Exercise Sheet 7 - Solution

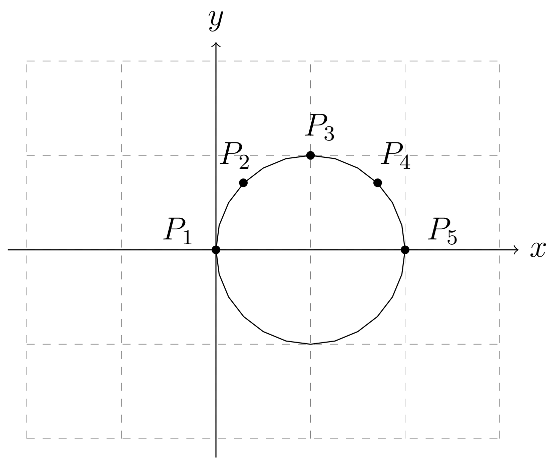
Exercise 1

a) It holds $\dot{\vec{x}}(t) = \frac{d\vec{x}}{dt} = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$. This function corresponds to the velocity vector of $\vec{x}(t)$ or the tangent vector of $\vec{x}(t)$.

b) Plugging in numbers for t.

$$
\begin{aligned}
t_1 &= 0 : & \vec{x}(t_1) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \dot{\vec{x}}(t_1) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
t_2 &= \frac{\pi}{4} : & \vec{x}(t_2) &= \begin{pmatrix} 1 - \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} & \dot{\vec{x}}(t_2) &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\
t_3 &= \frac{\pi}{2} : & \vec{x}(t_3) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \dot{\vec{x}}(t_3) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
t_4 &= \frac{3\pi}{4} : & \vec{x}(t_4) &= \begin{pmatrix} 1 + \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} & \dot{\vec{x}}(t_4) &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\
t_5 &= \pi : & \vec{x}(t_5) &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \dot{\vec{x}}(t_5) &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\end{aligned}
$$

c) The function $\vec{x}(t)$ can be drawn by decomposing $\vec{x}(t) = \begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or with the help of the points computed in b). See sketch below, P_i are the points corresponding to t_i.



Exercise 2 (4 points)

a) (1 point)

$$\vec{v}(t) = \dot{\vec{x}}(t) = \begin{pmatrix} 6t \\ 2 \\ 3t^2 - 1 \end{pmatrix} \qquad v(t) = \left|\dot{\vec{x}}(t)\right| = \sqrt{9t^4 + 30t^2 + 5}$$

b) (1 point)

$$\vec{a}(t) = \ddot{\vec{x}}(t) = \begin{pmatrix} 6 \\ 0 \\ 6t \end{pmatrix} \qquad a(t) = \left|\ddot{\vec{x}}(t)\right| = \sqrt{36t^2 + 36} = 6\sqrt{t^2 + 1}$$

c) (2 points) We look for the minimum of the function $v(t)$. The function $v(t)$ is defined for $t \geq 0$ and $v(t)$ is minimal if the function of the root is minimal. So:

$$f(t) = 9t^4 + 30t^2 + 5 \qquad \dot{f}(t) = 36t^3 + 60t \qquad \ddot{f}(t) = 108t^2 + 60$$

$$\dot{f}(t) = 0 \quad \Rightarrow 6t(6t^2 + 10) = 0 \quad \Rightarrow t = 0 \text{ , for } 6t^2 + 10 = 0 \text{ no solution}$$

$$\text{so:} \quad \Rightarrow \quad t = 0 \quad \ddot{f}(0) = 60 > 0 \quad \Rightarrow \quad \text{relative minimum}$$ (1 point)

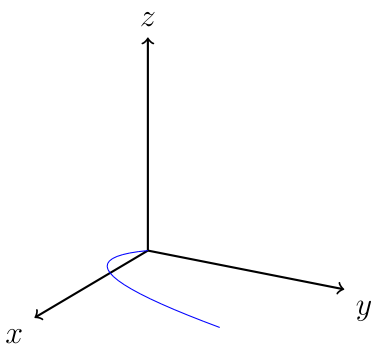
$$v(0) = \sqrt{5}$$

$t = 0$ is a boundary point! The function $f(t)$ is differentiable everywhere and for $t \to \infty$ we have $f(t) \to \infty$.
So we have an absolute minimum at $t = 0$. (1 point)

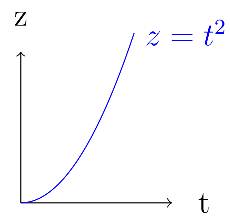
Exercise 3 (6 points)

a) We use the relations $x_1(t) = x(t)$, $x_2(t) = y(t)$ and $x_3(t) = z(t)$

- (1 point) With $x(t) = t$ and $y(t) = t(t-1) = t^2 - t$ we obtain the function $y = x(x-1) = x^2 - x$ by elimination of the parameter t. This is a parabola as can be seen from the graph on the right.



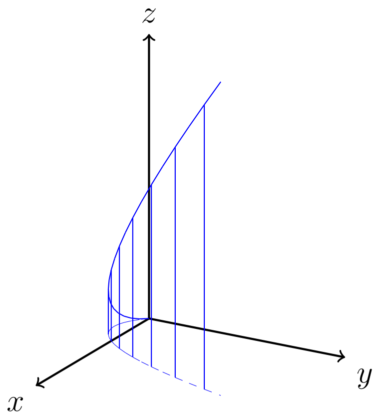
- (1 point) $z(t) = t^2$ is a parabola. Its value increases quadratically.



- (2 points) Plugging in values yields:

$$\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{x}(1) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{x}(2) = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(1 point)



(1 point)

b) (1 point)

$$\dot{\vec{x}}(t) = \frac{d\vec{x}(t)}{dt} = \begin{pmatrix} 1 \\ 2t-1 \\ 2t \end{pmatrix} \qquad \left|\dot{\vec{x}}(t)\right| = \sqrt{1+(2t-1)^2+(2t)^2} = \sqrt{8t^2-4t+2}$$

$$\dot{\vec{x}}(0) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \left|\dot{\vec{x}}(0)\right| = \sqrt{2}$$

$$\dot{\vec{x}}(1) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad \left|\dot{\vec{x}}(1)\right| = \sqrt{6}$$

$$\dot{\vec{x}}(2) = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \qquad \left|\dot{\vec{x}}(2)\right| = \sqrt{26}$$

c) (1 point) With $\left|\dot{\vec{x}}(t)\right| = f(t) = \sqrt{8t^2-4t+2}$ we get, as in subtask 1b): If $g(t) = 8t^2 - 4t + 2$ has an extremum then $f(t)$ has an extremum as well. The first and second derivatives of $g(t)$ are:

$$\dot{g}(t) = 16t - 4 \qquad \ddot{g}(t) = 16$$

From $\dot{g}(t) = 0$ one obtains $t = \frac{1}{4}$. Since $\ddot{g}(t) > 0$ for all $t \in \mathbb{R}$ there is a minimum at $t = \frac{1}{4}$. We get $\left|\dot{\vec{x}}(\frac{1}{4})\right| = \sqrt{\frac{3}{2}}$.
We need to check the boundary points.
Boundary point for $t = 0$: $\left|\dot{\vec{x}}(0)\right| = \sqrt{2}$.
Boundary point for $t = 2$: $\left|\dot{\vec{x}}(2)\right| = \sqrt{26}$.

Summarizing we can say that for $t = \frac{1}{4}$ the velocity is minimal and for $t = 2$ the velocity is maximal.

Exercise 4 (2 points)

a) (1 point)

$$\vec{x}(t) = \begin{pmatrix} 3t \\ t^2 - 2t \\ 4 - t^2 \end{pmatrix} \qquad \dot{\vec{x}}(t) = \begin{pmatrix} 3 \\ 2t - 2 \\ -2t \end{pmatrix}$$

$$\text{Tangent T:} \quad \vec{x}(t) = \vec{x}(2) + t \cdot \dot{\vec{x}}(2) = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$$

b) (1 point) The condition $\dot{\vec{x}}(t_1) \cdot \dot{\vec{x}}(t_2) = 0$ has to be satisfied, where t_1 and t_2 have to be in $[1, 3]$.

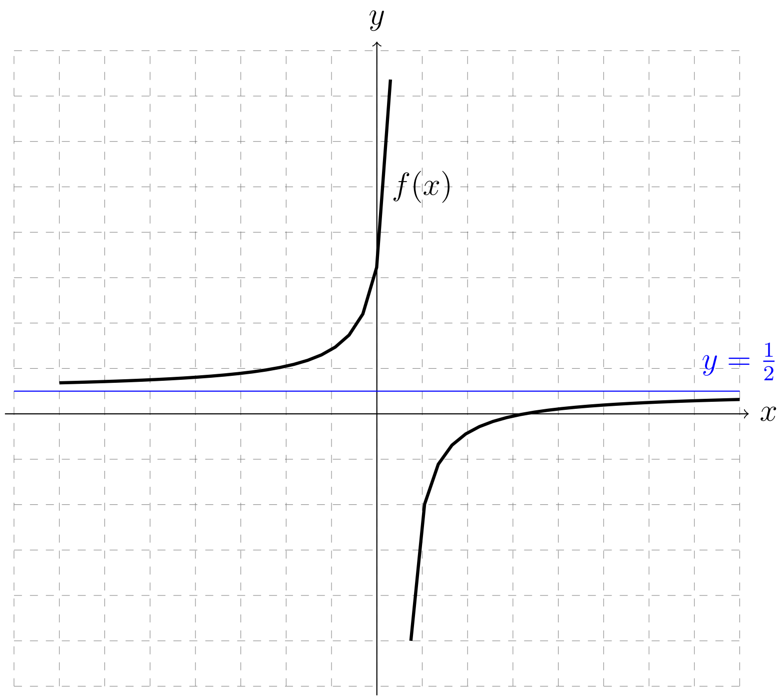
$$\begin{pmatrix} 3 \\ 2t_1 - 2 \\ -2t_1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2t_2 - 2 \\ -2t_2 \end{pmatrix} = 0 \quad \Leftrightarrow \quad 8t_1t_2 - 4t_1 - 4t_2 + 13 = 0 \quad \Leftrightarrow \quad t_2 = \frac{4t_1 - 13}{8t_1 - 4}$$

We consider t_2 as a function of t_1 and need to know the range of t_2 for t_1 in the interval $[1, 3]$.

Consider $y = \frac{4x-13}{8x-4}$ where $1 \leq x \leq 3$. We find

- y has a root at $x = \frac{13}{4}$
- y has a pole at $x = \frac{1}{2}$
- y has for $x \to \pm\infty$ the asymptote $y = \frac{1}{2}$

As y moves from the pole in $-\infty$ towards the root $x = \frac{13}{4}$ the values of y are always ≤ 0. So y is not contained in the interval $[1, 3]$. Also see graphical representation.



$\Rightarrow$ There is no solution since for $1 < t_1 \leq 3$ t_2 is not in the interval $[1, 3]$.

Exercise 5 (7 points)

a) (1 point) Find $\dot{\vec{r}}(t)$:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3t^2 - 1 \\ 3t^3 - t \end{pmatrix} \qquad \Rightarrow \qquad \dot{\vec{r}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 6t \\ 9t^2 - 1 \end{pmatrix}$$

b) (1 point) <u>y-intercept</u>: x = 0

$$3t^2 - 1 = 0 \quad \Rightarrow \quad t_{1,2} = \pm\frac{1}{\sqrt{3}} \qquad \Rightarrow \quad \vec{r}(t_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{r}(t_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The point $P(0/0)$ is passed twice. It must be investigated seperately.

<u>x-intercept</u>: y = 0

$$3t^3 - t = 0 \quad \Rightarrow \quad t_{1,2} = \pm\frac{1}{\sqrt{3}} \text{ (already computed)} \quad t_3 = 0 \qquad \Rightarrow \quad \vec{r}(t_3) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

c) (1 point) The tangent is vertical if $\dot{x}(t) = 0$ and $\dot{y}(t) \neq 0$.

$$6t = 0 \quad \Rightarrow \quad t_3 = 0 \text{ , siehe oben} \quad \Rightarrow \quad \vec{r}(t_3) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad , \quad \dot{\vec{r}}(t_3) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

d) (1 point) The tangent is horizontal if $\dot{y}(t) = 0$ and $\dot{x}(t) \neq 0$.

$$9t^2 - 1 = 0 \quad \Rightarrow \quad t_{4,5} = \pm\frac{1}{3}$$

$$\Rightarrow \quad \vec{r}(t_4) = \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{9} \end{pmatrix} \quad , \quad \dot{\vec{r}}(t_4) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \vec{r}(t_5) = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{9} \end{pmatrix} \quad , \quad \dot{\vec{r}}(t_5) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

e) (1 point) The point $P(0/0)$ is passed at $t_1 = \frac{1}{\sqrt{3}}$ and at $t_2 = -\frac{1}{\sqrt{3}}$.

$$\dot{\vec{r}}(t_1) = \begin{pmatrix} 6 \cdot \frac{1}{\sqrt{3}} \\ 9 \cdot \frac{1}{3} - 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot \sqrt{3} \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

$$\dot{\vec{r}}(t_2) = \begin{pmatrix} -6 \cdot \frac{1}{\sqrt{3}} \\ 9 \cdot \frac{1}{3} - 1 \end{pmatrix} = \begin{pmatrix} -2 \cdot \sqrt{3} \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$$

f) (1 point)

$$\vec{r}(-1) = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \qquad \vec{r}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \vec{r}(1) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

g) (1 point)

We start with $\vec{r}(-1)$. The direction vector is $\dot{\vec{r}}(-1) = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$. From this points one moves towards the origin.

Subsequently one reaches the origin, computed point $\vec{r}(-\frac{1}{\sqrt{3}})$, with direction $\dot{\vec{r}}(-\frac{1}{\sqrt{3}}) = 2 \cdot \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$.

One moves on towards the point with horizontal tangent $\vec{r}(-\frac{1}{3}) = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{9} \end{pmatrix}$ with direction vector $\dot{\vec{r}}(-\frac{1}{3}) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$.

Afterwards one reaches the point with vertical tangent $\vec{r}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ with direction vector $\dot{\vec{r}}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Accordingly one moves towards the point $\vec{r}(\frac{1}{3})$, to the point $\vec{r}(\frac{1}{\sqrt{3}})$ and end at the point $\vec{r}(1)$.
The curve can be drawn as follows: