Exercise Sheet 3 - Solution

Exercise 1

a)
- $(\overrightarrow{a} + \overrightarrow{b}) \cdot \overrightarrow{c} = \overrightarrow{a} \cdot \overrightarrow{c} + \overrightarrow{b} \cdot \overrightarrow{c}$

$$(\overrightarrow{a} + \overrightarrow{b}) \cdot \overrightarrow{c} = \left(\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \right) \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = -5 + 7 + 3 = 5$$

$$\overrightarrow{a} \cdot \overrightarrow{c} + \overrightarrow{b} \cdot \overrightarrow{c} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3 + 2 = 5$$

- $(\overrightarrow{a} + \overrightarrow{b}) \times \overrightarrow{c} = \overrightarrow{a} \times \overrightarrow{c} + \overrightarrow{b} \times \overrightarrow{c}$

$$(\overrightarrow{a} + \overrightarrow{b}) \times \overrightarrow{c}) = \left(\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \right) \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7-3 \\ -3-5 \\ 5-(-7) \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 12 \end{pmatrix}$$

$$\overrightarrow{a} \times \overrightarrow{c} + \overrightarrow{b} \times \overrightarrow{c} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-4 \\ -4-3 \\ 3-(-2) \end{pmatrix} + \begin{pmatrix} 5-(-1) \\ 1-2 \\ 2-(-5) \end{pmatrix} = \begin{pmatrix} -2 \\ -7 \\ 5 \end{pmatrix} + \begin{pmatrix} 6 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 12 \end{pmatrix}$$

- $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -2-20 \\ 8-(-3) \\ 15-4 \end{pmatrix} = \begin{pmatrix} -22 \\ 11 \\ 11 \end{pmatrix}$$

$$-\overrightarrow{b} \times \overrightarrow{a} = -\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = -\begin{pmatrix} 20-(-2) \\ -3-8 \\ 4-15 \end{pmatrix} = -\begin{pmatrix} 22 \\ -11 \\ -11 \end{pmatrix} = \begin{pmatrix} -22 \\ 11 \\ 11 \end{pmatrix}$$

b) It holds:

$$\vec{r} = \overrightarrow{OA} + t \cdot \overrightarrow{AB} = \overrightarrow{OA} + t \cdot \left(\overrightarrow{OB} - \overrightarrow{OA} \right) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} 2-1 \\ 5-2 \\ 9-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

c) It holds:

$$\vec{n} \cdot \left(\vec{r} - \overrightarrow{r_A}\right) = 0 \quad \Leftrightarrow \quad \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix} \right) = 0$$

$$\Rightarrow \quad \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix} = 0 \quad \Rightarrow \quad 3x + y + 2z - 26 = 0$$

Exercise 2 (4 points)

a) (2 points)

$$\text{Direction vector of the line:} \quad \overrightarrow{UV} = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$$

$$\text{Normal vector of the plane:} \quad \vec{n} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$ (1 point)

Computation of the scalar product $\overrightarrow{UV} \cdot \vec{n} = |\overrightarrow{UV}| \cdot |\vec{n}| \cdot \cos(\varphi)$

$$\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = \sqrt{11} \cdot \sqrt{14} \cdot \cos(\varphi) \quad \Leftrightarrow \quad 8 = \sqrt{154} \cdot \cos(\phi) \quad \Rightarrow \varphi = 49.86^o$$

The sought angle is the complementary angle between normal vector and direction vector, that is $90^o - 49.865^o = 40.14^o$. (1 point)

b) (2 points) Setting $z = t$ we solve for x and y:

$$9x + 6y = 12 - 3t \qquad (1)$$

$$2x + y = 2 - t \qquad (2)$$ (1 point)

A short computation yields $x = -t$ and $y = t + 2$. The equation of the line is

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ t+2 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$ (1 point)

Exercise 3 (5 points)

a) (3 points) We set $\lambda_1 \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_2 \cdot \begin{pmatrix}3\\-2\\1\end{pmatrix} = \begin{pmatrix}5\\2\\7\end{pmatrix}$. Then we have: (1 point)

$$\begin{array}{l} (1) \\ (2) \end{array} \left| \begin{array}{r} \lambda_1 + 3\lambda_2 = 5 \\ 2\lambda_1 - 2\lambda_2 = 2 \end{array} \right| \begin{array}{l} \\ \div 2 \end{array} \quad \Rightarrow \quad \begin{array}{l} (1) \\ (2) \end{array} \left| \begin{array}{r} \lambda_1 + 3\lambda_2 = 5 \\ \lambda_1 - \lambda_2 = 1 \end{array} \right|$$

$(1) - (2) \quad \Rightarrow \quad 4\lambda_2 = 4 \quad \Rightarrow \quad \lambda_2 = 1$ and from (1) we find: $\lambda_1 = 2$

$\lambda_1 = 2$ and $\lambda_2 = 1$ satisfy the z component: $3 \cdot 2 + 1 = 7$ $\surd$ (1 point)

So: $\begin{pmatrix}2\\4\\6\end{pmatrix} + \begin{pmatrix}3\\-2\\1\end{pmatrix} = \begin{pmatrix}5\\2\\7\end{pmatrix}$ (1 point)

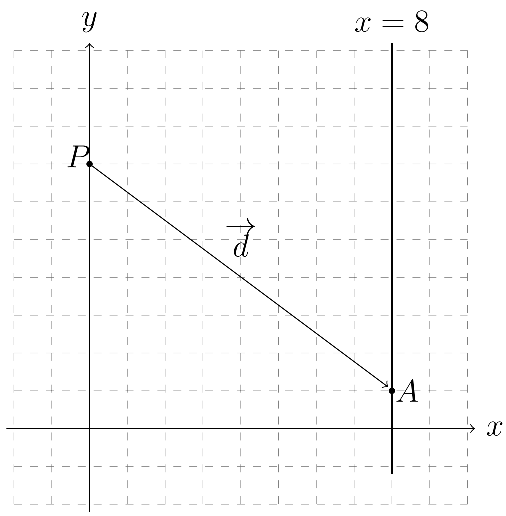
b) (2 points) We set $\lambda_1 \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_2 \cdot \begin{pmatrix}3\\-2\\1\end{pmatrix} = \begin{pmatrix}10\\-4\\7\end{pmatrix}$. Then we have: (1 point)

$$\begin{array}{l} (1) \\ (2) \end{array} \left| \begin{array}{r} \lambda_1 + 3\lambda_2 = 10 \\ 2\lambda_1 - 2\lambda_2 = -4 \end{array} \right| \begin{array}{l} \\ \div 2 \end{array} \quad \Rightarrow \quad \begin{array}{l} (1) \\ (2) \end{array} \left| \begin{array}{r} \lambda_1 + 3\lambda_2 = 10 \\ \lambda_1 - \lambda_2 = -2 \end{array} \right|$$

$(1) - (2) \quad \Rightarrow \quad 4\lambda_2 = 12 \quad \Rightarrow \quad \lambda_2 = 3$ and from (1) we find: $\lambda_1 = 1$

$\lambda_1 = 1$ and $\lambda_2 = 3$ do not satisfy the z component: $3 \cdot 1 + 3 = 6 \neq 7$. Hence $\overrightarrow{b}$ is not a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. (1 point)

Exercise 4 (3 points)



From the graph we find the condition $d = |\overrightarrow{PA}|$. (1 point)

$$\vec{d} = \overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = \begin{pmatrix} 8 \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ y-7 \end{pmatrix}$$

$$\Rightarrow \quad d = \sqrt{64 + (y-7)^2}$$ (1 point)

$$\begin{aligned} \sqrt{64 + (y-7)^2} &= 10 \qquad | \, ()^2 \\ 64 + (y-7)^2 &= 100 \qquad | -64 \\ (y-7)^2 &= 36 \qquad | \sqrt{\ldots} \\ y - 7 &= \pm 6 \qquad | \end{aligned}$$

We get two solutions: $y_1 = 13$ and $y_2 = 1$. (1 point)

Exercise 5 (3 points)
In order to determine the direction of the angle bisectors of the lines with direction vectors $\vec{a}$ and $\vec{b}$ these need to have the same length.

The vectors $|\vec{b}| \cdot \vec{a}$ and $|\vec{a}| \cdot \vec{b}$ have the same length. (1 point)

The direction of the angle bisectors of the lines with direction vectors $\vec{a}$ and $\vec{b}$ are then given by the vectors $\overrightarrow{w_1} = |\vec{b}| \cdot \vec{a} + |\vec{a}| \cdot \vec{b}$ and $\overrightarrow{w_2} = |\vec{b}| \cdot \vec{a} - |\vec{a}| \cdot \vec{b}$ (1 point)

We have to show $\overrightarrow{w_1} \cdot \overrightarrow{w_2} = 0$ and compute

$$\overrightarrow{w_1} \cdot \overrightarrow{w_2} = \left(|\vec{b}| \cdot \vec{a} + |\vec{a}| \cdot \vec{b}\right) \cdot \left(|\vec{b}| \cdot \vec{a} - |\vec{a}| \cdot \vec{b}\right) = |\vec{b}|^2 \cdot \vec{a}^2 \underbrace{- |\vec{a}||\vec{b}| \cdot \vec{a}\vec{b} + |\vec{a}||\vec{b}| \cdot \vec{b}\vec{a}}_{0 \text{ , da } \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}} - |\vec{a}|^2 \cdot \vec{b}^2$$

$$= |\vec{b}|^2 \cdot \vec{a}^2 - |\vec{a}|^2 \cdot \vec{b}^2 = |\vec{b}|^2 \cdot |\vec{a}|^2 - |\vec{a}|^2 \cdot |\vec{b}|^2 = 0 \qquad \text{QED}$$ (1 Punkt)