2}^{-\pi/2} \frac{1}{r}(-2\sin(t) + 4\cos(t))dt = \frac{1}{r}\Big[2\cos(t) + 4\sin(t)\Big]_{\pi/2}^{-\pi/2} \\ &= \frac{1}{r}\Big[(0-4) - (0+4)\Big] = -\frac{8}{r}\end{aligned}$$

Exercise 4 (4 points)

a) (2 points) It holds:

$$\vec{x}(t) = t\cdot\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \dot{\vec{x}}(t) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \text{with} \quad 0 \le t \le 1 \qquad \text{(1 point)}$$

The integral is:

$$\int_C \vec{F}(\vec{x})d\vec{x} = \int_0^1 \begin{pmatrix} 5t \\ t^2 \\ 6t^3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} dt = \int_0^1 \left(6t^3 + 2t^2 + 15t\right) dt$$

$$= \left[\frac{6t^4}{4} + \frac{2t^3}{3} + \frac{15t^2}{2}\right]_0^1 = \frac{3}{2} + \frac{2}{3} + \frac{15}{2} = 9 + \frac{2}{3} = 9\frac{2}{3}$$ (1 point)

b) (2 points) It holds:

$$\vec{x}(t) = \begin{pmatrix} 3t \\ 2t^3 \\ t^2 \end{pmatrix} \quad \text{and} \quad \dot{\vec{x}}(t) = \begin{pmatrix} 3 \\ 6t^2 \\ 2t \end{pmatrix} \quad \text{with} \quad 0 \leq t \leq 1$$ (1 point)

The integral is:

$$\int_C \vec{F}(\vec{x})d\vec{x} = \int_0^1 \begin{pmatrix} 3t + 2t^3 \\ t^4 \\ 6t^6 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6t^2 \\ 2t \end{pmatrix} dt = \int_0^1 \left(12t^7 + 6t^6 + 6t^3 + 9t\right) dt$$

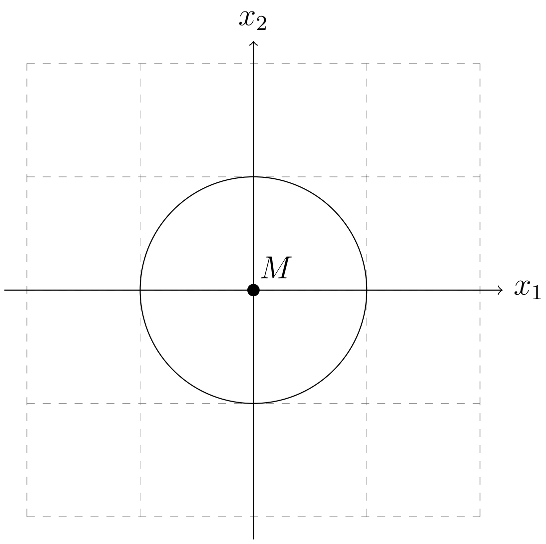
$$= \left[\frac{12t^8}{8} + \frac{6t^7}{7} + \frac{6t^4}{4} + \frac{9t^2}{2}\right]_0^1 = \frac{3}{2} + \frac{6}{7} + \frac{3}{2} + \frac{9}{2} = \frac{15}{2} + \frac{6}{7}$$

$$= \frac{105}{14} + \frac{12}{14} = \frac{117}{14} = 8\frac{5}{14}$$ (1 point)

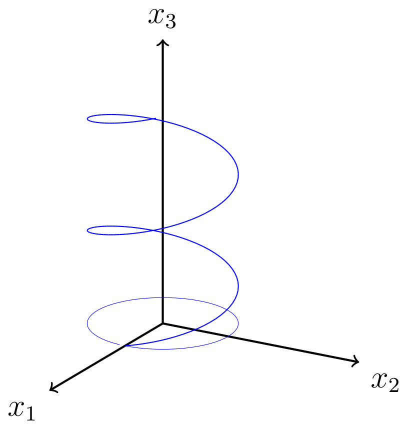
Exercise 5 (6 points)

a) (1 point) We first consider the course of $\vec{x}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix}$ in the x_1-x_2 plane.

The curve $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ is a circle that starts at the point (1/0) and moves counterclockwise. The circle is traversed twice.



In x_3 direction we move according to $x_3(t) = t$. This is a liniar motion with constant velocity.



b) (2 points)

$$\vec{x}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix} \quad \text{and} \quad \dot{\vec{x}}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix} \quad \text{with} \quad 0 \le t \le 4\pi \qquad \text{(1 point)}$$

The integral is:

$$\int_C \vec{F}(\vec{x})d\vec{x} = \int_0^{4\pi} \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix} dt$$

$$= \int_0^{4\pi} \Big(-\sin(t)\cos(t) + \sin(t)\cos(t) + t \Big) dt = \int_0^{4\pi} t dt$$

$$= \left[\frac{t^2}{2} \right]_0^{4\pi} = 8\pi^2 \qquad \text{(1 point)}$$

c) (3 points)

$$\int_C \vec{F} d\vec{x} = \int_{-\infty}^{+\infty} \frac{1}{\cos^2(t) + \sin^2(t) + t^2} \begin{pmatrix} t\sin(t) \\ -t\cos(t) \\ t+1 \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix} dt \qquad \text{(1 point)}$$

$$= \int_{-\infty}^{+\infty} \frac{-t\sin^2(t) - t\cos^2(t) + t + 1}{1+t^2} dt = \int_{-\infty}^{+\infty} \frac{-t[\sin^2(t) + \cos^2(t)] + t + 1}{1+t^2} dt$$

$$= \int_{-\infty}^{+\infty} \frac{-t+t+1}{1+t^2} dt = \int_{-\infty}^{+\infty} \frac{1}{1+t^2} dt \qquad \text{(1 point)}$$

$$= \lim_{t\to\infty} \left[\int_{-t}^{+t} \frac{1}{1+t^2} \right] dt = \lim_{t\to\infty} \Big[\arctan(t) \Big]_{-t}^{+t}$$

$$= \lim_{t\to\infty} [\arctan(t)] - \lim_{t\to\infty} [\arctan(-t)] = \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \pi \qquad \text{(1 point)}$$

Hint: Since $\frac{1}{1+t^2}$ is symmetric one can compute the integral as

$$\int_{-\infty}^{+\infty} \frac{1}{1+t^2} dt = 2 \int_0^{+\infty} \frac{1}{1+t^2} dt = 2 \cdot \lim_{t\to\infty} \left[\int_0^{+t} \frac{1}{1+t^2} \right] dt.$$