Exercise Sheet 4 - Solution

Exercise 1

x	x_0	$\Delta x = x - x_0$	$f(x) = x^3$	$f(x_0) = x_0^3$	$\Delta f = f(x) - f(x_0)$	$\frac{\Delta f}{\Delta x}$
2	1	1	8	1	7	7
1.5	1	0.5	3.375	1	2.375	4.75
1.3	1	0.3	2.197	1	1.197	3.99
1.1	1	0.1	1.331	1	0.331	3.31
1.05	1	0.05	1.157625	1	0.157625	3.1525
1.01	1	0.01	1.030301	1	0.030301	3.0301
1.001	1	0.001	1.003003001	1	0.003003001	3.003001

Exercise 2 (4 points)

a) (2 points) Polynomial division yields:

$$(x^4 - x_0^4) : (x - x_0) = x^3 + x^2 x_0 + x x_0^2 + x_0^3 \qquad \text{(1 point)}$$

$$\begin{array}{l}
\underline{-(x^4 - x^3 x_0)} \\
\quad x^3 x_0 - x_0^4 \\
\quad \underline{-(x^3 x_0 - x^2 x_0^2)} \\
\qquad x^2 x_0^2 - x_0^4 \\
\qquad \underline{-(x^2 x_0^2 - x x_0^3)} \\
\qquad\quad x x_0^3 - x_0^4 \\
\qquad\quad \underline{-(x x_0^3 - x_0^4)} \\
\qquad\qquad 0
\end{array}$$

So: $f'(x_0) = \lim_{x \to x_0} \frac{x^4 - x_0^4}{x - x_0} = \lim_{x \to x_0} (x^3 + x^2 x_0 + x x_0^2 + x_0^3) = 4x_0^3$ (1 point)

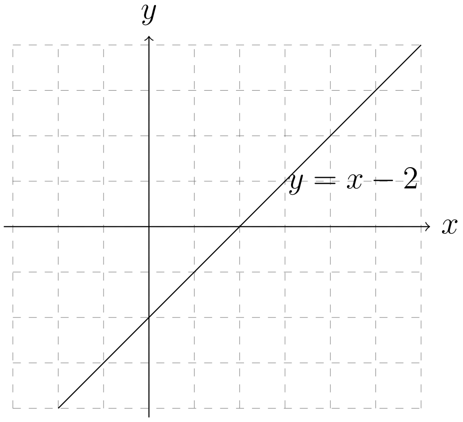
b) (2 points)

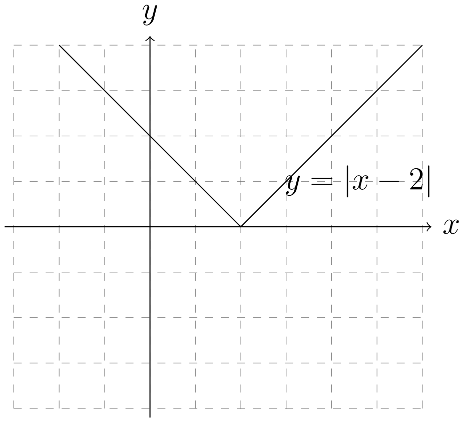
$$f'(x_0) = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{(\sqrt{x})^2 - (\sqrt{x_0})^2} = \lim_{x \to x_0} \frac{\cancel{\sqrt{x} - \sqrt{x_0}}}{(\sqrt{x} + \sqrt{x_0})\cancel{(\sqrt{x} - \sqrt{x_0})}}$$

$$= \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{\sqrt{x_0} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}$$

Exercise 3 (4 points)

a) (1 point) We draw the grph of the function $y = f(x)$ by first drawing the graph of the function $y = g(x) = x - 2$!





As can be seen from the plots, the function $f(x)$ has different slopes. For $x < 2$ the slope is -1!

b) (1 point)

$$f(x) = \begin{cases} x - 2, & x > 2 \\ 0, & x = 2 \\ -x + 2, & x < 2 \end{cases}$$

c) (2 points) For the two regions $x < 2$ and $x > 2$ we get the following differential quotients:

$$x < 2 \text{ and } x_0 < 2 \quad \Rightarrow \quad f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{(-x+2) - (-x_0+2)}{x - x_0}$$
$$= \lim_{x \to x_0} \frac{-x + 2 + x_0 - 2}{x - x_0} = \lim_{x \to x_0} \frac{-x + x_0}{x - x_0}$$
$$= \lim_{x \to x_0} \frac{-(x_0 - x)}{x - x_0} = -1$$
$$x > 2 \text{ and } x_0 > 2 \quad \Rightarrow \quad f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{(x-2) - (x_0-2)}{x - x_0}$$
$$= \lim_{x \to x_0} \frac{x - 2 - x_0 + 2}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$$

At the point $x = 2$ the differential quotient is not defined. The derivative does not exist at this point!

Exercise 4 (5 points)

For reasons of simplicity we slightly abuse notation here: when checking differentiability in this solution we write $f'(x) = ...$ followed by a curly bracket (various cases) where differentiability needs to be checked at the transmission points. Strictly speaking, the function $f'(x)$ already exists when we write $f'(x)$ (otherwise we are not allowed to write that) and thus the function is also differentiable at x_0. Still we here use this notation and mean that the function is differentiable in all open subintervals where the derivative is the given function.

The functions that appear in this lecture course allow to check for differentiabilty by using the descriptive criterion below (checking the derivatives from the left and from the right).

a) (1 point)

$$f(x) = \begin{cases} \frac{x^3}{x} = x^2, & x > 0 \\ 0, & x = 0 \\ \frac{x^3}{-x} = -x^2, & x < 0 \end{cases} \qquad f'(x) = \begin{cases} 2x, & x > 0 \\ 0, & x = 0 \\ -2x, & x < 0 \end{cases}$$

$$\lim_{x \uparrow 0} f(x) = \lim_{x \downarrow 0} f(x) = 0 \qquad \lim_{x \uparrow 0} f'(x) = \lim_{x \downarrow 0} f'(x) = 0$$

$$\Rightarrow f(x) \text{ is continuous} \qquad \Rightarrow f(x) \text{ is differentiable}$$

b) (1 point)

$$f(x) = |e^{3x-1}(x^2 - 4x + 4)| = |e^{3x-1}(x-2)^2| = e^{3x-1}(x-2)^2$$

The functions e^{3x-1} and $(x-2)^2$ are both continuous everywhere and differentiable. Thus the function $f(x)$ is continuous and differentiable everywhere.

$$\begin{aligned} f'(x) &= 3e^{3x-1}(x-2)^2 + 2e^{3x-1}(x-2) = \big(3(x-2)+2\big)e^{3x-1}(x-2) \\ &= e^{3x-1}(x-2)(3x-4), \quad \text{so} \quad f'(3) = 5e^8 \end{aligned}$$

c) (1 point)

$$f(x) = \begin{cases} e^{-x}, & x \le 0 \\ e^{x}, & x > 0 \end{cases}, \; x_0 = 0 \qquad f'(x) = \begin{cases} -e^{-x}, & x < 0 \\ e^{x}, & x > 0 \end{cases}$$

$$\lim_{x \uparrow 0} f(x) = \lim_{x \downarrow 0} f(x) = 1 \qquad \lim_{x \uparrow 0} f'(x) = -1 \neq \lim_{x \downarrow 0} f'(x) = 1$$

$$\Rightarrow f(x) \text{ ist continuous} \qquad \Rightarrow f(x) \text{ is not differentiable}$$

d) (1 point)

$$f(x) = \begin{cases} \sin(x), & x \le 0 \\ x^3 + 2x^2 + x, & x > 0 \end{cases}, x_0 = 0 \qquad f'(x) = \begin{cases} \cos(x), & x \le 0 \\ 3x^2 + 4x + 1, & x > 0 \end{cases}$$

$$\lim_{x \uparrow 0} f(x) = \lim_{x \downarrow 0} f(x) = 0 \qquad \lim_{x \uparrow 0} f'(x) = \lim_{x \downarrow 0} f'(x) = 1$$

$$\Rightarrow f(x) \text{ is continuous} \qquad \Rightarrow f(x) \text{ is differentiable}$$

e) (1 point)

$$f(x) = \begin{cases} \frac{1}{x}, & 0 < x < 1 \\ 1 - x, & x \geq 1 \end{cases} \quad , x_0 = 1$$

$$\lim_{x \uparrow 1} f(x) = 1 \neq \lim_{x \downarrow 1} f(x) = 0$$

$\Rightarrow f(x)$ is not continuous $\quad\quad \Rightarrow f(x)$is not differentiable

Exercise 5 (3 points)

a) (1 point) We deduce the derivative from the difference quotient by computing the limit $x \to x_0$.

b) (1 point)

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\frac{x}{1-x} - \frac{x_0}{1-x_0}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\frac{x(1-x_0) - x_0(1-x)}{(1-x)(1-x_0)}}{x - x_0} = \lim_{x \to x_0} \frac{\frac{x - xx_0 - x_0 + xx_0}{(1-x)(1-x_0)}}{x - x_0} = \lim_{x \to x_0} \frac{\frac{x - x_0}{(1-x)(1-x_0)}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\cancel{x - x_0}}{(1-x)(1-x_0)\cancel{(x - x_0)}} = \lim_{x \to x_0} \frac{1}{(1-x)(1-x_0)}$$

c) (1 point)

$$f'(x_0) = \lim_{x \to x_0} \frac{1}{(1-x)(1-x_0)} = \frac{1}{(1-x_0)(1-x_0)} = \frac{1}{(1-x_0)^2}$$