Exercise Sheet 6 - Solution

Exercise 1

a) Let $f(x)$ be a differentiable function on the interval I. Then:

- $f'(x) > 0$ for all $x \in I \quad \Rightarrow \quad f(x)$ is increasing on I
- $f'(x) < 0$ for all $x \in I \quad \Rightarrow \quad f(x)$ is decreasing on I

b) Let $f : D(f) \to \mathbb{R}$.

- f has an absolute maximum at x_0 , if
$$f(x_0) \geq f(x) \text{ for all } x \in \mathbb{D}(f).$$
- f has an absolute minimum at x_0 , if
$$f(x_0) \leq f(x) \text{ for all } x \in \mathbb{D}(f).$$
- The function f has a relative maximum at $x_0 \in \mathbb{D}(f)$ if there exists a ϵ-neighbourhood $U_\epsilon(x_0)$ such that
$$f(x_0) \geq f(x) \text{ for all } x \in \mathbb{D}(f) \cap U_\epsilon(x_0) \ .$$
- The function f has a relative minimum at $x_0 \in \mathbb{D}(f)$ if there exists a ϵ-neighbourhood $U_\epsilon(x_0)$ such that
$$f(x_0) \leq f(x) \text{ for all } x \in \mathbb{D}(f) \cap U_\epsilon(x_0) \ .$$

See also Storrer page 84/85.

c) The function f has an inflection point at x_0 if $f''(x_0) = 0$ and f'' changes the sign in x_0.

d) The saddle point is a horizontal point of inflection. So it additionally holds that $f'(x_0) = 0$.

Exercise 2 (9 points)

a) (1 point) The e function only takes on positive values so the function $y = x + 1$ descides. It holds:
$f(x) < 0$ for $x < -1$ and $f(x) > 0$ for $x > -1$

b) (1 point) $f(x) = (x+1) \cdot e^{-x} = 0 \quad \Leftrightarrow \quad x + 1 = 0 \quad \Rightarrow \quad x = -1$

c) (2 points)

$f'(x) = 1 \cdot e^{-x} + (x+1) \cdot e^{-x} \cdot (-1) = e^{-x} - x \cdot e^{-x} - e^{-x} = -x \cdot e^{-x}$ (1 point)

It follows: the function is increasing ($f'(x) > 0$) for $x < 0$ and the function is decreasing ($f'(x) < 0$) for $x > 0$. (1 point)

d) (1 point) From $f'(x) = 0$ we get $x = 0$. We need the second derivative.

$$f''(x) = (-1) \cdot e^{-x} + (-x) \cdot e^{-x} \cdot (-1) = -e^{-x} + x \cdot e^{-x} = (x-1) \cdot e^{-x}$$

Since $f''(0) = -e^{-0} = -1 < 0$, the point (0/1) is a relative maximum. This is also the absolute maximum.

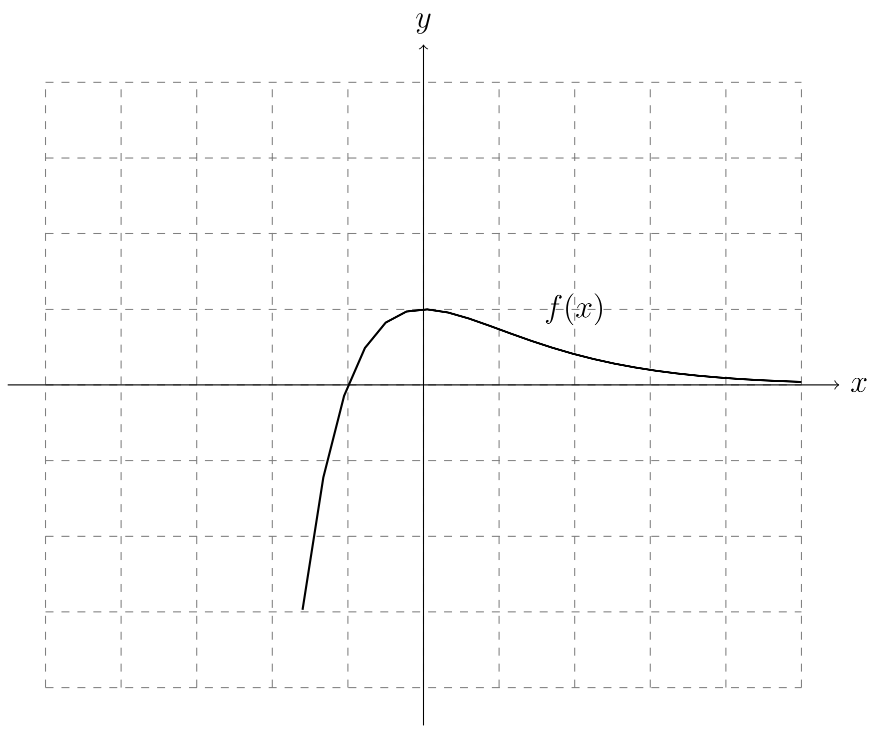
e) (2 points)

Concave: $f''(x) > 0 \qquad x > 1$ (1 point)

Convex: $f''(x) < 0 \qquad x < 1$ (1 point)

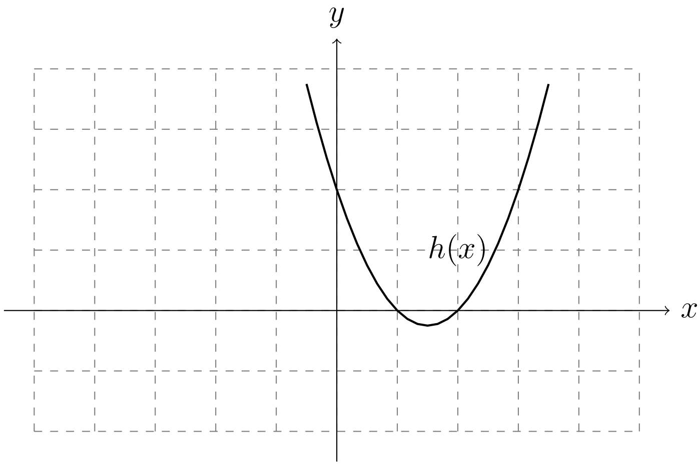
f) (1 point) $f''(0) = 0 \qquad \Rightarrow \qquad x = 1, y = \frac{2}{e}$

g) (1 point)



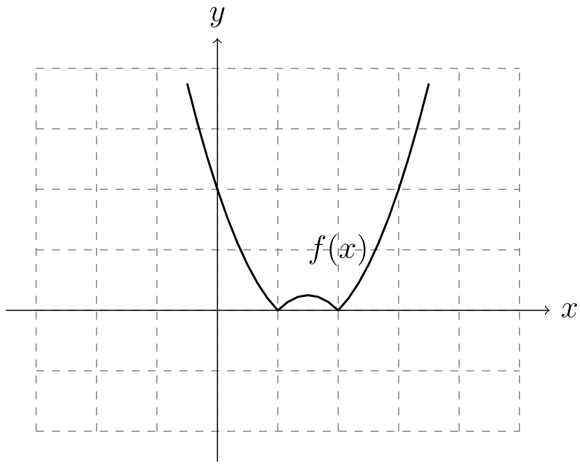
Exercise 3 (5 points)

a) We first consider the function $h(x) = x^2 - 3x + 2$. Since this is a simple function - a parabola of degree two opening to the top - and since $h(x) = (x-2)(x-1)$ with roots at $x_1 = 2$ and $x_2 = 1$, we can quickly draw a qualitative graph. Due to symmetry the vertex (extremum) is at $x_s = \frac{x_1+x_2}{2} = 1.5$. (1 point)



(1 point)

So we find the sought values. The roots of $f(x)$ are $x_1 = 2$ and $x_2 = 1$. At $x_s = 1.5$ there is a local (or relative) maximum with $f(x) = 0.25$.
There are global minima at the roots. (1 point)



(1 point)

b) (1 point) Consider now the function on $[0, 2]$ we have the same roots but the extrema change. At $x = 0$ there is a global maximum while at $x_s = 1.5$ there is still a local (or realtive) maximum. There are global minima at the roots.

Exercise 4 (4 points)

We set $y = ax^2 + bx + c$. (1 point)

The derivative is $y' = 2ax + b$. (1 point)

The conditions given imply the equations: (1 point)

$$\begin{aligned} a + b + c &= 3 \qquad (1) \\ 2a + b \quad &= 1 \qquad (2) \\ 4a + b \quad &= 5 \qquad (3) \end{aligned}$$

From (3)-(2) we get $2a = 4$ and thus $a = 2$. From (2) we get directly $b = -3$ and from (3) $c = 4$.

The sought function is $y = 2x^2 - 3x + 4$. (1 point)

Exercise 5 (2 points) With $y = f(x) = \frac{1}{x^2}$ and $y' = f'(x) = -\frac{2}{x^3}$ we find the linearized function at $x_0 = 0,01$

$$\begin{aligned} y = g(x) =& f(0,01) + f'(0,01)(x - 0,01) = 10^4 - 2 \cdot 10^6 (x - 0,01) \\ =& 10^4 - 2 \cdot 10^6 \cdot x + 2 \cdot 10^4 = 3 \cdot 10^4 - 2 \cdot 10^6 \cdot x \end{aligned}$$

$$\begin{aligned} g(0,0001) =& 3 \cdot 10^4 - 2 \cdot 10^6 \cdot 10^{-4} = 3 \cdot 10^4 - 2 \cdot 10^2 = 30000 - 200 = 29800 \\ f(0.0001) =& \frac{1}{(10^{-4})^2} = \frac{1}{10^{-8}} = 10^8 = 100'000'000 \end{aligned}$$ (1 point)

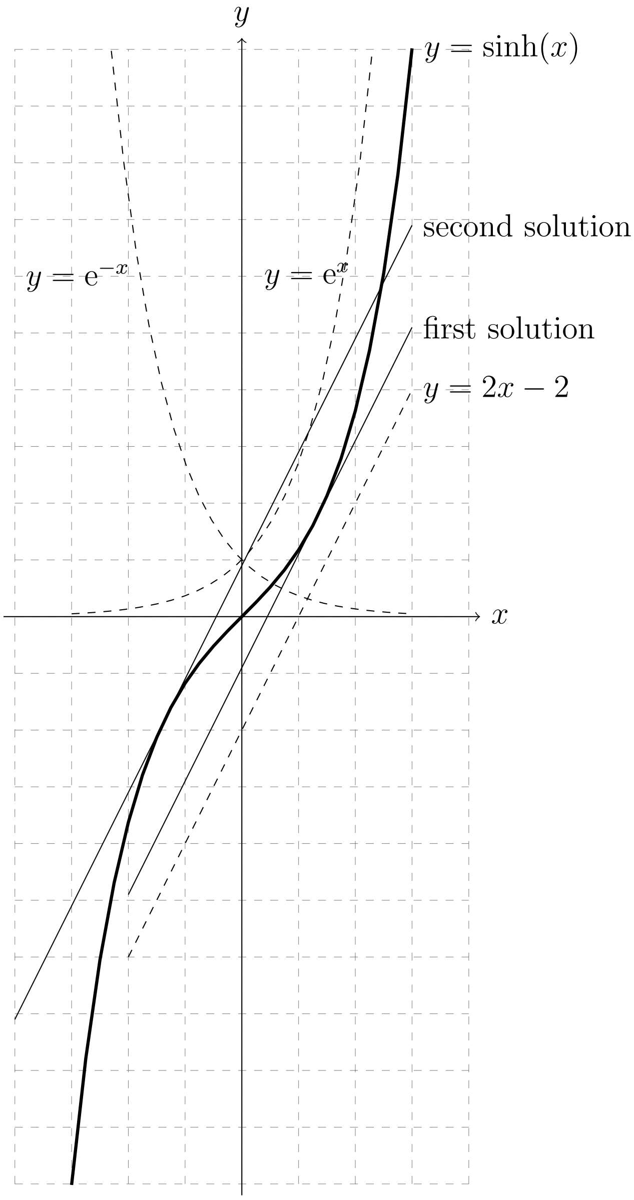
The result of the linearized function $g(x)$ is far from the correct value. Still, linearization is often used in natural sciences and engineering.

Conclusion: when computing values in a neighborhood of a linearized function the function itself is not allowed to have a large curvature. (1 point)

Exercise 6 (3 points)
From $f(x) = \frac{1}{2}(e^x - e^{-x})$ we get $f'(x) = \frac{1}{2}(e^x + e^{-x})$.
From $f'(x) = 2$ we find the equation $\frac{1}{2}(e^x + e^{-x}) = 2$. Substituting $z = e^x$ yields a quadratich equation $z^2 - 4z + 1 = 0$.
We find the two solutions $z_{1,2} = 2 \pm \sqrt{3}$
$e^x = 2 + \sqrt{3} \approx 3.732$ implies $x_1 \approx 1.32$
$e^x = 2 - \sqrt{3} \approx 0.268$ implies: $x_2 \approx -1.32$ (1 point)

From $f'(x) = \frac{1}{2}$ we find $\frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}$. The same substitution yields the quadratic equation $z^2 - z + 1 = 0$. This equation has no solution! (1 point)

Graphical representation. (1 point)



y
$y = \sinh(x)$
second solution
$y = \mathrm{e}^{-x}$
$y = \mathrm{e}^{x}$
first solution
$y = 2x - 2$
x