APPENDIX A

Descent Theory

1. History and motivation

The theory of descent in modern algebraic geometry was introduced by Grothendieck in the Séminaire Bourbaki [**40**], with details and proofs offered in SGA 1 [**43**, Exposé VIII]. The origins of the subject go back at least to Weil, although his (less general) results predate the modern language of schemes. This Appendix gives a self-contained treatment of some of the more important results which are generally gathered under the heading, "theory of descent." Some of the easier steps are left as exercises, but all of these are solved in the Answers.

The idea behind descent is that, under appropriate hypotheses, objects and morphisms over a scheme can be described locally. An object is described (uniquely up to canonical isomorphism) by an object on some cover, plus a gluing map satisfying a cocycle condition. A morphism between two objects thus specified is determined by giving a morphism locally (i.e., on the cover), which is compatible with the gluing maps.

Let us spell this out in the particular case of vector bundles on schemes, and for simplicity, we take our covers to be Zariski covers, fine enough that they give local trivializations. So let T be a scheme, and $\mathcal{E}$ a locally free sheaf of $\mathcal{O}_T$-modules of some finite rank r. Then there exists a Zariski open cover (U_i) of T and isomorphisms

$$\lambda_i\colon \mathcal{E}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus r} \tag{1}$$

of $\mathcal{O}_{U_i}$-modules for each i. If we set $U_{ij} = U_i \cap U_j$, then for any pair i and j, the isomorphisms λ_i and λ_j determine isomorphisms $\varphi_{ij}\colon \mathcal{O}_{U_{ij}}^{\oplus r} \to \mathcal{O}_{U_{ij}}^{\oplus r}$ via the diagram

$$\begin{array}{ccc} & \overset{\lambda_i|_{U_{ij}}}{\nearrow} & \mathcal{O}_{U_{ij}}^{\oplus r} \\ \mathcal{E}|_{U_{ij}} & & \downarrow{\scriptstyle \varphi_{ij}} \\ & \underset{\lambda_j|_{U_{ij}}}{\searrow} & \mathcal{O}_{U_{ij}}^{\oplus r} \end{array} \tag{2}$$

Note that specifying the transition mappings φ_{ij} is the same as giving GL_r-valued transition functions on each U_{ij}. The φ_{ij} satisfy the *cocycle condition*: φ_{ii} is the identity map for every i, and for every triple i, j, k, if we set $U_{ijk} = U_i \cap U_j \cap U_k$, then we have

$$(\varphi_{jk}|_{U_{ijk}}) \circ (\varphi_{ij}|_{U_{ijk}}) = \varphi_{ik}|_{U_{ijk}}. \tag{3}$$

(Note that the condition that each φ_{ii} be the identity follows from the latter condition, applied to the triple i, i, i.)

Now *descent* for locally free sheaves in the Zariski topology is a collection of assertions which imply that, given an open cover (U_i), and a collection of morphisms φ_{ij} satisfying the cocycle condition (3), then there exists a locally free sheaf $\mathcal{E}$ (unique up to canonical isomorphism) together with local trivializations (1) such that the diagram (2) commutes for all i and j. There is a similar assertion for morphisms, to the effect that if the locally free sheaf $\mathcal{F}$ also admits local trivializations $\mu_i\colon \mathcal{F}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus s}$ and transition maps ψ_{ij}, then there is a bijection between morphisms $h\colon \mathcal{E} \to \mathcal{F}$ of locally free sheaves and collections of morphisms $h_i\colon \mathcal{O}_{U_i}^{\oplus r} \to \mathcal{O}_{U_i}^{\oplus s}$ for all i such that the diagram

$$\begin{array}{ccc} \mathcal{O}_{U_{ij}}^{\oplus r} & \xrightarrow{h_i|_{U_{ij}}} & \mathcal{O}_{U_{ij}}^{\oplus s} \\ {\scriptstyle \varphi_{ij}}\downarrow & & \downarrow{\scriptstyle \psi_{ij}} \\ \mathcal{O}_{U_{ij}}^{\oplus r} & \xrightarrow[h_j|_{U_{ij}}]{} & \mathcal{O}_{U_{ij}}^{\oplus s} \end{array}$$

commutes, for all i and j.

The assertions just spelled out are artificially restrictive. Indeed it is not necessary for $\mathcal{E}$ to be trivialized on the cover (U_i). In fact we need not restrict to locally free sheaves; the same considerations work in the context of arbitrary quasi-coherent sheaves. Given a Zariski open covering $\{U_i\}$ of T, and a collection $\mathcal{E}_i$ of quasi-coherent sheaves on U_i, with isomorphisms $\varphi_{ij}\colon \mathcal{E}_i|_{U_{ij}} \to \mathcal{E}_j|_{U_{ij}}$ of sheaves of $\mathcal{O}_{U_{ij}}$-modules, satisfying the cocycle condition (3), then there is a quasi-coherent sheaf $\mathcal{E}$ on T, with isomorphisms $\mathcal{E}|_{U_i} \to \mathcal{E}_i$, giving rise to these transition homomorphisms. And there is a similar version of descent for morphisms between quasi-coherent sheaves: if $\mathcal{E}$ comes from $\mathcal{E}_i$ and φ_{ij}, and $\mathcal{F}$ comes from $\mathcal{F}_i$ and ψ_{ij}, then there is a canonical bijection between morphisms $h\colon \mathcal{E} \to \mathcal{F}$ and collections $h_i\colon \mathcal{E}_i \to \mathcal{F}_i$ of morphisms such that $\psi_{ij} \circ h_i|_{U_{ij}} = h_j|_{U_{ij}} \circ \varphi_{ij}$ for all i, j. We will use this fact, which is a basic construction in algebraic geometry; a reference is [**EGA** 0.3.3.1].

These assertions can be stated more succinctly, avoiding all the indices, by defining T' to be the disjoint union of the open sets U_i, which comes with a canonical mapping $T' \to T$. The sheaves $\mathcal{E}_i$ determine a sheaf $\mathcal{E}'$ on T'. The transition functions φ_{ij} amount to an isomorphism

$$\varphi\colon p_1^*(\mathcal{E}') \to p_2^*(\mathcal{E}')$$

on $T' \times_T T'$, where p_1 and p_2 are the projections from $T' \times_T T'$ to T'. The cocycle condition asserts that $p_{23}^*(\varphi) \circ p_{12}^*(\varphi) = p_{13}^*(\varphi)$ on $T' \times_T T' \times_T T'$, where

$$p_{ij}\colon T' \times_T T' \times_T T' \to T' \times_T T'$$

are the projections to the corresponding factors.

A key feature of Grothendieck's descent theory is that it extends from Zariski coverings to the more general étale and smooth coverings that are required for the theory of stacks. In fact, the appropriate morphisms to use are quite general flat morphisms.

Notation A.1. *Given a morphism $f\colon T' \to T$ of schemes, set $T'' = T' \times_T T'$, with its two projections p_1 and p_2 from T'' to T'. Let $T''' = T' \times_T T' \times_T T'$, which comes* *with three projections p_{12}, p_{13}, and p_{23} from T''' to T''. We also have three projections q_1, q_2, and q_3 from T''' to T', with $q_i = p_1 \circ p_{ij}$ and $q_j = p_2 \circ p_{ij}$, $1 \le i < j \le 3$.*

Descent for objects says that an object specified on a cover, together with a patching isomorphism satisfying a cocycle condition, determines an object defined on the base, and this object is unique up to canonical isomorphism. In more traditional terminology, every **descent datum** (pair consisting of an object defined on the cover, with a patching isomorphism satisfying the cocycle condition) is **effective** (determines an object on the base); the object on the base that realizes this effectivity is called a **solution** to the descent problem posed by the given datum.

Descent for morphisms says that, if we are given two sets of descent data, together with respective objects on the base (solutions to the descent data), then to give a morphism between these objects is the same as to give a morphism between the objects on the cover, subject to a compatibility condition.

The next theorem spells this out in the case of quasi-coherent sheaves on schemes. Following the statement are detailed explanations of its assertions. The next two sections are devoted to the proof of the theorem, while the rest of this appendix will discuss applications to other descent situations, especially those involving schemes instead of quasi-coherent sheaves.

THEOREM A.2. *Let $f\colon T' \to T$ be a flat morphism of schemes. Assume, further, that f is surjective and T admits an open covering by affine subsets, each the image of some quasi-compact open subset of T'.* (a) *Let $\mathcal{E}'$ be a quasi-coherent sheaf on T' and $\varphi\colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ an isomorphism on T'' such that*

$$p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$$

on T'''. Then there exists a quasi-coherent sheaf $\mathcal{E}$ on T and an isomorphism $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ on T' satisfying

$$p_2^*\lambda = \varphi \circ p_1^*\lambda$$

on T''. Moreover the pair consisting of the sheaf $\mathcal{E}$ and the isomorphism λ is unique up to canonical isomorphism.
(b) *With notation as in* (a), *suppose $(\mathcal{F}', \psi)$ is another descent datum with solution given by $\mathcal{F}$ and μ. Then, for every morphism $h'\colon \mathcal{E}' \to \mathcal{F}'$ on T' satisfying*

$$p_2^*h' \circ \varphi = \psi \circ p_1^*h'$$

on T'', there is a unique morphism $h\colon \mathcal{E} \to \mathcal{F}$ on T such that $\mu \circ f^*h = h' \circ \lambda$ on T'.

The hypotheses on the morphism f (flat, surjective, etc.) will be discussed in Section 3. The hypothesis in (a), that $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$, means that the diagram

$$\begin{array}{ccccc}
p_{12}^*p_1^*\mathcal{E}' & \xrightarrow{p_{12}^*\varphi} & p_{12}^*p_2^*\mathcal{E}' & = & p_{23}^*p_1^*\mathcal{E}' \\
\| & & & & \downarrow{\scriptstyle p_{23}^*\varphi} \\
p_{13}^*p_1^*\mathcal{E}' & \xrightarrow[p_{13}^*\varphi]{} & p_{13}^*p_2^*\mathcal{E}' & = & p_{23}^*p_2^*\mathcal{E}'
\end{array}$$

commutes. The three equal signs denote canonical isomorphisms coming from the equalities $p_1 \circ p_{jk} = q_j = p_2 \circ p_{ij}$.

The conclusion in (a), that $p_2^*\lambda = \varphi \circ p_1^*\sigma$, means that the diagram

$$\begin{array}{ccc} p_1^* f^* \mathcal{E} & \xrightarrow{p_1^*\lambda} & p_1^* \mathcal{E}' \\ \Vert & & \downarrow{\scriptstyle \varphi} \\ p_2^* f^* \mathcal{E} & \xrightarrow[p_2^*\lambda]{} & p_2^* \mathcal{E}' \end{array}$$

commutes.

We clarify what it means in (a) for the solution to be unique up to canonical isomorphism. Precisely, it means that if $\mathcal{F}$ is another quasi-coherent sheaf on T, and $\mu\colon f^*\mathcal{F} \to \mathcal{E}'$ is an isomorphism on T' satisfying $p_2^*\mu = \varphi \circ p_1^*\mu$ on T'', i.e., the diagram

$$\begin{array}{ccc} p_1^* f^* \mathcal{F} & \xrightarrow{p_1^*\mu} & p_1^* \mathcal{E}' \\ \Vert & & \downarrow{\scriptstyle \varphi} \\ p_2^* f^* \mathcal{F} & \xrightarrow[p_2^*\mu]{} & p_2^* \mathcal{E}' \end{array}$$

commutes, then there is a unique isomorphism $h\colon \mathcal{E} \to \mathcal{F}$ such that $\mu \circ f^*h = \lambda$ on T', i.e., the diagram

$$\begin{array}{ccc} f^*\mathcal{E} & \xrightarrow{f^*h} & f^*\mathcal{F} \\ & {\scriptstyle \lambda}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & \mathcal{E}' & \end{array}$$

commutes. This uniqueness claim is in fact a special case of (b), applied to the identity morphism on $\mathcal{E}'$.

The hypothesis in (b), that $p_2^*h' \circ \varphi = \psi \circ p_1^*h'$, means that the diagram

$$\begin{array}{ccc} p_1^* \mathcal{E}' & \xrightarrow{\varphi} & p_2^* \mathcal{E}' \\ {\scriptstyle p_1^*h'}\downarrow & & \downarrow{\scriptstyle p_2^*h'} \\ p_1^* \mathcal{F}' & \xrightarrow[\psi]{} & p_2^* \mathcal{F}' \end{array}$$

commutes.

Finally, the conclusion in (b), that $\mu \circ f^*h = h' \circ \lambda$, means that the diagram

$$\begin{array}{ccc} f^*\mathcal{E} & \xrightarrow{f^*h} & f^*\mathcal{F} \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{E}' & \xrightarrow[h']{} & \mathcal{F}' \end{array}$$

commutes.

2. The affine case

The general case of Theorem A.2 will be reduced to the affine case, which amounts to some elementary commutative algebra. This algebra is worked out in this section. No Noetherian or finiteness conditions on either rings or modules are required.

We are concerned with an arbitrary homomorphism $A \to A'$ of commutative rings with unit, which corresponds to a morphism $f\colon T' \to T$, with $T = \mathrm{Spec}(A)$ and $T' = \mathrm{Spec}(A')$. Let $A'' = A' \otimes_A A'$, and $A''' = A' \otimes_A A' \otimes_A A'$, so we have identifications $T'' = \mathrm{Spec}(A'')$ and $T''' = \mathrm{Spec}(A''')$. The projections p_1 and p_2 from T'' to T' correspond to the homomorphisms $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$ from A' to $A' \otimes_A A'$. Similarly, the projections p_{12}, p_{13}, and p_{23} from T''' to T'' correspond to the mappings from $A' \otimes_A A'$ to $A''' = A' \otimes_A A' \otimes_A A'$ that take $x \otimes y$ to $x \otimes y \otimes 1$, $x \otimes 1 \otimes y$, and $1 \otimes x \otimes y$, respectively. Projections q_1, q_2, and q_3 from T''' to T' correspond to mappings $A' \to A'''$ given by $x \mapsto x \otimes 1 \otimes 1$, $x \mapsto 1 \otimes x \otimes 1$, and $1 \otimes 1 \otimes x$, respectively.

Definition A.3. A homomorphism $A \to A'$ of commutative rings with unit is **flat** if, for any exact sequence $M_1 \to M_2 \to M_3$ of A-modules, the induced sequence $A' \otimes_A M_1 \to A' \otimes_A M_2 \to A' \otimes_A M_3$ (of A'-modules) is exact. The homomorphism is called **faithfully flat** if it is flat and the corresponding map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is surjective.

Exercise A.1. (1) Show that a flat homomorphism $A \to A'$ is faithfully flat if and only if, for any nonzero A-module M, $A' \otimes_A M \neq 0$. (2) Show that a homomorphism $A \to A'$ is faithfully flat if and only if the following condition is satisfied: a sequence $M' \to M \to M''$ of A-modules is exact if and only if the sequence $A' \otimes_A M' \to A' \otimes_A M \to A' \otimes_A M''$ is exact.

Exercise A.2. Suppose $A \to A'$ is faithfully flat. (1) Show that a homomorphism $M \to N$ of A-modules is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if the homomorphism $A' \otimes_A M \to A' \otimes_A N$ is a monomorphism (resp. epimorphism, resp. isomorphism). (2) Show that an A-module M is finitely generated (resp. finitely presented, resp. flat, resp. locally free of finite rank n) if and only if the A'-module $A' \otimes_A M$ is finitely generated (resp. finitely presented, resp. flat, resp. locally free of finite rank n).

For any homomorphism $A \to A'$, and any A-module M, there is a canonical homomorphism $\gamma\colon M \to A' \otimes_A M$, taking u to $1 \otimes u$. There are two canonical homomorphisms $A' \otimes_A M \to A' \otimes_A A' \otimes_A M$, taking $x \otimes u$ to $x \otimes 1 \otimes u$ and $1 \otimes x \otimes u$, corresponding to the two projections p_1 and p_2.

Lemma A.4. *Let M be an A-module. If $A \to A'$ is faithfully flat, then*

$$M \xrightarrow{\gamma} A' \otimes_A M \rightrightarrows A' \otimes_A A' \otimes_A M$$

is exact, that is, the canonical homomorphism γ maps M isomorphically to the set of elements in $A' \otimes_A M$ that have the same image in $A' \otimes_A A' \otimes_A M$ by the two projection homomorphisms. Equivalently, if one defines a homomorphism δ from $A' \otimes_A M$ to

$A' \otimes_A A' \otimes_A M$ by the formula $\delta(x \otimes u) = 1 \otimes x \otimes u - x \otimes 1 \otimes u$, then the sequence

$$0 \to M \xrightarrow{\gamma} A' \otimes_A M \xrightarrow{\delta} A' \otimes_A A' \otimes_A M$$

of A-modules is exact.

Proof. By Exercise A.2 (2), it suffices to show that the sequence becomes exact after tensoring it (on the left) over A by A', i.e., that the sequence

$$0 \longrightarrow A' \otimes_A M \xrightarrow{A'\otimes\gamma} A' \otimes_A A' \otimes_A M \xrightarrow{A'\otimes\delta} A' \otimes_A A' \otimes_A A' \otimes_A M$$

is exact. Let $\mu\colon A' \otimes_A A' \to A'$ be the multiplication map, $\mu(x \otimes y) = xy$. The injectivity of the first map $A' \otimes \gamma$ is now immediate, since the mapping $\mu \otimes M\colon A' \otimes_A A' \otimes_A M \to A' \otimes_A M$ gives a left inverse to it. Suppose an element $\sum x_i \otimes y_i \otimes u_i$ is in the kernel of $A' \otimes \delta$, i.e.

$$\sum x_i \otimes 1 \otimes y_i \otimes u_i = \sum x_i \otimes y_i \otimes 1 \otimes u_i.$$

Applying μ to the first two factors yields

$$\sum x_i \otimes y_i \otimes u_i = \sum x_i y_i \otimes 1 \otimes u_i,$$

and $\sum x_i y_i \otimes 1 \otimes u_i$ is the image of $\sum x_i y_i \otimes u_i$ in $A' \otimes_A A' \otimes_A M$, as required. □

The proof of this lemma is a common one in descent theory: one makes a faithfully flat base extension to achieve the situation where the covering map $T' \to T$ has a section, in which case the assertion proves itself.

Although we don't need it, a natural generalization of this lemma is true:

Exercise A.3. Define the *Amitsur complex* $T^\bullet = T^\bullet(A'/A)$ for a homomorphism $A \to A'$ by setting $T^0 = A$, and, for $n \geq 1$, T^n is the tensor product of n copies of A' over A. Define $\delta^n\colon T^n \to T^{n+1}$ by: δ^0 is the given map from A to A', and $\delta^n = \sum_{i=0}^{n} (-1)^i \epsilon_i$, where $\epsilon_i(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_n$. This is a complex of A-modules. Show that, for any A-module M, if $A \to A'$ is faithfully flat, the complex $T^\bullet \otimes_A M$ is exact.

Descent for morphisms of modules amounts to the following easy consequence of the preceding lemma:

Lemma A.5. *If $A \to A'$ is faithfully flat, and M and N are A-modules, then the sequence*

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{A'}(A' \otimes_A M, A' \otimes_A N) \rightrightarrows \mathrm{Hom}_{A'\otimes_A A'}(A' \otimes_A A' \otimes_A M, A' \otimes_A A' \otimes_A N)$$

is exact.

Proof. The exactness of Lemma A.4, applied to N, together with the left exactness of Hom, gives the exactness of

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, A' \otimes_A N) \rightrightarrows \mathrm{Hom}_A(M, A' \otimes_A A' \otimes_A N).$$

Using the identifications $\mathrm{Hom}_A(M, P) = \mathrm{Hom}_B(B \otimes_A M, P)$ for any homomorphism $A \to B$ and any B-module P, first for $B = A'$ and then for $B = A' \otimes_A A'$, translates this exact sequence into the exact sequence of the lemma. □

Now let M' be an A'-module. We have, as we recall, projection maps p_1 and p_2 from $\operatorname{Spec}(A'')$ to $\operatorname{Spec}(A')$, where $A'' = A' \otimes_A A'$. Hence we have pullbacks $p_1^*(M') = A'' \otimes_{p_1, A'} M'$ and $p_2^*(M') = A'' \otimes_{p_2, A'} M'$. The two pullbacks $p_1^*(M')$ and $p_2^*(M')$ can be identified with $M' \otimes_A A'$ and $A' \otimes_A M'$ respectively, where the actions of A'' on these modules are given by $(x \otimes y)\cdot(u \otimes z) = xu \otimes yz$ and $(x \otimes y)\cdot(z \otimes u) = xz \otimes yu$ respectively, with x, y, and z in A' and u in M'. Similarly, the three pullbacks of M' by q_1, q_2, and q_3 to A''' can be identified with $M' \otimes_A A' \otimes_A A'$, $A' \otimes_A M' \otimes_A A'$, and $A' \otimes_A A' \otimes_A M'$, respectively, again with the diagonal actions of $A''' = A' \otimes_A A' \otimes_A A'$.

Suppose $\varphi\colon M' \otimes_A A' = p_1^*(M') \to p_2^*(M') = A' \otimes_A M'$ is an isomorphism of A''-modules. This determines by the three pullbacks p_{ij}, isomorphisms

$$\varphi_{ij} = p_{ij}^*(\varphi) : \; q_i^*(M') = p_{ij}^*(p_1^*(M')) \;\to\; p_{ij}^*(p_2^*(M')) = q_j^*(M').$$

For example, φ_{12} is the map from $M' \otimes_A A' \otimes_A A'$ to $A' \otimes_A M' \otimes_A A'$ that takes $u \otimes x \otimes y$ to $\varphi(u \otimes x) \otimes y$; that is, if $\varphi(u \otimes x) = \sum x_i \otimes u_i$, then $\varphi_{12}(u \otimes x \otimes y) = \sum x_i \otimes u_i \otimes y$. Similarly, $\varphi_{13}(u \otimes y \otimes x) = \sum x_i \otimes y \otimes u_i$, and $\varphi_{23}(y \otimes u \otimes x) = \sum y \otimes x_i \otimes u_i$.

Descent for modules amounts to the following assertion:

LEMMA A.6. *Suppose $A \to A'$ is faithfully flat, M' is an A'-module, and $\varphi\colon M' \otimes_A A' \to A' \otimes_A M'$ is an isomorphism of A''-modules such that $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$ from $q_1^*(M')$ to $q_3^*(M')$. Define the A-module M by*

$$M = \{\, u \in M' \mid \varphi(u \otimes 1) = 1 \otimes u \,\}.$$

Then the canonical homomorphism $\lambda\colon A' \otimes_A M \to M'$, $x \otimes u \mapsto x\cdot u$, is an isomorphism.

PROOF. Let $\tau\colon M' \to A' \otimes_A M'$ be defined by $\tau(u) = 1 \otimes u - \varphi(u \otimes 1)$. We have an exact sequence

$$0 \to M \to M' \xrightarrow{\tau} A' \otimes_A M'$$

Tensoring this on the right with A' over A gives the top row of the following diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & M \otimes_A A' & \longrightarrow & M' \otimes_A A' & \longrightarrow & A' \otimes_A M' \otimes_A A' \\
& \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle A' \otimes \varphi} \\
0 \longrightarrow & M' & \longrightarrow & A' \otimes_A M' & \longrightarrow & A' \otimes_A A' \otimes_A M'
\end{array}$$

The bottom row is the exact sequence from Lemma A.4, applied to the A-module M'. The map ψ is defined by $\psi(u \otimes x) = x \cdot u$, and we want to show ψ is an isomorphism. Since the rows are exact, and the right two vertical maps are isomorphisms, this conclusion will follow if we verify that the diagram is commutative.

The left square commutes since, for u in M and x in A', $\varphi(u \otimes x) = (1 \otimes x)\varphi(u \otimes 1) = (1 \otimes x)(1 \otimes u) = 1 \otimes xu$. To prove that the right diagram commutes, we must show that, for any u in M' and x in A', the element $u \otimes x$ in $M' \otimes_A A'$ has the same image by either route around the square. Let $\varphi(u \otimes 1) = \sum y_i \otimes v_i$, with $y_i \in A'$ and $v_i \in M'$. Then

$$\varphi(u \otimes x) = (1 \otimes x)\varphi(u \otimes 1) = \sum y_i \otimes x v_i,$$

so the image of $u \otimes x$ by the lower route is

$$\sum 1 \otimes y_i \otimes x v_i - \sum y_i \otimes 1 \otimes x v_i.$$

On the upper route, $u \otimes x$ maps to the right to $1 \otimes u \otimes x - \varphi(u \otimes 1) \otimes x = 1 \otimes u \otimes x - \sum y_i \otimes v_i \otimes x$, which maps down to

$$1 \otimes \varphi(u \otimes x) - \sum y_i \otimes \varphi(v_i \otimes x) = \sum 1 \otimes y_i \otimes xv_i - \sum y_i \otimes \varphi(v_i \otimes x).$$

We are therefore reduced to verifying that

$$\sum y_i \otimes \varphi(v_i \otimes x) = \sum y_i \otimes 1 \otimes xv_i.$$

But this is exactly the assertion that $\varphi_{23}(\varphi_{12}(u \otimes 1 \otimes x)) = \varphi_{13}(u \otimes 1 \otimes x)$. □

To complete the proof that the construction of this lemma solves the descent problem for modules, i.e., that it solves case (a) of Theorem A.2, we must verify that the identity $\varphi \circ p_1^*\lambda = p_2^*\lambda$ is satisfied. This amounts to verifying that the diagram

$$\begin{array}{ccc} A' \otimes_A M \otimes_A A' & \xrightarrow{p_1^*\lambda} & M' \otimes_A A' \\ \kappa \downarrow & & \downarrow \varphi \\ A' \otimes_A A' \otimes_A M & \xrightarrow{p_2^*\lambda} & A' \otimes_A M' \end{array}$$

commutes, where $\kappa(x \otimes u \otimes y) = x \otimes y \otimes u$. This amounts to the identity $x \otimes \lambda(y \otimes u) = \varphi(\lambda(x \otimes u) \otimes y)$, i.e., $x \otimes yu = \varphi(xu \otimes y)$, or $(x \otimes y)(1 \otimes u) = (x \otimes y)\varphi(u \otimes 1)$, which follows from the fact that u is in M.

Similarly, we want Lemma A.5 to give a proof of (b) of Theorem A.2 in the affine case. This means that we have A'-modules M' and N', with isomorphisms

$$\varphi\colon M' \otimes_A A' \to A' \otimes_A M' \quad \text{and} \quad \psi\colon N' \otimes_A A' \to A' \otimes_A N',$$

and we have A-modules M and N, with isomorphisms $\lambda\colon A' \otimes_A M \to M'$ and $\mu\colon A' \otimes_A N \to N'$, satisfying $\varphi \circ p_1^*\lambda = p_2^*\lambda$ and $\psi \circ p_1^*\mu = p_2^*\mu$. We are given a homomorphism $h'\colon M' \to N'$ of A'-modules, satisfying the identity $p_2^*(h') \circ \varphi = \psi \circ p_1^*(h')$. We must show that there is a unique homomorphism $h\colon M \to N$ of A-modules such that $\mu \circ (A' \otimes h) = h' \circ \lambda$. Set $g' = \mu^{-1} \circ h' \circ \lambda\colon A' \otimes_A M \to A' \otimes_A N$. If we show that $p_1^*(g') = p_2^*(g')$, then Lemma A.5 will produce a unique homomorphism $h\colon M \to N$ such that $g' = A' \otimes h$. This says that $h' \circ \lambda = \mu \circ (A' \otimes h)$, as required. To conclude the proof, we calculate:

$$\begin{aligned} p_1^*(g') = p_1^*(\mu^{-1} \circ h' \circ \lambda) &= p_1^*(\mu)^{-1} \circ p_1^*(h' \circ \lambda) = p_2^*(\mu)^{-1} \circ \psi \circ p_1^*(h') \circ p_1^*(\lambda) \\ &= p_2^*(\mu)^{-1} \circ p_2^*(h') \circ \varphi \circ p_1^*(\lambda) = p_2^*(\mu^{-1} \circ h') \circ p_2^*(\lambda) = p_2^*(g'), \end{aligned}$$

as required. The uniqueness assertion in (a) is a special case of (b), so the theorem is proved in the affine case.

The overall structure of the proofs in this section is worth noting, as it will be repeated below in the proof of Theorem A.2. First, we proved descent for morphisms in the case of objects pulled back from the base (Lemma A.5). Then we showed that every descent datum is effective (Lemma A.6). We saw as a formal consequence that descent for morphisms holds in the case of an arbitrary pair of descent data, each admitting a solution, and from this that the solution to any descent problem is unique up to canonical isomorphism.

3. The general case

In this section, we complete the proof of Theorem A.2. Recall that a morphism $f\colon T' \to T$ of schemes is **faithfully flat** if it is flat and surjective. It is not enough to assume f is faithfully flat for the conclusions of the theorem to hold, as we'll see below in Exercise A.6. To pass from the affine case (Lemmas A.5 and A.6) to the case of general schemes we'll need some additional hypothesis on the morphism f. There are two additional hypotheses that one may impose, the one that appears in Grothendieck's exposés, and a more general, local version appearing in [**101**], following a suggestion of S. Kleiman.

(i) f is **fpqc**, that is, faithfully flat and quasi-compact. (French: *fidèlement plat, quasi-compact.*) This means that the pre-image, under f, of any affine open subset of the base is covered by finitely many affine open subsets.

(ii) f is **locally fpqc**, meaning that f is faithfully flat and T can be covered by affine open subsets, each the image of some quasi-compact open subset of T'.

In applications in this book, we are mainly interested in étale or smooth coverings. These all fall under case (ii):

Proposition A.7. *Let $f\colon T' \to T$ be a faithfully flat morphism, locally of finite presentation. Then T admits an affine open covering by subsets, each the image by f of some affine open subset of T'. In particular, f is locally fpqc as in (ii), above.*

Proof. Since f is flat and locally of finite presentation, it is an open morphism (cf. the Glossary). We start with an affine open covering (T_i), $i \in I$, of T. Taking $(T'_{i,j})$, $j \in J_i$ to be an affine open covering of $f^{-1}(T_i)$, for each i, their images $T_{i,j} = f(T'_{i,j})$ are open and cover T. We claim that for any affine open $U \subset T_{i,j}$, $f^{-1}(U) \cap T'_{i,j}$ is affine. Since the $T_{i,j}$ cover T, we have a covering of T by such open subsets U.

Since $T_{i,j}$ is a subscheme of T_i, it is separated. By [**EGA** II.1.6.2], any morphism from an affine scheme to a separated scheme is affine. So the morphism $T'_{i,j} \to T_{i,j}$ obtained by restricting f is affine, hence the pre-image in $T'_{i,j}$ of any affine open subset of $T_{i,j}$ is affine. □

As described at the end of the previous section, to prove Theorem A.2, it suffices to prove descent for morphisms of objects pulled back from the base and to show that every descent datum is effective. In other words, Theorem A.2 follows from the following pair of assertions.

Lemma A.8. *Assume $f\colon T' \to T$ is* (i) *fpqc or* (ii) *locally fpqc. Let $\mathcal{E}$ and $\mathcal{F}$ be quasi-coherent sheaves on T. Then, for every morphism $h'\colon f^*\mathcal{E} \to f^*\mathcal{F}$ on T' such that $p_1^*h' = p_2^*h'$ on T'' there is a unique morphism $h\colon \mathcal{E} \to \mathcal{F}$ on T such that $f^*h = h'$.*

Lemma A.9. *Assume $f\colon T' \to T$ is* (i) *fpqc or* (ii) *locally fpqc. Let $\mathcal{E}'$ be a quasi-coherent sheaf on T' and $\varphi\colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ an isomorphism on T'' such that $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$ on T'''. Then there exists a quasi-coherent sheaf $\mathcal{E}$ on T and an isomorphism $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ on T' such that $p_2^*\lambda = \varphi \circ p_1^*\lambda$ on T''.*

Let us say that f satisfies **descent for morphisms** if the conclusion of Lemma A.8 is valid for f. Let us say that f satisfies **effective descent** if both the conclusion of

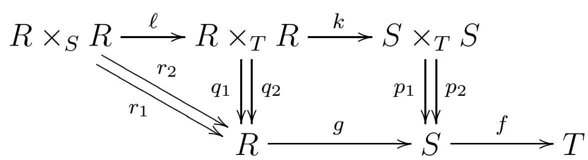
Lemma A.8 and of Lemma A.9 are valid for f. We have proved in the previous section that every faithfully flat morphism of affine schemes satisfies effective descent. We saw in the first section that every Zariski open covering satisfies effective descent. These two facts will be combined to deduce what is claimed in Lemmas A.8 and A.9, namely that every morphism that is fpqc, or locally fpqc, satisfies effective descent.

The argument rests on the following two claims. Let $f\colon S \to T$ and $g\colon R \to S$ be morphisms of schemes.

First claim: Suppose g satisfies descent for morphisms, and suppose for any morphism $g'\colon R' \to S'$ obtained from g by a base change with respect to an arbitrary morphism $S' \to S$ and any pair of quasi-coherent sheaves $\mathcal{E}'$ and $\mathcal{F}'$ on S', the map induced by pullback $g'^*\colon \mathrm{Hom}(\mathcal{E}', \mathcal{F}') \to \mathrm{Hom}(g'^*\mathcal{E}', g'^*\mathcal{F}')$ is injective. Then f satisfies descent for morphisms if and only if $f \circ g$ satisfies descent for morphisms.

To prove this, we consider the following diagram:

(4)



Given quasi-coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on T, if $h''\colon g^*f^*\mathcal{E} \to g^*f^*\mathcal{F}$ satisfies $q_1^*h'' = q_2^*h''$, then $r_1^*h'' = \ell^*q_1^*h'' = \ell^*q_2^*h'' = r_2^*h''$, so by descent for morphisms for g there exists a unique $h'\colon f^*\mathcal{E} \to f^*\mathcal{F}$ such that $g^*h' = h''$. The morphism k factors as $R \times_T R \to R \times_T S \to S \times_T S$, a pair of morphisms each obtained from g by base change. Now since $k^*p_1^*h' = k^*p_2^*h'$ it follows that $p_1^*h' = p_2^*h'$. If descent for morphisms holds for f, it follows that there exists a unique morphism $h\colon \mathcal{E} \to \mathcal{F}$ such that $f^*h = h'$, and hence descent for morphisms holds for $f \circ g$. Conversely, suppose $f \circ g$ satisfies descent for morphisms. If we are given quasi-coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on T, and if $h'\colon f^*\mathcal{E} \to f^*\mathcal{F}$ satisfies $p_1^*h' = p_2^*h'$, then $h'' := g^*h'$ satisfies $q_1^*h'' = q_2^*h''$, so there exists a morphism $h\colon \mathcal{E} \to \mathcal{F}$ satisfying $g^*f^*h = h''$, and hence $f^*h = h'$.

Exercise A.4. Use this first claim to show that every *affine* faithfully flat morphism of schemes satisfies descent for morphisms.

Second claim: Suppose g satisfies effective descent, and suppose any morphism obtained from g by base change satisfies descent for morphisms. Then f satisfies effective descent if and only if $f \circ g$ satisfies effective descent.

We refer to the diagram (4). For the "only if" portion of the claim, we suppose f satisfies effective descent. Now suppose we are given a quasi-coherent sheaf $\mathcal{E}''$ on R together with an isomorphism $\varphi'\colon q_1^*\mathcal{E}'' \to q_2^*\mathcal{E}''$ satisfying the cocycle condition

$$\pi_{13}^*\varphi' = \pi_{23}^*\varphi' \circ \pi_{12}^*\varphi' \tag{5}$$

where $\pi_{ij}\colon R \times_T R \times_T R \to R \times_T R$ denote the various projections. By pulling back (5) by the morphism $R \times_S R \times_S R \to R \times_T R \times_T R$, we obtain the cocycle identity for the cover g. So, by effective descent for the morphism g, there exists a sheaf $\mathcal{E}'$ on S together with an isomorphism $\lambda'\colon g^*\mathcal{E}' \to \mathcal{E}''$ such that $r_2^*\lambda' = k^*\varphi' \circ r_1^*\lambda'$. Now we claim there exists a morphism $\varphi\colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ such that $q_2^*\lambda' \circ h^*\varphi = \varphi' \circ q_1^*\lambda'$. By the first claim, k (a composite of two pullbacks of g, as we saw in the proof of the first claim) satisfies descent for morphisms. Now consider the morphism $q_2^*\lambda'^{-1} \circ \varphi' \circ q_1^*\lambda'\colon k^*p_1^*\mathcal{E}' \to k^*p_2^*\mathcal{E}'$. For the existence of φ as promised we must check the agreement of the two pullback to $(R \times_T R) \times_{S\times_T S} (R \times_T R)$. This fiber product is identified, via the map which on points is given by $(w, x, y, z) \mapsto (w, y, z, x)$, with the fiber product $R \times_S R \times_T R \times_S R$, whereupon the agreement of the two pullbacks reduces to the identity

$$\pi_{14}^*\varphi' = \pi_{34}^*\varphi' \circ \pi_{23}^*\varphi' \circ \pi_{12}^*\varphi'. \tag{6}$$

In fact (6) is the pullback of a similar identity on $R \times_T R \times_T R \times_T R$, and the latter is deduced by combining the pullback of (5) by π_{123} with the pullback of (5) by π_{134} (here π_{ij} and π_{ijk} denote projections from quadruple fiber products). Now φ satisfies the cocycle condition for the covering map f, since the map $R\times_T R\times_T R \to S\times_T S\times_T S$ can be written as a composite of three morphisms, each obtained from g by base change, and via this map the cocycle condition we are claiming pulls back to (5). By effective descent for f there exists a quasi-coherent sheaf $\mathcal{E}$ on T with an isomorphism $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ satisfying $p_2^*\lambda = \varphi \circ p_1^*\lambda$. Hence effective descent holds for $f \circ g$.

Exercise A.5. Show, conversely, that under the hypotheses of the second claim, if $f \circ g$ satisfies effective descent, then effective descent holds for f.

We now complete the proof of Lemmas A.8 and A.9. Suppose we are in case (i) of the lemmas, that is, f is faithfully flat and quasi-compact. We let (T_i), $i \in I$, be an affine open cover of T. For each i, we let $(T'_{i,j})$, $j \in J_i$ be an affine open cover of $f^{-1}(T_i)$. By the hypothesis, we may suppose the set J_i to be finite, for every i. Now for each $i \in I$, the map $f_i\colon \coprod_{j\in J_i} T'_{i,j} \to T_i$ is a faithfully flat morphism of affine schemes.

We consider the following commutative diagram

$$\begin{array}{ccc} \coprod_{i,j} T'_{i,j} & \xrightarrow{\coprod f_i} & \coprod_i T_i \\ \downarrow & & \downarrow \\ T' & \xrightarrow{f} & T \end{array} \tag{7}$$

The vertical maps are Zariski open coverings, and for such maps we know effective descent holds. By the affine case (Lemmas A.5 and A.6), effective descent holds for each morphism f_i, hence as well for the top map in (7). By Exercise A.4, any morphism obtained from the latter by base change satisfies descent for morphisms. So, by the second claim, the composite map $\coprod T'_{i,j} \to T$ in (7) satisfies effective descent. Again invoking the second claim, we conclude that effective descent holds for f.

We turn to case (ii) of the lemmas, where f is locally fpqc. Now we let (T_i), $i \in I$, be an affine open cover of T, such that each T_i is the image of quasi-compact open $T'_i \subset T'$, and in particular the morphism $f_i\colon T'_i \to T_i$, obtained by restricting f, is quasi-compact. There is no loss of generality in supposing that the T'_i cover T', for we may cover $f^{-1}(T_i)$ by affines $T'_{i,j}$, $j \in J_i$, replace the indexing set I by $\coprod J_i$, and replace T'_i by $T'_i \cup T'_{i,j}$ to ensure that this is so. By case (i) of the assertions, then, each map $T'_i \to T_i$ satisfies effective descent. Now, to conclude, we consider a square as in (7) but with $\coprod_i T'_i$ in the upper left-hand corner, and we reason as above except we appeal to case (i) at the second step of the deduction.

EXERCISE A.6. Show that Theorem A.2 (a) fails for the covering map

$$\coprod_p \operatorname{Spec} \mathbb{Z}_p \to \operatorname{Spec} \mathbb{Z}.$$

Note that f is faithfully flat, but is not fpqc, nor even locally fpqc.

4. Categorical formulation

There is a category-theoretic approach to stating the above descent results. The proposition in this section outlines how the results appear in this language; one often sees them expressed this way in the literature.

Fix schemes T and T' and a morphism $f\colon T' \to T$. Let $\mathcal{C}(T)$ be the category of quasi-coherent sheaves on T, with their usual morphisms as sheaves of $\mathcal{O}_T$-modules. Let $\mathcal{C}(T'/T)$ be the category whose objects are pairs $(\mathcal{E}', \varphi)$ of descent data, with a morphism from $(\mathcal{E}', \varphi)$ to $(\mathcal{F}', \psi)$ being a homomorphism $h'\colon \mathcal{E}' \to \mathcal{F}'$ such that $p_2^* h' \circ \varphi = \psi \circ p_1^* h'$. There is a canonical functor $\mathcal{C}(T) \to \mathcal{C}(T'/T)$, taking a quasi-coherent sheaf $\mathcal{E}$ on T to the pair consisting of the sheaf $f^*\mathcal{E}$ and the canonical isomorphism $\mathrm{can}\colon p_1^* f^* \mathcal{E} \cong (f \circ p_1)^*(\mathcal{E}) = (f \circ p_2)^*(\mathcal{E}) \cong p_2^* f^* \mathcal{E}$. (The cocycle condition $p_{13}^*\mathrm{can} = p_{13}^*\mathrm{can} \circ p_{23}^*\mathrm{can}$ amounts to the compatibility of the canonical isomorphisms among pullbacks to T'''.)

PROPOSITION A.10. *If $f\colon T' \to T$ is locally fpqc then the induced functor from the category $\mathcal{C}(T)$ of quasi-coherent sheaves of $\mathcal{O}_T$-modules to the category $\mathcal{C}(T'/T)$ of descent data is an equivalence of categories.*

PROOF. Let $(\mathcal{E}', \varphi)$ be an object of $\mathcal{C}(T'/T)$. To give an isomorphism $(f^*\mathcal{E}, \mathrm{can}) \xrightarrow{\sim} (\mathcal{E}', \varphi)$ is, by definition, the same as to give an isomorphism $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ satisfying $p_2^*\lambda = \varphi \circ p_1^*\lambda$. So essential surjectivity of the functor is equivalent to the condition in Theorem A.2(a).

If we have isomorphisms $(f^*\mathcal{E}, \mathrm{can}) \xrightarrow{\sim} (\mathcal{E}', \varphi)$ and $(f^*\mathcal{F}, \mathrm{can}) \xrightarrow{\sim} (\mathcal{F}', \psi)$ then Theorem A.2(b) is the assertion that the map

$$\mathrm{Hom}_{\mathcal{C}(T)}(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{C}(T'/T)}((\mathcal{E}', \varphi), (\mathcal{F}', \psi))$$

(obtained by applying the functor and composing with the isomorphisms) is bijective. In the case of identity morphisms $1_{(f^*\mathcal{E}, \mathrm{can})}$ and $1_{(f^*\mathcal{F}, \mathrm{can})}$, this is the condition for the functor to be fully faithful. □

REMARK A.11. There is a larger category $\mathcal{C}_0(T'/T)$ whose objects consist of pairs $(\mathcal{E}', \varphi)$ where $\mathcal{E}'$ is a quasi-coherent sheaf on T' and $\varphi\colon p_1^*\mathcal{E}' \to p_2\mathcal{E}'$ is an isomorphism. Morphisms in $\mathcal{C}_0(T'/T)$ are defined just as in $\mathcal{C}(T'/T)$, making $\mathcal{C}(T'/T)$ a full subcategory of $\mathcal{C}_0(T'/T)$. This subcategory has the property that, given an object $(\mathcal{E}', \varphi)$ of $\mathcal{C}(T'/T)$, if $(\mathcal{E}', \varphi) \to (\mathcal{F}', \psi)$ is an isomorphism in $\mathcal{C}_0(T'/T)$ then $(\mathcal{F}', \psi)$ must also lie in $\mathcal{C}(T'/T)$. (One says then that $\mathcal{C}(T'/T)$ is a *saturated* subcategory of $\mathcal{C}_0(T'/T)$.) The verification of this fact is by a diagram chase. This fact tells us that if the descent problem corresponding to an object $(\mathcal{F}', \psi)$ of $\mathcal{C}_0(T'/T)$ admits a solution (meaning that $\mathcal{F}'$ is isomorphic to a sheaf $f^*\mathcal{F}$ compatibly with ψ) then $(\mathcal{F}', \psi)$ lies in $\mathcal{C}(T'/T)$, i.e., ψ must satisfy the cocycle condition.

5. Faithfully flat descent

In this section we give some of the descent statements that are important for the theory of stacks. Most of these results are rather quick consequences of Theorem A.2. More challenging applications will be given in the last section. First we have a descent result for vector bundles.

PROPOSITION A.12. *Let $f\colon T' \to T$ be a locally fpqc morphism of schemes. Then:*
(a) *Given a locally free sheaf of finite type $\mathcal{E}'$ on T' and an isomorphism $\varphi\colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ such that $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$, there exists a locally free sheaf of finite type $\mathcal{E}$ on T and an isomorphism $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ satisfying $p_2^*\lambda = \varphi \circ p_1^*\lambda$, and these are unique up to canonical isomorphism.*
(b) *With notation as in* (a)*, suppose $(\mathcal{F}', \psi)$ is another descent datum with solution given by $\mathcal{F}$ and μ. Then, for every morphism $h'\colon \mathcal{E}' \to \mathcal{F}'$ satisfying $p_2^*h' \circ \varphi = \psi \circ p_1^*h'$ there is a unique morphism $h\colon \mathcal{E} \to \mathcal{F}$ such that $\mu \circ f^*h = h' \circ \lambda$.*

PROOF. This follows from Theorem A.2, coupled with Exercise A.2. □

Next we turn to descent for affine schemes.

PROPOSITION A.13. *Let $f\colon T' \to T$ be a locally fpqc morphism of schemes.* (a) *Given an affine morphism of schemes $P' \to T'$ and an isomorphism $\varphi\colon P' \times_T T' \to T' \times_T P'$ over T'' satisfying the cocycle condition, there exists an affine morphism $P \to T$ and isomorphism $\lambda\colon T' \times_T P \to P'$ over T', unique up to canonical isomorphism, such that $T' \times_T \lambda = \varphi \circ (\lambda \times_T T')$.*
(b) *With notation as in* (a)*, suppose (Q', ψ) is another descent datum with solution given by $Q \to T$ and μ. Then, for every morphism $h'\colon P' \to Q'$ over T' satisfying $(T' \times_T h') \circ \varphi = \psi \circ (h' \times_T T')$ there is a unique morphism $h\colon P \to Q$ such that $\mu \circ (T' \times_T h) = h' \circ \lambda$.*

We will see that descent for morphisms reduces to the statement that the functor $\mathrm{Hom}(-, X)$ satisfies the sheaf axiom for any locally fpqc covering, which holds for an arbitrary scheme X.

PROPOSITION A.14. *Let $f\colon T' \to T$ be a locally fpqc morphism of schemes. Let X be a scheme. If $g\colon T' \to X$ is a morphism of schemes such that $g \circ p_1 = g \circ p_2$, then there is a unique morphism $h\colon T \to X$ such that $h \circ f = g$.*

The proof of this proposition requires some preparatory results. Below we denote by $f^{\#}\colon \mathcal{O}_T \to f_*\mathcal{O}_{T'}$ the morphism of structure sheaves induced by a morphism of schemes $f\colon T \to T'$. Let $p\colon T'' \to T$ be the composition $f \circ p_1 = f \circ p_2$.

LEMMA A.15. *Suppose $f\colon T' \to T$ is locally fpqc. Then the sequence*

$$0 \longrightarrow \mathcal{O}_T \xrightarrow{f^{\#}} f_*\mathcal{O}_T \xrightarrow{f_*p_1^{\#} - f_*p_2^{\#}} p_*\mathcal{O}_{T''}$$

is exact.

Proof. By Theorem A.2(b) applied to $\mathcal{E} = \mathcal{F} = \mathcal{O}_T$ and adjointness of pushfoward and pullback, the sequence

$$(8)\qquad 0 \longrightarrow \Gamma(T, \mathcal{O}_T) \xrightarrow{f^\#} \Gamma(T, f_*\mathcal{O}_{T'}) \xrightarrow{f_*p_1^\# - f_*p_2^\#} \Gamma(T, p_*\mathcal{O}_{T''})$$

is exact. The sequence (8) with T replaced by any open subscheme of T is still exact, so the sequence of sheaves is exact. □

Lemma A.16. *Suppose $f\colon T' \to T$ is locally fpqc. Let Z be a subset of T such that $f^{-1}(Z)$ is closed in T'. Then Z is a closed subset of T.*

Proof. Since closedness of Z can be checked Zariski locally on T, we are reduced to the case that T' and T are affine. Let us write $T' = \operatorname{Spec} A'$ and $T = \operatorname{Spec} A$. Since f is surjective, it is sufficient to show that the hypothesis that $f^{-1}(Z)$ is closed implies

$$(9)\qquad f^{-1}(\overline{Z}) = f^{-1}(Z),$$

where $\overline{Z}$ denotes the closure of Z. We introduce the ideals

$$I = \bigcap_{\mathfrak{p}\in Z} \mathfrak{p} \qquad \text{and} \qquad I' = \bigcap_{\mathfrak{p}'\in f^{-1}(Z)} \mathfrak{p}',$$

corresponding to closed subsets $\overline{Z} \subset T$ and $f^{-1}(Z) \subset T'$, respectively. We have $I' \cap A = I$ (viewing A as a subring of A'). In other words, I fits into an exact sequence

$$(10)\qquad 0 \longrightarrow I \longrightarrow A \longrightarrow A'/I'.$$

Tensoring (10) by A' identifies $I \otimes_A A'$ with the kernel of the composite of $A' \to A' \otimes_A A'$, $x \mapsto 1 \otimes x$, with the quotient map by the ideal $I' \otimes_A A'$. The ideal $A' \otimes_A I'$ has the same radical as the ideal $I' \otimes_A A'$, since $(f \circ p_1)^{-1}(Z)$ is the closed subset of T'' associated with both. Thus $\sqrt{I \otimes_A A'} = I'$, and (9) is established. □

Exercise A.7. If $f\colon T' \to T$ is any surjective morphism of schemes, then for any points x and y in T such that $f(x) = f(y)$, there exists $z \in T''$ such that $p_1(z) = x$ and $p_2(z) = y$.

Proof of Proposition A.14. By Exercise A.7, the map h that we are required to produce is completely determined on the set-theoretic level. The map h is continuous by Lemma A.16. Finally, the required map of structure sheaves $h^\#\colon \mathcal{O}_X \to h_*\mathcal{O}_T$ is determined uniquely by applying h_* to the exact sequence from Lemma A.15. □

Proof of Proposition A.13. To prove (a), we need to show is the existence of the solution to a descent problem. To give a scheme, affine over T, is the same as giving a quasi-coherent sheaf of $\mathcal{O}_T$-algebras. This sheaf (as a sheaf of modules) is constructed by descent for quasi-coherent sheaves, and is given the structure of $\mathcal{O}_T$-algebra (multiplication map) using descent for morphisms of quasi-coherent sheaves.

For (b), the exactness of

$$(11)\qquad \operatorname{Hom}_T(P, Q) \to \operatorname{Hom}_{T'}(T' \times_T P, T' \times_T Q) \rightrightarrows \operatorname{Hom}_{T''}(T'' \times_T P, T'' \times_T Q)$$

is the same as the exactness of

$$\operatorname{Hom}_T(P, Q) \to \operatorname{Hom}_T(T' \times_T P, Q) \rightrightarrows \operatorname{Hom}_T(T'' \times_T P, Q),$$

which follows from Proposition A.14.

As in the previous sections, the existence of the solution to a descent problem plus exactness of (11) imply the full assertions of both statements of this proposition. □

Torsors for an affine group scheme provide an important example of affine morphisms of schemes. We recall that if G is a group scheme over T then a left G-torsor (for the étale topology) is specifying by giving a scheme E (the total space) with a map $E \to T$ (the structure map), together with a left G-action $a\colon G \times E \to E$ which upon pullback by some étale cover $T' \to T$ becomes isomorphic to the trivial G-torsor $G \times T'$ (with action of G on itself by left multiplication). Examples are any unramified two-sheeted cover (for $G = \mathbb{Z}/2$, i.e., the constant group scheme $T \times \mathbb{Z}/2 \to T$ over any T) and the complement of the zero section of a line bundle (for the multiplicative group $\mathbb{G}_m$). A consequence of Proposition A.13 is that effective descent holds for G-torsors whenever G is an affine group scheme over the base scheme. Note that the action of G on E is given by a map of affine schemes, to which descent of morphisms applies.

Corollary A.17. *Let $f\colon T' \to T$ be a locally fpqc morphism. Let G be an affine group scheme over T. Then:* (a) *Given a G-torsor E' on T' and an isomorphism $\varphi\colon p_1^*E' \to p_2^*E'$ over T'' satisfying the cocycle condition over T''' there exists a G-torsor E on T and G-equivariant isomorphism $\lambda\colon f^*E \to E'$ over T', unique up to canonical isomorphism, such that $p_2^*\lambda = \varphi \circ p_1^*\lambda$.*
(b) *Let notation be as in* (a)*, and suppose (F', ψ) is another descent datum with solution given by F and μ. Then, for every G-equivariant isomorphism $h'\colon E' \to F'$ over T' satisfying $p_2^*h' \circ \varphi = \psi \circ p_1^*h'$ there is a unique G-equivariant isomorphism $h\colon E \to F$ over T such that $\mu \circ f^*h = h' \circ \lambda$.*

6. Non-effective descent: an example

In this section we show how descent can fail for proper morphisms. In the next section we will see how, with projective morphisms and suitable additional data, it is possible to overcome this problem.

Let T be a smooth projective threefold over the complex numbers which has a 2-to-1 étale cover $f\colon T' \to T$, such that there exists a nodal curve Z in T whose pre-image in T' consists of the union of two smooth curves E and F meeting transversely at two points that we denote P and Q.

Now form X' by modifying T' along $E \cup F$. Near P, we first blow up E, and then we blow up the proper transform of F. Near Q, we first blow up F, and then the proper transform of E. Away from $\{P, Q\}$, the order of blow-up is irrelevant, so we can glue these together to make a scheme X'.

Since T' is a 2-to-1 cover of T it has an involution that respects the map to T. Because of the order in which we performed the blow-ups, this involution actually extends to an involution of X'. Both the involution of T' and that of X' are without fixed points. We can express the problem of trying to form the quotient of X' by the involution as a descent problem. The pair consisting of the object $X' \to T'$ (in the category of schemes over T'), together with the isomorphism

$$X' \times_T T' \to T' \times_T X'$$

which is the identity map over the identity component of T'' and the involution over the other component, is a descent datum. This descent datum, we claim, is non-effective, i.e., there is no scheme quotient of X' by its involution.

Indeed, suppose X is a scheme over T with a map $\pi\colon X' \to X$ making

$$\begin{array}{ccc} X' & \longrightarrow & T' \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \longrightarrow & T \end{array}$$

a cartesian diagram. Consider, in X', the pre-images $A \cup B$ of P and $C \cup D$ of Q, where each of A, B, C, D is a rational curve. Now we make a calculation in the ring of cycles modulo algebraic equivalence [**31**, §10.3] on X. Denoting this ring by $B^*(X)$, we have, with a suitable labeling of the curves, equations $[B] = [C] + [D]$ and $[D] = [A] + [B]$ in $B^*(X')$, and hence

$$[A] + [C] = 0 \tag{12}$$

in $B^*(X)$. Since π is finite and flat of degree 2, we find from (12) that

$$2\pi_*[A] = \pi_*([A] + [C]) = 0 \tag{13}$$

in $B^*(X)$. This is impossible if X is a scheme. Indeed, if U denotes an affine neighborhood in X of the generic point of $\pi(A)$, and if we take Y to be a a generically chosen hypersurface of U, then the closure $\overline{Y}$ of Y in X meets $\pi(A)$ properly in at least one point. This means that $[\overline{Y}] \cdot \pi_*[A]$ is a zero-cycle class of positive degree, which is a contradiction to (13).

Of course, the quotient of X' by the involution exists as an analytic space. This analytic space quotient is, in effect, Hironaka's example of an algebraic space which is not a scheme (cf. [**52**, Exa. B.3.4.1]). So, effective descent fails for general schemes. The category of algebraic spaces, which contains quasi-separated schemes as a full subcategory, has the advantage over the category of schemes in that effective descent holds in a quite general setting (ast least for faithfully flat morphisms, locally of finite presentation). We remark that this descent property, stated in this text as Proposition **??**[1], relies on Artin's criterion for a stack to be algebraic (Theorem **??**[2]), whose proof is beyond the scope of these notes.[3]

Exercise A.8. Construct a non-effective descent datum with T a threefold over the real numbers and f the map induced by base change via $\mathbb{R} \to \mathbb{C}$. This demonstrates that there exist an algebraic space, separated and of finite type over a field, which is not a scheme, but which becomes a scheme after a finite extension of the base field.

[1] A reference to a statement in Part II of the book, which might not appear for a while.

[2] Another reference to Part II of the book.

[3] When the contents of Part II of this book are more settled, change this comment either to refer to discussion of proof in this book (if it will be written) or to some more helpful brief statement here with precise reference to Artin's paper.

7. Further descent results

Despite the failure of effective fpqc descent for general morphisms of schemes, there are restricted classes of morphisms of schemes for which effective descent is known to hold. We saw that affine morphisms form one such class of morphisms.

It is important for the theory of stacks that quasi-affine and (polarized) quasi-projective morphisms make up two other such classes. Let us recall that a morphism is quasi-affine if it can be factored as a quasi-compact open inclusion followed by an affine morphism. If $f\colon X \to Y$ is any separated quasi-compact morphism of schemes, then in the canonical factorization

$$X \xrightarrow{g} \operatorname{Spec} f_*\mathcal{O}_X \xrightarrow{h} Y, \tag{14}$$

g is an open inclusion if and only if $f = h \circ g$ is quasi-affine [**EGA** II.5.1.6]. Quasi-projective morphisms enjoy a similar characterization, factoring through $\operatorname{Proj}(\bigoplus f_*\mathcal{O}_X(n))$. The Proj construction relies on a choice of relative ample invertible sheaf $\mathcal{O}_X(1)$, which must be included as part of the descent datum.

PROPOSITION A.18. *Let $f\colon T' \to T$ be a locally fpqc morphism of schemes. Then:*
(a) *Given a quasi-affine morphism of schemes $P' \to T'$ and an isomorphism $\varphi\colon P' \times_T T' \to T' \times_T P'$ over T'' satisfying the cocycle condition, there exists a quasi-affine morphism $P \to T$ and isomorphism $\lambda\colon T' \times_T P \to P'$ over T', unique up to canonical isomorphism, such that $T' \times_T \lambda = \varphi \circ (\lambda \times_T T')$.*
(b) *With notation as in* (a)*, suppose (Q', ψ) is another descent datum with solution given by $Q \to T$ and μ. Then, for every morphism $h'\colon P' \to Q'$ over T' satisfying $(T' \times_T h') \circ \varphi = \psi \circ (h' \times_T T')$ there is a unique morphism $h\colon P \to Q$ such that $\mu \circ (T' \times_T h) = h' \circ \lambda$.*

PROOF. Let t' denote the morphism $P' \to T'$. We have the canonical factorization

$$P' \to \operatorname{Spec}(t'_*\mathcal{O}_{P'}) \to T'. \tag{15}$$

Set $\mathcal{E}' = t'_*\mathcal{O}_{P'}$ and $\overline{P}' = \operatorname{Spec}\mathcal{E}'$. Since f is flat, we have a canonical isomorphism $p_1^*\mathcal{E}' \xrightarrow{\sim} (t' \times_T T')_*\mathcal{O}_{P'\times_T T'}$. Under this isomorphism, the morphisms we obtain by pulling back (15) by p_1,

$$P' \times_T T' \to \overline{P}' \times_T T' \to T'', \tag{16}$$

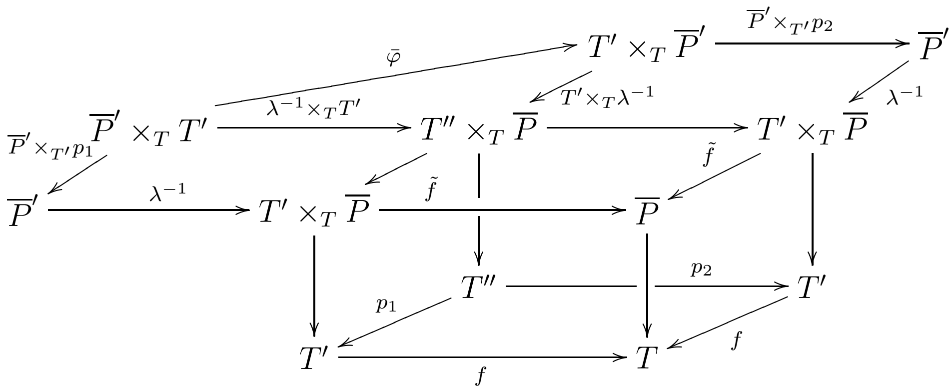
constitute the canonical factorization of $P' \times_T T' \to T''$. Similarly,

$$T' \times_T P' \to T' \times_T \overline{P}' \to T'', \tag{17}$$

gives the canonical factorization of $T' \times_T P' \to T''$, under the canonical isomorphism $p_2^*\mathcal{E}' \xrightarrow{\sim} (T' \times_T t')_*\mathcal{O}_{T'\times_T P'}$.

The isomorphism $\varphi\colon P' \times_T T' \to T' \times_T P'$ determines an isomorphism $\bar\varphi\colon \overline{P}' \times_T T' \to T' \times_T \overline{P}'$. Since φ satisfies the cocycle condition, so does $\bar\varphi$. Now by Proposition A.13, there is an affine morphism $\overline{P} \to T$ and an isomorphism $\lambda\colon T' \times_T \overline{P} \to \overline{P}'$ satisfying $T' \times_T \lambda = \bar\varphi \circ (\lambda \times_T T')$.

Since $\overline{P}'$ is isomorphic to $T' \times_T P'$, the morphism $\overline{P}' \to \overline{P}$ is locally fpqc. Moreover we can canonically identify $\overline{P}' \times_{\overline{P}} \overline{P}'$ with $\overline{P}' \times_T T'$. To do this, we start with the cube with cartesian faces and extend the top face with cartesian squares involving the isomorphism λ, as shown in the following diagram, where $\tilde{f}$ is used to denote the second projection from $T' \times_T \overline{P}$.



By the condition on λ, the upper triangle commutes. Using Lemma A.16, we see that there is a one-to-one correspondence between open subschemes $U \subset \overline{P}$ and open subschemes $U' \subset \overline{P}'$ satisfying

$$(\overline{P}' \times_{T'} p_1)^{-1}(U') = \bar{\varphi}^{-1}\big((\overline{P}' \times_{T'} p_2)^{-1}(U')\big). \tag{18}$$

In (15) we have P' realized as an open subscheme of $\overline{P}'$. The pre-image of P' by $\overline{P}' \times_{T'} p_1$, respectively by $\overline{P}' \times_{T'} p_2$, is the image of the open inclusion in (16), respectively (17). Now (18) holds since we have a commutative diagram

$$\begin{array}{ccc} P' \times_T T' & \longrightarrow & \overline{P}' \times_T T' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\bar\varphi} \\ T' \times_T P' & \longrightarrow & T' \times_T \overline{P}' \end{array}$$

So there is a unique open subscheme $P \subset \overline{P}$ satisfying $(\tilde{f} \circ \lambda^{-1})^{-1}(P) = P'$. Now the scheme P and the restriction of λ to $T' \times_T P$ constitute a solution to the descent problem posed by P' and φ. □

The large diagram in the proof of this proposition illustrates a general principle. To give the descent datum (P', φ) is equivalent to giving a scheme P'' with morphism to T'' and an *equivalence relation*

$$(\tilde{p}_1, \tilde{p}_2) \colon P'' \to P' \times P'$$

compatible with $(p_1, p_2) \colon T'' \to T' \times T'$. The compatibility condition is that the diagram

$$\begin{array}{ccc} P'' & \xrightarrow{\tilde{p}_i} & P' \\ \downarrow & & \downarrow \\ T'' & \xrightarrow{p_i} & T' \end{array}$$

is cartesian for $i = 1, 2$. (To be an equivalence relation means that $(\tilde{p}_1, \tilde{p}_2)$ is a locally closed embedding[4] satisfying conditions that generalize the usual conditions when S is a set for a subset of $S \times S$ to be an equivalence relation.) We take P'' to be $P' \times_T T'$, with $\tilde{p}_1$ the projection map to P' and $\tilde{p}_2$ the composite of φ and the projection $T' \times_T P' \to P'$. In the language of equivalence relations, effectivity amounts to providing a scheme P over T and a map $P' \to P$ such that $P'' \cong P' \times_P P'$. Many descent problems can be stated in the language of equivalence relations (see [**43**]). In this appendix we stick to the language of descent data, though in the next result, descent for quasi-projective schemes, we employ the notation for the maps that we have just introduced:

$$\tilde{p}_1 \colon P' \times_T T' \to P', \tag{19}$$

$$\tilde{p}_2 \colon P' \times_T T' \xrightarrow{\sim} T' \times_T P' \to P'. \tag{20}$$

Proposition A.19. *Let $f \colon T' \to T$ be a locally fpqc morphism of schemes. Given a quasi-projective morphism of schemes $P' \to T'$, a relatively ample invertible sheaf $\mathcal{L}'$ on P', an isomorphism $\varphi \colon P' \times_T T' \to T' \times_T P'$ over T'' satisfying the cocycle condition, and an isomorphism $\omega \colon \tilde{p}_1^* \mathcal{L}' \to \tilde{p}_2^* \mathcal{L}'$ satisfying the cocycle condition on $P' \times_T T''$, where $\tilde{p}_1$ and $\tilde{p}_2$ are the maps of* (19)–(20), *there exists a scheme P with quasi-projective morphism $P \to T$ and relatively ample invertible sheaf $\mathcal{L}$, an isomorphism $\lambda \colon T' \times_T P \to P'$ over T' and, with $\tilde{f} \colon P' \to P$ the composition of λ^{-1} and projection, an isomorphism $\chi \colon \tilde{f}^* \mathcal{L} \to \mathcal{L}'$; these satisfy $T' \times_T \lambda = \varphi \circ (\lambda \times_T T')$ and $\tilde{p}_2^* \chi = \omega \circ \tilde{p}_1^* \chi$. The solution to the descent problem is unique up to canonical isomorphism.*

As above, we set $P'' = P' \times_T T'$. If we further define $P''' = P' \times_T T''$ then the usual cocycle condition on φ is expressed by the commutativity of the triangle

$$\begin{array}{ccc} P''' & \longrightarrow & T' \times_T P'' \\ & \searrow \quad \swarrow & \\ & T'' \times_T P' & \end{array}$$

where the maps are the ones obtained from φ by base change. There are projection maps $\tilde{p}_{12}$ and $\tilde{p}_{13}$ (obtained from p_{12} and p_{13} by base change) and $\tilde{p}_{23}$ (the composite $P''' \to T' \times_T P'' \to P''$). Now the cocycle condition on ω is that the diagram

$$\begin{array}{ccccc} \tilde{p}_{12}^* \tilde{p}_1^* \mathcal{L}' & \xrightarrow{\tilde{p}_{12}^* \omega} & \tilde{p}_{12}^* \tilde{p}_2^* \mathcal{L}' & = & \tilde{p}_{23}^* \tilde{p}_1^* \mathcal{L}' \\ \| & & & & \downarrow {\scriptstyle \tilde{p}_{23}^* \omega} \\ \tilde{p}_{13}^* \tilde{p}_1^* \mathcal{L}' & \xrightarrow{\tilde{p}_{13}^* \omega} & \tilde{p}_{13}^* \tilde{p}_2^* \mathcal{L}' & = & \tilde{p}_{23}^* \tilde{p}_2^* \mathcal{L}' \end{array}$$

commutes.

[4] The correct condition is really *monomorphism*, but the more restrictive condition suffices for the discussion of effectivity since the diagonal morphism of any scheme is a locally closed embedding.

The condition on χ is commutativity of the diagram

$$\begin{array}{ccc} \tilde{p}_1^*\tilde{f}^*\mathcal{L} & \xrightarrow{\tilde{p}_1^*\chi} & \tilde{p}_1^*\mathcal{L}' \\ \| & & \downarrow{\scriptstyle\omega} \\ \tilde{p}_2^*\tilde{f}^*\mathcal{L} & \xrightarrow{\tilde{p}_2^*\chi} & \tilde{p}_2^*\mathcal{L}' \end{array}$$

where the equality $\tilde{f}\circ\tilde{p}_1 = \tilde{f}\circ\tilde{p}_2$ is a consequence of the condition on λ (as detailed in the large commutative diagram in the proof of Proposition A.18).

Before we give the proof of this result, we recall that a quasi-projective morphism is a morphism of finite type which factors as an open embedding followed by a map of the form $\mathrm{Proj}(\mathcal{S}) \to X$ where $\mathcal{S}$ is a graded sheaf of quasi-coherent $\mathcal{O}_X$-algebras. Associated to a separated morphism of finite type of schemes $f\colon X \to Y$ and an invertible sheaf $\mathcal{L}$ on X is a graded sheaf of algebras $\mathcal{S} := \bigoplus_{n\geq 0} f^*(\mathcal{L}^{\otimes n})$, open subscheme $U \subset X$, and factorization of the restriction of f to U as

$$U \to \mathrm{Proj}(\mathcal{S}) \to Y.$$

Now [**EGA** II.4.6.3] states that the map f is quasi-projective if and only if $U = X$ and $X \to \mathrm{Proj}(\mathcal{S})$ is an open embedding. Further, for a morphism of schemes to be of finite type is a Zariski local condition, and this is a condition that holds for any morphism if it holds after fpqc base change ([**EGA** IV.2.7.1(v)]).

Proof of Proposition A.19. We introduce $P'' = P' \times_T T'$ as above, with morphism $t''\colon P'' \to P'$. Consider the composite isomorphism

$$p_1^* t'_* \mathcal{L}' \cong t''_* \tilde{p}_1^* \mathcal{L}' \xrightarrow{t''_*\omega} t''_* \tilde{p}_2^* \mathcal{L}' \cong p_2^* t'_* \mathcal{L}'$$

of two base-change isomorphisms and the pushforward of ω. We claim that $t'_*\mathcal{L}'$, together with this isomorphism, constitutes a descent datum, and hence determines by Theorem A.2 a quasi-coherent sheaf $\mathcal{S}_1$ on T. Verifying the cocycle condition amounts to writing down a large diagram whose commutativity results by (i) naturality of the base change isomorphism, (ii) the property that a composite of base change morphisms resulting from two commuting squares glued together equals the base change morphism coming from the large outer diagram (see the Glossary), and (iii) the cocycle condition on ω.

The same consideration applies as well to $\mathcal{L}'^{\otimes n}$ yielding a sheaf $\mathcal{S}_n$ on T, for all $n \geq 0$. So, we get a graded quasi-coherent sheaf $\mathcal{S} = \oplus_{n\geq 0}\mathcal{S}_n$ on T which is given an algebra structure by using descent for morphisms of quasi-coherent sheaves.

The remainder of the argument exactly parallels the proof of Proposition A.18. We have the canonical factorization of $P' \to T'$ through $\overline{P}' := \mathrm{Proj}(\bigoplus t'_*\mathcal{L}'^{\otimes n})$, with descent datum $\overline{\varphi}\colon \overline{P}' \times_T T' \to T' \times_T \overline{P}'$. A solution is given by $\overline{P} := \mathrm{Proj}(\mathcal{S})$. As before there is a uniquely determined open subscheme $P \subset \overline{P}$ whose pullback is the image of $P' \to \overline{P}'$. Now $P \to T$ with $T' \times_T P \to P'$ and the restriction of the invertible sheaf $\mathcal{O}_{\overline{P}}(1)$ to P constitute a solution to the descent problem. □

Proposition A.19 is used to show that various families of curves determine stacks. It is also are used to show that other moduli problems, such as abelian varieties with various kinds of polarization, give rise to stacks.

REMARK A.20. The proof of Proposition A.18 is in fact the special case $\mathcal{L}' = \mathcal{O}_{P'}$ of the proof just given. In fact there is a common generalization of Propositions A.18 and A.19. This is the statement that effective descent holds for schemes equipped with relatively ample invertible sheaves. The proof is obtained by copying the proof of Proposition A.19 and changing "of finite type" to "quasi-compact" throughout.

Locally quasi-finite separated morphisms are another class of morphisms which enjoy an effective descent property. For this we restrict to fppf covers, since we will exploit the fact that fppf morphisms are open (cf. the Glossary) in the proof.

PROPOSITION A.21. *Let $f\colon T' \to T$ be an fppf morphism of schemes. Then:* (a) *Given a locally quasi-finite separated morphism of schemes $P' \to T'$ and an isomorphism $\varphi\colon P' \times_T T' \to T' \times_T P'$ over T'' satisfying the cocycle condition, there exists a locally quasi-finite separated morphism $P \to T$ and isomorphism $\lambda\colon T' \times_T P \to P'$ over T', unique up to canonical isomorphism, such that $T' \times_T \lambda = \varphi \circ (\lambda \times_T T')$.*
(b) *With notation as in* (a)*, suppose (Q', ψ) is another descent datum with solution given by $Q \to T$ and μ. Then, for every morphism $h'\colon P' \to Q'$ over T' satisfying $(T' \times_T h') \circ \varphi = \psi \circ (h' \times_T T')$ there is a unique morphism $h\colon P \to Q$ such that $\mu \circ (T' \times_T h) = h' \circ \lambda$.*

PROOF. Let (T_i), $i \in I$, be an affine open cover of T. Since fppf morphisms are open, each T_i is covered by the images of finitely many affine open subsets of $f^{-1}(T_i)$. So, as in the proof of Lemmas A.8 and A.9, we may reduce to the case that f is an fppf morphism of *affine* schemes.

Suppose, now, P' is a quasi-compact scheme. Then since T' is affine, the morphism $P' \to T'$ is quasicompact, hence is quasi-finite and separated. According to [**EGA** IV.18.12.12], any quasi-finite separated morphism is quasi-affine. So Proposition A.18 may be applied to produce quasi-affine $P \to T$ and isomorphism $\lambda : T' \times_T P \to P'$ solving the descent problem.

Now the proof is completed by covering P' by quasi-compact open subschemes P'_α with the property $\varphi(P'_\alpha \times_T T') = T' \times_T P'_\alpha$ and gluing the corresponding T-schemes P_α (see Exercise A.9, below). Such a covering exists, since given any quasi-compact open $U \subset P'$, the image V of $\varphi(U \times_T T')$ under the projection morphism $T' \times_T P' \to P'$ satisfies: (i) $V \subset P'$ is quasi-compact open; (ii) $U \subset V$; and (iii) $\varphi(V \times_T T') = T' \times_T V$. We have (i) since V is the image under an fppf morphism of the quasi-compact scheme $\varphi(U \times_T T')$, and we have $U \subset V$ since by the cocycle condition, the evident sections of $P' \times_T T' \to P'$ and $T' \times_T P' \to P'$ are compatible with φ. For (iii), given $p' \in V$, say, with $\varphi(p, t_2) = (t_1, p')$ for some $p \in P'$ over $t_1 \in T'$ and $(t_1, t_2) \in T''$, if we consider some $(p', t_3) \in V \times_T T'$ and write $\varphi(p, t_3) = (t_1, p'')$, then the cocycle condition gives $\varphi(p', t_3) = (t_2, p'') \in T' \times_T V$, and one inclusion is established. The other inclusion is established by a similar argument. □

Exercise A.9. In the setting of the proof of Proposition A.21, having reduced to the case T and T' are affine, let P'_α be quasi-compact opens covering P' and having the property stated in the proof. Suppose that $P_\alpha \to T$ and $\lambda_\alpha : T' \times_T P_\alpha \to P'_\alpha$ solve the descent problem posed by P'_α.

(i) For α and β show that $P'_{\alpha\beta} := P'_\alpha \cap P'_\beta$ satisfies: $P'_{\alpha\beta} = \lambda_\alpha(T' \times_T U_{\alpha\beta})$ for a unique open subscheme $U_{\alpha\beta} \subset P_\alpha$.

(ii) Show that there is a unique isomorphism $\vartheta_{\beta\alpha} \colon U_{\alpha\beta} \to U_{\beta\alpha}$ fitting into a commutative diagram

$$\begin{array}{ccc} T' \times_T U_{\alpha\beta} & \xrightarrow{\lambda_\beta^{-1}\circ\lambda_\alpha} & T' \times_T U_{\beta\alpha} \\ \downarrow & & \downarrow \\ U_{\alpha\beta} & \longrightarrow & U_{\beta\alpha} \end{array}$$

(iii) Show that P_α, $U_{\alpha\beta}$, and $\vartheta_{\beta\alpha}$ satisfy the compatibility condition for *recollement* to be applied (cf. the Glossary) to produce P with open embeddings $\varphi_\alpha \colon P_\alpha \to P$, and that P admits a unique morphism to T restricting to the given $P_\alpha \to T$ for every α.

(iv) Conclude that the λ_α glue to give λ such that P and λ satisfy the requirements of the proposition.

A more recent descent result – which is not needed in this book – stems from the study of principal bundles on curves. Consider a scheme T with a covering by two Zariski open subsets. Then, the cocycle condition on the transition mappings is vacuous, so any isomorphism of objects on the overlap determines an object on T. One might expect a similar result for the cover consisting of the formal neighborhood of a divisor on T and the complement of the divisor. So, for instance, a vector bundle on a curve C over a field k should be determined uniquely up to isomorphism by a vector bundle on the complement of a k-rational point x, a vector bundle on $\operatorname{Spec} \widehat{\mathcal{O}}_{x,C}$, and an isomorphism on the overlap. Here is the precise result:

Proposition A.22. *Let $T = \operatorname{Spec} A$ be an affine scheme, and let $f \colon T' \to T$ be the cover given by $T' = T'_1 \amalg T'_2$, where $T'_1 \subset T$ is the complement of the divisor corresponding to a non-zero-divisor $a \in A$ and T'_2 is* Spec *of the completion of A with respect to the a-adic topology. Let $T'' = T'_1 \times_T T'_2$ with projections p_i to T'_i. Given a quasi-coherent sheaves $\mathcal{E}'_i$ on T'_i, for $i = 1$ and 2 such that $\mathcal{E}'_2$ is f-regular (i.e., such that multiplication by f induces an injective map $\mathcal{E}'_2 \to \mathcal{E}'_2$) and an isomorphism $\varphi \colon p_1^*\mathcal{E}'_1 \to p_2^*\mathcal{E}'_2$, there exists a locally free sheaf $\mathcal{E}$ on T, unique up to canonical isomorphism, with an isomorphism $\lambda' \colon f^*\mathcal{E} \to \mathcal{E}'$ satisfying $p_2^*\lambda = \varphi \circ p_1^*\lambda$.*

We remark that Proposition A.22 does not follow from faithfully flat descent. In fact, the map f is not even flat in general (it is when A is Noetherian). What is true is that f is faithful, i.e., we have $f^*\mathcal{E} = 0$ if and only if $\mathcal{E} = 0$ for quasi-coherent $\mathcal{E}$. This "faithful descent" result is proved by Beauville and Laszlo in [**10**] and has important applications in conformal field theory (see [**9**] for a survey) and in the geometric Langlands program.

Answers to Exercises

A.1. (1) For $\Rightarrow$, a nonzero element of M determines an inclusion $A/I \to M$, hence an inclusion $A'/IA' \to A' \otimes_A M$. With $\mathfrak{m}$ any maximal ideal containing I, it suffices to show $A'/\mathfrak{m}A' \neq 0$, and this holds by surjectivity of $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$. For $\Leftarrow$, the crucial fact is that $\mathfrak{p} \in \mathrm{Spec}\, A$ implies $A/\mathfrak{p} \to A'/\mathfrak{p}A'$ is injective. Indeed, if the image in A' of some $a \in A \smallsetminus \mathfrak{p}$ lies in $\mathfrak{p}A'$ then $(\mathfrak{p}+aA)/\mathfrak{p}$ would be a nonzero A-module becoming zero under $A' \otimes_A -$. Now any maximal ideal of the localization $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_{A/\mathfrak{p}} A'/\mathfrak{p}A'$ gives an element of $\mathrm{Spec}\, A'$ that maps to $\mathfrak{p}$. The condition in (2) is readily shown to be equivalent to that given in (1); a reference is [**17**, Proposition I.3.1.1].

A.2. (1) If ρ is the homomorphism, look at the exact sequence

$$0 \to \mathrm{Ker}(\rho) \to M \to N \to \mathrm{Coker}(\rho) \to 0.$$

(2) If $A' \otimes_A M$ is finitely generated, one can find a finitely generated free A-module F and a morphism $F \to M$ such that $A' \otimes_A F \to A' \otimes_A M$ is surjective. Then (1) shows that $F \to M$ is surjective. The same argument on the kernel of $F \to M$ gives the corresponding assertion for finitely presented. The flat case follows directly from the definitions, and the last follows from the fact that locally free of finite rank is equivalent to flat and finitely presented. A reference for this last fact is [**17**, Corollary II.5.2.2].

A.3. It suffices to prove that $A' \otimes_A T^\bullet \otimes_A M$ is exact. One can prove this as in the lemma, or, more elegantly, by defining a chain homotopy $h^n \colon A' \otimes_A T^n \otimes_A M \to A' \otimes_A T^{n-1} \otimes_A M$ by the formula $h^n(x \otimes x_1 \otimes \cdots \otimes x_n \otimes m) = x \cdot x_1 \otimes x_2 \cdots \otimes x_n \otimes m$, and verifying that $h^{n+1} \circ \delta^n + \delta^{n-1} \circ h^n = 1_{A' \otimes T^n \otimes M}$.

A.4. Cover T by affines T_i, and let $S_i = f^{-1}(T_i)$. Descent for morphisms holds for each $S_i \to T_i$ by the affine case, hence as well for $\coprod S_i \to \coprod T_i$. Since Zariski coverings satisfy descent for morphisms, we may deduce descent for morphisms for $\coprod S_i \to T$, and then for $T' \to T$.

A.5. Given $\mathcal{E}'$ on S and $\varphi \colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ satisfying the cocycle condition, pull back the cocycle condition via $R \times_T R \times_T R \to S \times_T S \times_T S$ and use effective descent for $f \circ g$ to conclude there exists $\mathcal{E}$ on T and $\lambda' \colon g^*f^*\mathcal{E} \to g^*\mathcal{E}'$ such that $q_2^*\lambda' = k^*\varphi \circ q_1^*\lambda'$. Since $k \circ \ell$ factors through the image of S in $S \times_T S$ (by the diagonal morphism), we have $r_2^*\lambda' = \ell^* q_2^*\lambda' = \ell^* q_1^*\lambda' = r_1^*\lambda'$, hence there exists $\lambda \colon f^*\mathcal{E} \to \mathcal{E}'$ such that $g^*\lambda = \lambda'$. Now $k^*p_2^*\lambda = q_2^*g^*\lambda = k^*\varphi \circ q_1^*g^*\lambda' = k^*(\varphi \circ p_1^*\lambda)$, hence $p_2^*\lambda = \varphi \circ p_1^*\lambda$.

A.6. A non-effective descent datum is given by multiplication by p/q on the rank 1 free module on $\mathrm{Spec}\, \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q$, for every pair p and q of prime numbers.

A.7. What is true more generally is that if T_1 and T_2 are any schemes mapping to T, with $x \in T_1$ and $y \in T_2$ mapping to the same point $t \in T$, then the fiber product $T_1 \times_T T_2$ contains a point z with $p_1(z) = x$ and $p_2(z) = y$. Localizing, we may suppose we are in the affine case with x, y, and t all closed points. Passing to closed subschemes we are reduced to the assertion that the tensor product of two fields over a third field is a nonzero ring and hence contains a prime ideal.

A.8. Repeat the given construction using an irreducible curve defined over $\mathbb{R}$ which becomes the union of two irreducible components (meeting at nodes) after extending the base field to $\mathbb{C}$.

A.9. For (i), we have by $U \mapsto \lambda_\alpha(T' \times_T U)$ a bijection between open subschemes $U \subset P_\alpha$ and open subschemes $U' \subset P'_\alpha$ satisfying $\varphi(U' \times_T T') = T' \times_T U'$. For (ii) and (iii), obtain the morphism $\vartheta_{\beta\alpha}$ and identitites $\vartheta_{\alpha\beta} \circ \vartheta_{\beta\alpha} = 1_{U_{\alpha\beta}}$, $\vartheta_{\beta\alpha} \circ \vartheta_{\alpha\beta} = 1_{U_{\beta\alpha}}$, and $\vartheta_{\gamma\beta} \circ \vartheta_{\beta\alpha} = \vartheta_{\gamma\alpha}$ (the first two to show $\vartheta_{\beta\alpha}$ is an isomorphism, the last for recollement) using Proposition A.14. The commutativity of the diagram in (ii) gives (iv).