Exercise Sheet 8 - Solution

Exercise 1

a)

$$\int (x^4 - 4x^3 + x - 1)dx = \frac{x^5}{5} - 4\frac{x^4}{4} + \frac{x^2}{2} - x + C = \frac{1}{5}x^5 - x^4 + \frac{1}{2}x^2 - x + C$$

b)

$$\int \left(\frac{2}{x} - \frac{t}{x^2}\right)dx = \int \left(\frac{2}{x} - tx^{-2}\right)dx = 2\ln|x| - t\frac{x^{-1}}{-1} + C = 2\ln|x| + \frac{t}{x} + C$$

c)

$$\int \left(\sqrt[4]{x} - \frac{1}{\sqrt[5]{x}}\right)dx = \int \left(x^{\frac{1}{4}} - x^{-\frac{1}{5}}\right)dx = \frac{x^{\frac{5}{4}}}{\frac{5}{4}} - \frac{x^{\frac{4}{5}}}{\frac{4}{5}} + C = \frac{4}{5}\sqrt[4]{x^5} - \frac{5}{4}\sqrt[5]{x^4} + C$$

Exercise 2 (4 points)

a) (1 point)

$$\int f(x)dx = \int (2x^6 - 2x^2 + 3x + 2)dx = \frac{2}{7}x^7 - \frac{2}{3}x^3 + \frac{3}{2}x^2 + 2x + C$$

b) (1 point)

$$\int f(x)dx = \int \left(\frac{4}{x^3} + \frac{2}{x}\right)dx = \int \left(4x^{-3} + 2\frac{1}{x}\right)dx = 4\frac{x^{-2}}{-2} + 2\ln|x| + C$$
$$= -\frac{2}{x^2} + 2\ln|x| + C$$

c) (1 point)

$$\int f(x)dx = \int \left(\frac{1}{\sqrt[5]{x}} - \sqrt[4]{x}\right)dx = \int \left(\frac{1}{x^{\frac{1}{5}}} - x^{\frac{1}{4}}\right)dx = \int \left(x^{-\frac{1}{5}} - x^{\frac{1}{4}}\right)dx$$
$$= \frac{x^{\frac{4}{5}}}{\frac{4}{5}} - \frac{x^{\frac{5}{4}}}{\frac{5}{4}} + C = \frac{5}{4}\sqrt[5]{x^4} - \frac{4}{5}\sqrt[4]{x^5} + C$$

d) (1 point)

$$\int f(x)dx = \int \frac{1}{\tan(x)} = \int \frac{1}{\frac{\sin(x)}{\cos(x)}}dx = \int \frac{\cos(x)}{\sin(x)} = \ln|\sin(x)| + C$$

Exercise 3 (6 points)

a) 1) (1 point) $f(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$ $x_0 = 4$, $y_0 = 2$

$$F(x) = \int f(x)dx = \int (x^{\frac{1}{2}} - x^{-\frac{1}{2}})dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3}\sqrt{x^3} - 2\sqrt{x} + C$$

$$F(4) = \frac{4}{3} + C = 2 \quad \Rightarrow \quad C = \frac{2}{3} \quad \Rightarrow \quad F(x) = \frac{2}{3}\sqrt{x^3} - 2\sqrt{x} + \frac{2}{3}$$

2) (1 point) $f(x) = x - \frac{1}{x}$ $x_0 = \sqrt{e}$, $y_0 = \frac{e}{2}$

$$F(x) = \int f(x)dx = \int (x - \frac{1}{x})dx = \frac{1}{2}x^2 - \ln|x| + C$$

$$F(\sqrt{e}) = \frac{1}{2}e - \frac{1}{2} + C = \frac{e}{2} \quad \Rightarrow \quad C = \frac{1}{2} \quad \Rightarrow \quad F(x) = \frac{1}{2}x^2 - \ln|x| + \frac{1}{2}$$

3) (1 point) $f(x) = 2\sin(x)\cos(x)$ $x_0 = \frac{\pi}{2}$, $y_0 = 0$

$$F(x) = \int f(x)dx = \int (2\sin(x)\cos(x))dx = \int \sin(2x)dx = -\frac{1}{2}\cos(2x) + C$$

$$F(\frac{\pi}{2}) = \frac{1}{2} + C = 0 \quad \Rightarrow \quad C = -\frac{1}{2} \quad \Rightarrow \quad F(x) = -\frac{1}{2}\cos(2x) - \frac{1}{2}$$

b) 1) (1 point)

$$\int_{-2}^{2} \left(x^2 - x + 1\right) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_{-2}^{2} = \left(\frac{8}{3} - \frac{4}{2} + 2\right) - \left(-\frac{8}{3} - \frac{4}{2} + (-2)\right)$$

$$= \frac{8}{3} - 2 + 2 + \frac{8}{3} + 2 + 2 = \frac{16}{3} + 4 = \frac{28}{3}$$

2) (1 point)

$$\int_{-2}^{2} \left(e^{2t} + e^{-2t}\right) dt = \left[\frac{e^{2t}}{2} + \frac{e^{-2t}}{-2}\right]_{-2}^{2} = \frac{1}{2}\left[e^{2t} - e^{-2t}\right]_{-2}^{2}$$

$$= \frac{1}{2}\left[\left(e^4 - e^{-4}\right) - \left(e^{-4} - e^4\right)\right] = \frac{1}{2}\left[e^4 - e^{-4} - e^{-4} + e^4\right]$$

$$= \frac{1}{2}\left[2e^4 - 2e^{-4}\right] = e^4 - e^{-4} \approx 54.57$$

3) (1 point) Hint:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \Rightarrow \quad \int \tan(x)dx = -\ln|\cos(x)| + C$$

With this, the integral can be computed as:

$$\begin{aligned}\int_0^{\pi/3} (\tan(x) + x)\,dx &= \Big[-\ln|\cos(x)| + \frac{x^2}{2} \Big]_0^{\pi/3} \\ &= \Big[-\ln\Big(\frac{1}{2}\Big) + \frac{\pi^2}{18} \Big] - \Big[-\ln(1) + 0 \Big] \\ &= \ln(2) + \frac{\pi^2}{18} \approx 1.24\end{aligned}$$

Exercise 4 (6 points)

a) (2 points)

$$f(x) = x^3 - 2x^2 - x + 2 \qquad f'(x) = 3x^2 - 4x - 1 \qquad f''(x) = 6x - 4$$

Finding the roots: Guessing yields $f(1) = 0$. Dividing $(x^3 - 2x^2 - x + 2) : (x - 1) = x^2 - x - 2 = (x - 2)(x + 1)$ yields the roots $f(-1) = 0$ and $f(2) = 0$.

Finding the extremas:

$$f'(x) = 0 \quad \Leftrightarrow \quad 3x^2 - 4x - 1 = 0 \quad \Rightarrow \quad x_{1,2} = \frac{2 \pm \sqrt{7}}{3}$$

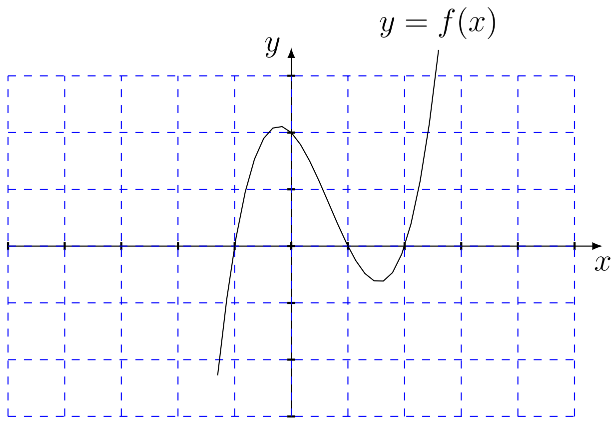
$$x_1 = \frac{2 + \sqrt{7}}{3} \approx 1.54 \quad f''(x_1) = 2\sqrt{7} > 0 \quad \Rightarrow \quad \text{minimum:} \quad f(x_1) = -0.63$$

$$x_2 = \frac{2 - \sqrt{7}}{3} \approx -0.22 \quad f''(x_2) = -2\sqrt{7} < 0 \quad \Rightarrow \quad \text{maximum:} \quad f(x_2) = 2.11$$

(1 point)

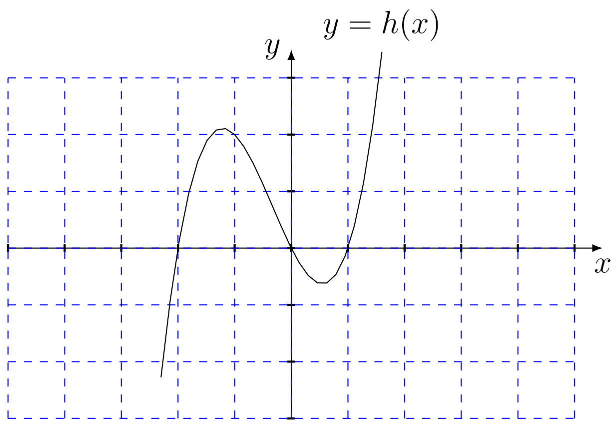
Finding the point of inflection:

$$f''(x) = 0 \quad \Leftrightarrow x_3 = \frac{2}{3} \qquad f'''(x) \neq 0 \quad \Rightarrow \quad \text{point of inflectiont:} \quad f(x_3) \approx 0.74$$



(1 point)

b) (1 point) $h(x)$ emerges when shifting $f(x)$ by one unit to the left.
It holds: $h(x) = f(x - (-1)) = f(x + 1)$. The roots, the extrema and the point of inflection are shifted accordingly!



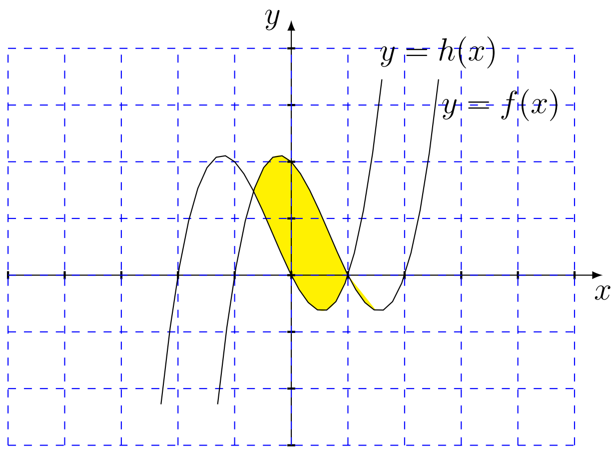
(1 point)

c) (1 point) After a short calculation one gets $h(x) = f(x + 1) = x^3 + x^2 - 2x$. From the representation $h(x) = x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x + 2)(x - 1)$ the roots -2, 0 and 1 are found. This can also be seen from the graphical representation of $h(x)$.

$$
\begin{aligned}
x^3 - 2x^2 - x + 2 &= x^3 + x^2 - 2x && \Big| - x^3 \\
-2x^2 - x + 2 &= x^2 - 2x && \Big| + 2x^2 \\
-x + 2 &= 3x^2 - 2x && \Big| + x \\
2 &= 3x^2 - x && \Big| - 2 \\
0 &= 3x^2 - x - 2
\end{aligned}
$$

We find two intersection points: $x_1 = -\frac{2}{3}$ and $x_2 = 1$.
The points are $S_1(-\frac{2}{3}/\frac{40}{27})$ and $S_2(1/0)$,

d) (2 points)



(1 point)

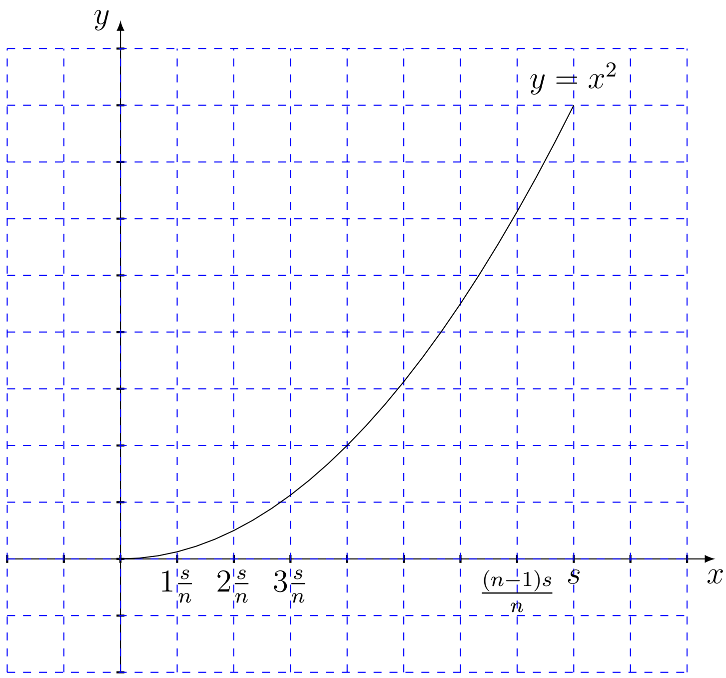
Now the yellow shaded area can be computed:

$$
\begin{aligned}
A &= \int_{-\frac{2}{3}}^{1} [f(x) - h(x)]dx = \int_{-\frac{2}{3}}^{1} [(x^3 - 2x^2 - x + 2) - (x^3 + x^2 - 2x)]dx \\
&= \int_{-\frac{2}{3}}^{1} [x^3 - 2x^2 - x + 2 - x^3 - x^2 + 2x]dx = \int_{-\frac{2}{3}}^{1} [-3x^2 + x + 2]dx \\
&= \left[-x^3 + \frac{x^2}{2} + 2x\right]_{-\frac{2}{3}}^{1} = \left[-1 + \frac{1}{2} + 2\right] - \left[-\left(-\frac{8}{27}\right) + \frac{4}{18} - \frac{4}{3}\right] \\
&= \left[\frac{3}{2}\right] - \left[\frac{8}{27} + \frac{4}{18} - \frac{4}{3}\right] = \frac{3}{2} - \frac{8}{27} - \frac{4}{18} + \frac{4}{3} \\
&= \frac{81}{54} - \frac{16}{54} - \frac{12}{54} + \frac{72}{54} = \frac{125}{54}
\end{aligned}
$$

(1 point)

Exercise 5 (4 points)

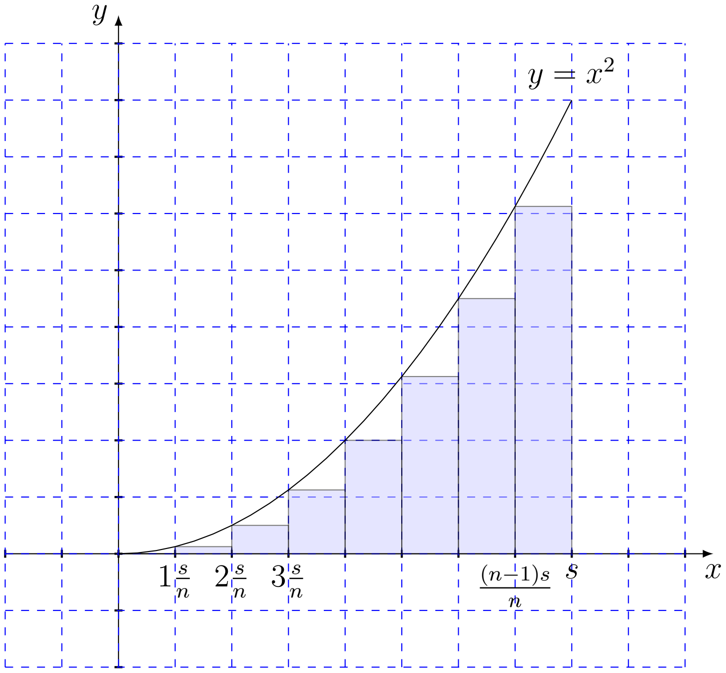
a) (1 point) Let each interval have a length of $\frac{s}{n}$ in x-direction. The first interval ends at $x = 1 \cdot \frac{s}{n}$, the second at $x = 2 \cdot \frac{s}{n}$, the third at $x = 3 \cdot \frac{s}{n}$. The second last interval ends at $x = (n-1) \cdot \frac{s}{n}$, the last at $x = n \cdot \frac{s}{n} = s$.



In the following representations the areas below and above the function are drawn as closed. In fact they consist of n rectangles!

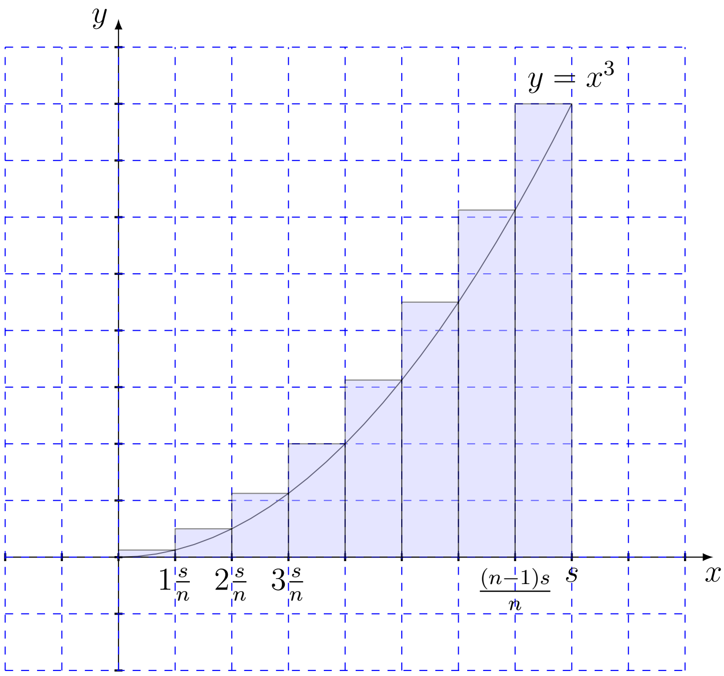
b) (1 point) See the following sketch with the rectangles below the function $y = x^2$.

$$\begin{aligned} U_n =& f(0\frac{s}{n}) \cdot \frac{s}{n} + f(1\frac{s}{n}) \cdot \frac{s}{n} + f(2\frac{s}{n}) \cdot \frac{s}{n} + \cdots + f((n-1)\frac{s}{n}) \cdot \frac{s}{n} = \frac{s}{n} \cdot \sum_{i=0}^{n-1} f\left(i \cdot \frac{s}{n}\right) \\ =& \frac{s}{n} \cdot \sum_{i=0}^{n-1} \left(i \cdot \frac{s}{n}\right)^2 = \frac{s}{n} \cdot \left(\frac{s}{n}\right)^2 \cdot \sum_{i=0}^{n-1} i^2 = \left(\frac{s}{n}\right)^3 \cdot \sum_{i=0}^{n-1} i^2 = \left(\frac{s}{n}\right)^3 \cdot \sum_{i=1}^{n-1} i^2 \\ =& \left(\frac{s}{n}\right)^3 \cdot \frac{(n-1)n(2n-1)}{6} = \frac{(n-1)(2n-1)s^3}{6n^2} \end{aligned}$$



c) (1 point) See the following sketch with the rectangles above the function $y = x^2$.

$$\begin{aligned} O_n =& f(1\frac{s}{n}) \cdot \frac{s}{n} + f(2\frac{s}{n}) \cdot \frac{s}{n} + f(3\frac{s}{n}) \cdot \frac{s}{n} + \cdots + f(s) \cdot \frac{s}{n} = \frac{s}{n} \cdot \sum_{i=1}^{n} f\left(i \cdot \frac{s}{n}\right) \\ =& \frac{s}{n} \cdot \sum_{i=1}^{n} \left(i \cdot \frac{s}{n}\right)^2 = \frac{s}{n} \cdot \left(\frac{s}{n}\right)^2 \cdot \sum_{i=1}^{n} i^2 = \left(\frac{s}{n}\right)^3 \cdot \sum_{i=1}^{n} i^2 \\ =& \left(\frac{s}{n}\right)^3 \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)s^3}{6n^2} \end{aligned}$$



d) (1 point) Thus we can now show:

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(n-1)(2n-1)s^3}{6n^2} = \lim_{n\to\infty} \frac{(1-\frac{1}{n})(2-\frac{1}{n})s^3}{6} = \frac{2s^3}{6} = \frac{s^3}{3}$$

$$\lim_{n\to\infty} O_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)s^3}{6n^2} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})(2+\frac{1}{n})s^3}{6} = \frac{2s^3}{6} = \frac{s^3}{3}$$

We have shown that

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} O_n \quad \text{holds and that} \quad \int_0^s x^2 dx = \frac{s^3}{3} \quad .$$