Exercise Sheet 2 - Solution

Exercise 1

a)

$$(1)\quad 2\cdot\vec{a} = 2\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix} \qquad (2)\quad 3\cdot\vec{b} = 3\cdot\begin{pmatrix}2\\-1\\-1\end{pmatrix} = \begin{pmatrix}6\\-3\\-3\end{pmatrix}$$

$$(3)\quad 4\cdot\vec{a}+3\cdot\vec{b} = 4\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} + 3\begin{pmatrix}2\\-1\\-1\end{pmatrix} = \begin{pmatrix}4\\4\\4\end{pmatrix} + \begin{pmatrix}6\\-3\\-3\end{pmatrix} = \begin{pmatrix}10\\1\\1\end{pmatrix}$$

$$(4)\quad 3\cdot\vec{a}-2\cdot\vec{b} = 3\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} - 2\begin{pmatrix}2\\-1\\-1\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix} - \begin{pmatrix}4\\-2\\-2\end{pmatrix} = \begin{pmatrix}-1\\5\\5\end{pmatrix}$$

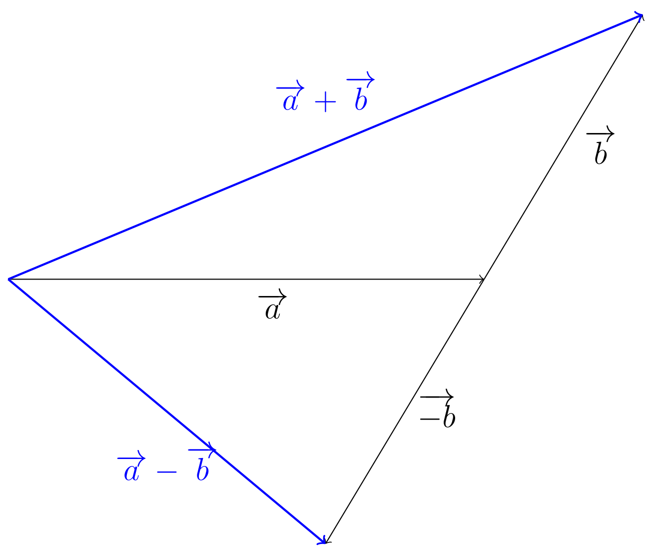
$$(5)\quad |\vec{a}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$(6)\quad |\vec{b}| = \sqrt{2^2+(-1)^2+(-1)^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$(7)\quad \vec{a}\cdot\vec{b} = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-1\end{pmatrix} = 2-1-1 = 0$$

$$(8)\quad \vec{a}\times\vec{b} = \begin{pmatrix}1\\1\\1\end{pmatrix}\times\begin{pmatrix}2\\-1\\-1\end{pmatrix} = \begin{pmatrix}-1-(-1)\\2-(-1)\\-1-2\end{pmatrix} = \begin{pmatrix}0\\3\\-3\end{pmatrix}$$

b) Hint: $\vec{a}-\vec{b} = \vec{a}+(-\vec{b})$



Exercise 2 (6 points)

a) (1 point) The parametric equation of the plane ADE is given by $\vec{r} = \overrightarrow{OA} + u \cdot \overrightarrow{AD} + v \cdot \overrightarrow{AE}$. We get:

$$\overrightarrow{OA} + \overrightarrow{AD} = \overrightarrow{OD} \quad \Rightarrow \quad \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ 8 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix} \parallel \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OA} + \overrightarrow{AE} = \overrightarrow{OE} \quad \Rightarrow \quad \overrightarrow{AE} = \overrightarrow{OE} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 8 \end{pmatrix} \parallel \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$$\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + u \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + v \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

b) (2 points) The coordinate equation is determined by the system of equations

$$\left| \begin{array}{c} x = 1 + u + 2v \\ y = 2 + u + v \\ z = 4v \end{array} \right|$$

The variable u can be eliminated from the $x-$ and $y-$component:
$x - y = -1 + v$ (1 point)

$$\left| \begin{array}{c} 4x - 4y = -4 + 4v \\ z = 4v \end{array} \right|$$

Thus: $4x - 4y - z = -4$ (1 point)

Hint: Another approach is to compute $\vec{n} \cdot (\vec{r} - \vec{r}_P)$ (Hesse normal form).

c) (1 point)

$$\begin{aligned}
& \overrightarrow{OM} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CE} \quad \text{with} \quad \overrightarrow{OC} \quad \text{and} \quad \overrightarrow{CE} \quad \text{as} \\
\Rightarrow \quad & \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{AD} \\
& = \overrightarrow{OA} + (\overrightarrow{OB} - \overrightarrow{OA}) + (\overrightarrow{OD} - \overrightarrow{OA}) \\
& = \overrightarrow{OA} + \overrightarrow{OB} - \overrightarrow{OA} + \overrightarrow{OD} - \overrightarrow{OA} = -\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OD} \\
\Rightarrow \quad & \overrightarrow{CE} = \overrightarrow{OE} - \overrightarrow{OC} = \overrightarrow{OE} + \overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OD}
\end{aligned}$$

From this we compute $\overrightarrow{OM}$:

$$\begin{aligned}\overrightarrow{OM} =& \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CE} \\ =& -\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OD} + \frac{1}{2}\Big(\overrightarrow{OE} + \overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OD}\Big) \\ =& -\frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OD} + \frac{1}{2}\overrightarrow{OE} \\ =& \frac{1}{2}\Big(-\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OD} + \overrightarrow{OE}\Big)\end{aligned}$$

Plugging in numbers we get

$$\overrightarrow{0M} = \frac{1}{2}\left[-\begin{pmatrix}1\\2\\0\end{pmatrix} + \begin{pmatrix}3\\0\\0\end{pmatrix} + \begin{pmatrix}7\\8\\0\end{pmatrix} + \begin{pmatrix}5\\4\\8\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}14\\10\\8\end{pmatrix} = \begin{pmatrix}7\\5\\4\end{pmatrix}$$

d) (1 point)

$$\overrightarrow{BM} = \overrightarrow{OM} - \overrightarrow{OB} = \begin{pmatrix}7\\5\\4\end{pmatrix} - \begin{pmatrix}3\\0\\0\end{pmatrix} = \begin{pmatrix}4\\5\\4\end{pmatrix}$$

$$\vec{r} = \overrightarrow{OB} + t \cdot \overrightarrow{BM} = \begin{pmatrix}3\\0\\0\end{pmatrix} + t \cdot \begin{pmatrix}4\\5\\4\end{pmatrix}$$

e) (1 point) From the parametric equation $\vec{r} = \overrightarrow{OB} + t \cdot \overrightarrow{BM}$ we get $x = 3 + 4t$, $y = 5t$ and $z = 4t$. Plugging this into the coordinate euqation of the plane ADE, $4x - 4y - z = -4$, we get

$$\begin{aligned}4(3 + 4t) - 4(5t) - 4t =& -4 \\ 12 + 16t - 20t - 4t =& -4 \\ 12 - 8t =& -4 \\ 16 =& 8t \\ t =& 2\end{aligned}$$

This yields the point of intersection (11/10/8).

Exercise 3 (4 points)

a) (1 point) The area computes according to $F = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AE}|$.

$$F = \frac{1}{2} \cdot \left| \begin{pmatrix}2\\-2\\0\end{pmatrix} \times \begin{pmatrix}4\\2\\8\end{pmatrix} \right| = \frac{1}{2} \cdot \left| \begin{pmatrix}-16\\-16\\12\end{pmatrix} \right| = \frac{1}{2} \cdot \left| 4 \begin{pmatrix}-4\\-4\\3\end{pmatrix} \right| = 2 \cdot \sqrt{41}$$

b) (1 point) The angle φ can be computed applying the scalar product.

$$\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}=\sqrt{2}\cdot\sqrt{21}\cdot\cos(\varphi) \quad\Leftrightarrow\quad 1=\sqrt{42}\cdot\cos(\varphi) \quad\Leftrightarrow\quad \cos(\varphi)=\frac{1}{\sqrt{42}}$$

We get $\varphi = 81.1^o$.

c) (2 points) It holds $\overrightarrow{RP}\cdot\overrightarrow{RQ}=0$.

$$\overrightarrow{RP}=\overrightarrow{OP}-\overrightarrow{OR}=\begin{pmatrix}3-x\\2\\-2\end{pmatrix} \qquad \overrightarrow{RQ}=\overrightarrow{OQ}-\overrightarrow{OR}=\begin{pmatrix}4-x\\1\\2\end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix}3-x\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}4-x\\1\\2\end{pmatrix}&=0\\(3-x)(4-x)+2-4&=0\\12-3x-4x+x^2-2&=0\\x^2-7x+10&=0\\(x-5)(x-2)&=0\end{aligned}$$

We find the points $R_1(5/0/0)$ and $R_2(2/0/0)$.

Exercise 4 (3 points)

a) (2 points) We find a simple counterexample. Assuming $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\vec{a}\times\left(\vec{b}\times\vec{c}\right)$ were true, we plug in $\vec{c}=\vec{b}$. Then we get $\left(\vec{a}\times\vec{b}\right)\times\vec{b}\neq\vec{a}\times\left(\vec{b}\times\vec{b}\right)$, (1 point)
since

- $\left(\vec{a}\times\vec{b}\right)\times\vec{b}$ is a vector in the plane determined by $\vec{a}$ and $\vec{b}$
- $\vec{a}\times\left(\vec{b}\times\vec{b}\right)=\vec{0}$ since $\vec{b}\times\vec{b}=\vec{0}$ and thus $\vec{a}\times\vec{0}=\vec{0}$! (1 point)

b) (1 point) Reformulating we get:

$$\left(\vec{a}\times\vec{b}\right)\times\vec{a}=-\vec{a}\times\left(\vec{a}\times\vec{b}\right)=-\vec{a}\times\left(-\vec{b}\times\vec{a}\right)=\vec{a}\times\left(\vec{b}\times\vec{a}\right) \qquad \checkmark$$

Exercise 5 (2 points)
To prove this idendity we use the following rules.

- $\cos(3x) = \cos(2x + x) = \cos(2x)\cos(x) - \sin(2x)\sin(x)$
- $\cos(2x) = 2\cos^2(x) - 1$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\sin^2(x) + \cos^2(x) = 1$

$$
\begin{aligned}
\frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x) =& \frac{3}{4}\cos(x) + \frac{1}{4}\cos(2x + x) \\
=& \frac{3}{4}\cos(x) + \frac{1}{4}\Bigg[\cos(2x)\cos(x) - \sin(2x)\sin(x)\Bigg] \\
=& \frac{3}{4}\cos(x) + \frac{1}{4}\Bigg[\Bigg(2\cos^2(x) - 1\Bigg)\cos(x) - 2\sin(x)\cos(x)\sin(x)\Bigg] \\
=& \frac{3}{4}\cos(x) + \frac{1}{4}\Bigg[2\cos^3(x) - \cos(x) - 2\sin^2(x)\cos(x)\Bigg] \\
=& \frac{3}{4}\cos(x) + \frac{1}{4}\Bigg[2\cos^3(x) - \cos(x) - 2\Bigg(1 - \cos^2(x)\Bigg)\cos(x)\Bigg] \\
=& \frac{3}{4}\cos(x) + \frac{1}{4}\Bigg[2\cos^3(x) - \cos(x) - 2\cos(x) + 2\cos^3(x)\Bigg] \\
=& \frac{3}{4}\cos(x) + \frac{1}{4}\Bigg[4\cos^3(x) - 3\cos(x)\Bigg] \\
=& \frac{3}{4}\cos(x) + \cos^3(x) - \frac{3}{4}\cos(x) \overset{QED}{=} \cos^3(x) \qquad \text{(2 points)}
\end{aligned}
$$