Exercise Sheet 5 - Solution

Exercise 1

a) Definition of continuity:

$$\lim_{x \to x_0} f(x) = f(x_0) \quad \Leftrightarrow \quad \lim_{x \uparrow x_0} f(x) = \lim_{x \downarrow x_0} f(x) = f(x_0)$$

b) A differentiable function is always continuous!

c)

(1) $f(x) = 0 \quad \Rightarrow \quad f'(x) = 0$ (2) $f(x) = 5 \quad \Rightarrow \quad f'(x) = 0$

(3) $f(x) = x \quad \Rightarrow \quad f'(x) = 1$ (4) $f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$

(5) $f(x) = e^x \quad \Rightarrow \quad f'(x) = e^x$ (6) $f(x) = \ln(x) \quad \Rightarrow \quad f'(x) = \frac{1}{x}$

(7) $f(x) = \sin(x) \quad \Rightarrow \quad f'(x) = \cos(x)$ (8) $f(x) = \cos(x) \quad \Rightarrow \quad f'(x) = -\sin(x)$

Exercise 2 (3 points) The function f is differentiable at x_0 if the conditions below are fulfilled.

- left- and right-hand function value are equal:

$$\lim_{x \uparrow x_0} f(x) = \lim_{x \downarrow x_0} f(x) = f(x_0) \quad \Rightarrow \text{ f is continuous at } x_0$$

- left- and right-hand derivatives are equal:

$$\lim_{x \uparrow x_0} f'(x) = \lim_{x \downarrow x_0} f'(x) \quad \Rightarrow \text{ f is differentiable at } x_0$$

See also Storrer page 59.

a) (1 point)

$$f'(x) = \begin{cases} 2x - a & , \quad x \leq x_0 \\ -2x & , \quad x > x_0 \end{cases} \quad ; \ x_0 = 1$$

These two conditions imply:

$$\left| \begin{array}{ccccc} 1 & - & a & = & -1 & + & b \\ 2 & - & a & = & -2 & & \end{array} \right|$$

Hence $a = 4$ and $b = -2$.

b) (1 point)

$$f'(x) = \begin{cases} \frac{1}{2} & , \quad x \le x_0 \\ a^2 x^{a-1} + b & , \quad x > x_0 \end{cases} \quad ; \; x_0 = 1$$

These two conditions imply:

$$\left| \begin{array}{rcrcr} \frac{3}{4} & = & a & + & b \\ \frac{1}{2} & = & a^2 & + & b \end{array} \right|$$

Hence $a = \frac{1}{2}$ and $b = \frac{1}{4}$.

c) (1 point)

$$f'((x) = \begin{cases} ae^{ax} & , \quad x \le x_0 \\ -\frac{b}{(x+2)^2} & , \quad x > x_0 \end{cases} \quad ; \; x_0 = 0$$

These two conditions imply:

$$\left| \begin{array}{rcrr} 1 & = & & \frac{b}{2} \\ a & = & - & \frac{b}{4} \end{array} \right|$$

Hence $a = -\frac{1}{2}$ and $b = 2$.

Exercise 3 (4 points)

a) (1 point) The result in the first row is enough to get all the points.

$$f'(x) = \left[\ln(x) \cdot x^n \right]' = \frac{1}{x} \cdot x^n + \ln(x) \cdot nx^{n-1} = x^{n-1} + n \cdot \ln(x) \cdot x^{n-1}$$

$$= x^{n-1} \left[1 + n \cdot \ln(x) \right] = x^{n-1} \left[\ln(e) + \ln(x^n) \right] = x^{n-1} \cdot \ln(e \cdot x^n)$$

b) (1 point) The result in the second row is enough to get all the points, also with negative exponent.

$$f'(x) = \frac{d}{dx} \left[\sqrt[4]{\frac{x^2 - 1}{x}} \right] = \frac{d}{dx} \left[\frac{x^2 - 1}{x} \right]^{\frac{1}{4}} = \frac{1}{4} \cdot \left[\frac{x^2 - 1}{x} \right]^{-\frac{3}{4}} \cdot \frac{2x \cdot x - (x^2 - 1) \cdot 1}{x^2}$$

$$= \frac{1}{4} \cdot \left[\frac{x}{x^2 - 1} \right]^{\frac{3}{4}} \cdot \frac{2x^2 - x^2 + 1}{x^2} = \frac{1}{4} \cdot \left[\frac{x}{x^2 - 1} \right]^{\frac{3}{4}} \cdot \frac{x^2 + 1}{x^2}$$

$$= \frac{1}{4} \cdot \sqrt[4]{\left[\frac{x}{x^2 - 1} \right]^3} \cdot \frac{x^2 + 1}{x^2}$$

c) (1 point)

$$\begin{aligned} f'(x) &= \frac{2\cos(x)(-\sin(x)) \cdot (e^{-x} - e^{x}) - \cos^2(x) \cdot (-e^{-x} - e^{x}))}{(e^{-x} - e^{x})^2} \\ &= \frac{-2\sin(x)\cos(x) \cdot (e^{-x} - e^{x}) + \cos^2(x) \cdot (e^{-x} + e^{x}))}{(e^{-x} - e^{x})^2} \\ &= \frac{-\sin(2x) \cdot (e^{-x} - e^{x}) + \cos^2(x) \cdot (e^{-x} + e^{x}))}{(e^{-x} - e^{x})^2} \end{aligned}$$

d) (1 point)

$$\begin{aligned} f'(x) &= [\cos(\cos(\cos(x)))]' = -\sin(\cos(\cos(x))) \cdot [\cos(\cos(x))]' \\ &= -\sin(\cos(\cos(x))) \cdot (-\sin(\cos(x))) \cdot [\cos(x)]' \\ &= \sin(\cos(\cos(x))) \cdot \sin(\cos(x)) \cdot (-\sin(x)) \\ &= -\sin(\cos(\cos(x))) \cdot \sin(\cos(x)) \cdot \sin(x) \end{aligned}$$

Exercise 4 (3 points)

a) (2 points) The derviative of the function $f(x)$ can be computed by applying the chain rule. The outer function has the form $\sqrt{x}$ with derivative $\frac{1}{2\cdot\sqrt{x}}$. (1 point)

$$\begin{aligned} f'(x) &= \left[\sqrt{\sqrt{x^2 - a^2} - a^2}\right]' = \frac{1}{2 \cdot \sqrt{\sqrt{x^2 - a^2} - a^2}} \cdot \left[\sqrt{x^2 - a^2} - a^2\right]' \\ &= \frac{1}{2 \cdot \sqrt{\sqrt{x^2 - a^2} - a^2}} \cdot \frac{1}{2 \cdot \sqrt{x^2 - a^2}} \cdot \left[x^2 - a^2\right]' \\ &= \frac{1}{2 \cdot \sqrt{\sqrt{x^2 - a^2} - a^2}} \cdot \frac{1}{2 \cdot \sqrt{x^2 - a^2}} \cdot 2x \\ &= \frac{x}{2 \cdot \sqrt{\sqrt{x^2 - a^2} - a^2} \cdot \sqrt{x^2 - a^2}} \end{aligned}$$

(1 point)

b) (1 point) The derviative of the function $f(x)$ can be computed by applying the chain rule. The outer function is $\tan x$ with derivative $1 + \tan^2 x = \frac{1}{\cos^2(x)}$.

$$\begin{aligned} f'(x) &= \left[\tan(\sqrt{x^2 - 1})\right]' = \left[1 + \tan^2(\sqrt{x^2 - 1})\right] \cdot \left[\sqrt{x^2 - 1}\right]' \\ &= \left[1 + \tan^2(\sqrt{x^2 - 1})\right] \cdot \frac{1}{2 \cdot \sqrt{x^2 - 1}} \cdot 2x = \left[1 + \tan^2(\sqrt{x^2 - 1})\right] \cdot \frac{x}{\sqrt{x^2 - 1}} \end{aligned}$$

Exercise 5 (3 points)
We set $h(x) = \sqrt{1-x^2}$ to enhance readability. With $f(x) = \arcsin(h(x))$ we find the derivative $f'(x) = \frac{1}{\sqrt{1-h^2(x)}} \cdot h'(x)$. (1 point)

$$
\begin{aligned}
f'(x) &= \left[\arcsin(\sqrt{1-x^2})\right]' = \left[\arcsin(h(x))\right]' \\
&= \frac{1}{\sqrt{1-h^2(x)}} \cdot \left[h(x)\right]' = \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \left[\sqrt{1-x^2}\right]' \\
&= \frac{1}{\sqrt{1-(1-x^2)}} \cdot \frac{1}{2 \cdot \sqrt{1-x^2}} \cdot (-2x) && \text{(1 point)} \\
&= \frac{1}{\sqrt{1-1+x^2}} \cdot \frac{1}{\sqrt{1-x^2}} \cdot (-x) = \frac{1}{\sqrt{x^2}} \cdot \frac{1}{\sqrt{1-x^2}} \cdot (-x) \\
&= -\frac{x}{|x|} \cdot \frac{1}{\sqrt{1-x^2}} && \text{(1 point)}
\end{aligned}
$$