Exercise Sheet 1 - Solution

Exercise 1 (5 points)

a) (1 point)

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot (-1) \cdot 8}}{2 \cdot (-1)} = \frac{-2 \pm \sqrt{4 + 32}}{-2} = \frac{-2 \pm \sqrt{36}}{-2} = \frac{-2 \pm 6}{-2}$$
$$\Rightarrow \quad x_1 = -2 \ , \ x_2 = 4$$

b) (1 point) The vertex of the parabola lies exactly between the roots (for symmetry reasons). We find $x_S = \frac{-2+4}{2} = 1$ and $y_S = 9$.

c) (2 points) We use the approach $y = mx + b$. We have two unknowns m and b and therefore need two equations.
The point (2/8) is on the line and on the parabola.
So we have: $8 = 2m + b$. (1 point)
In addition, the line and the parabola touch at the point (2/8), so the equation $-x^2 + 2x + 8 = mx + b$ has only one solution. Rearranged to $x^2 + (m-2)x + (b-8) = 0$ and solved for x we get:

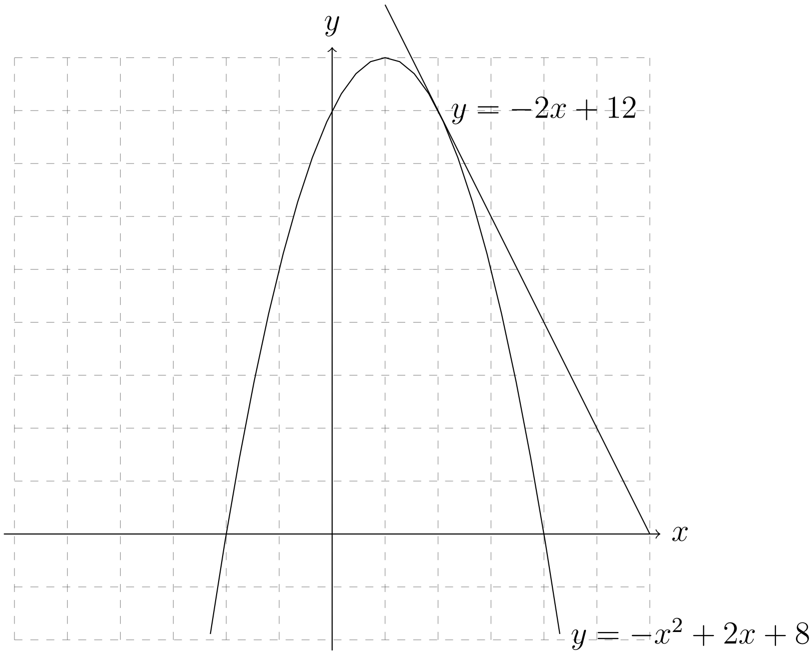
$$x_{1,2} = \frac{-(m-2) \pm \sqrt{(m-2)^2 - 4 \cdot 1 \cdot (b-8)}}{2 \cdot 1}$$

There is one solution for x if the root disappears. This is our second equation. Setting

$$x_1 = x_2 = \frac{-(m-2)}{2} = 2 \qquad \text{yields directly} \qquad m = -2$$

$8 = 2m + b$ yields $b = 12$.
The tangent equation is $y = -2x + 12$. (1 point)

d) (1 point) The graphs are illustrated below.



Exercise 2 (6 points)

(a) (2 points) The domain of the equation is $D = \mathbb{R}\backslash\{0\}$

$$\begin{aligned}
3x^2 + 6x &= \tfrac{4}{3} + \tfrac{8}{3x} && \mid \\
3x(x+2) &= \tfrac{4}{3}(1 + \tfrac{2}{x}) && \mid \cdot(3x) \\
9x^2(x+2) &= 4(x+2) && \mid -4(x+2) \\
9x^2(x+2) - 4(x+2) &= 0 && \mid \\
(x+2)\cdot(9x^2-4) &= 0 && \mid
\end{aligned}$$

(1 point)

This leads to the solutions $x_1 = -2$ and $x_{2,3} = \pm\frac{2}{3}$.

Thus $L = \{-2, \frac{2}{3}, -\frac{2}{3}\}$. (1 point)

(b) (2 points) By substituting $5^{2x} := z$ the equation simplifies to

$$z^2 - 7z + 10 = 0.$$

Factorizing $(z-2)(z-5) = 0$ yields the solutions $z_1 = 2$ and $z_2 = 5$. (1 point)

From $z_1 = 2$ we get $5^{2x} = 2 \quad \Rightarrow \quad x_1 = \frac{\log(2)}{2\cdot\log(5)} \approx 0.215$

From $z_2 = 5$ we get $5^{2x} = 5 \quad \Rightarrow \quad x_2 = \frac{1}{2}$ (1 point)

(c) (2 points) Guessing leads the first solution $x_1 = 1$ of the equation. Polynomial division yields

$$\begin{array}{l} (x^3 - 2x^2 - 5x + 6) : (x-1) = x^2 - x - 6 \\ \underline{-(x^3 - x^2)} \\ \quad - x^2 - 5x + 6 \\ \quad \underline{-(-x^2 + x)} \\ \qquad - 6x + 6 \\ \qquad \underline{-(-6x + 6)} \\ \qquad\quad 0 \end{array}$$

(1 point)

Furthermore we can factorize $x^2 - x - 6 = (x+2)(x-3)$ and easily solve the equation $(x-1)(x+2)(x-3) = 0$.

We find $L = \{1, -2, 3\}$. (1 point)

Exercise 3 (4 points)

(a) (2 points)

$$\begin{aligned} 3x + 2y - z &= 3 && (1) \\ 2x - y + z &= 4 && (2) \\ 4x + 2y - 3z &= -4 && (3) \end{aligned}$$

We eliminate z by combining (1) & (2) and (1) & (3)

$$\begin{aligned} 3x + 2y - z &= 3 && (1) \\ 2x - y + z &= 4 && (2) \\ \Rightarrow \quad 5x + y &= 7 && (4) \end{aligned} \qquad \begin{aligned} 9x + 6y - 3z &= 9 && (1) \cdot 3 \\ 4x + 2y - 3z &= -4 && (3) \\ \Rightarrow \quad 5x + 4y &= 13 && (5) \end{aligned}$$

(1 point)

Now x and y can be determined by combining (4) & (5)

$$\begin{aligned} 5x + y &= 7 && (4) \\ 5x + 4y &= 13 && (5) \\ \Rightarrow \quad 3y &= 6 && \Rightarrow \quad y = 2 \end{aligned}$$

From (4) we get $x = 1$ and from (1) we get $z = 4$. (1 point)

(b) (2 points) From the system of equations

$$\begin{aligned} 6x + y - 2z &= 0 && (1) \\ 2x - y + z &= 4 && (2) \\ 4x + 2y - 3z &= -4 && (3) \end{aligned}$$

the variable y can be eliminated.

$$\begin{aligned} (1)+(2) &\Rightarrow \quad 8x - z = 4 \qquad (4) \\ (2)\cdot 2+(3) &\Rightarrow \quad 8x - z = 4 \qquad (5) \end{aligned}$$

(1 point)

Obviously equations (4) and (5) are identical. Plugging $x = t$ in (4) we get $z = 8t - 4$. The variable y is determined by equation (1). We find $y = 10t - 8$. The solution of the system is the equation of a straight line

$$\vec{r} = \begin{pmatrix} t \\ 10t-8 \\ 8t-4 \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \\ -4 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 10 \\ 8 \end{pmatrix}$$

(1 point)

Exercise 4 (6.5 points)

a) ($\frac{1}{2}$ point for each subtask)

1) $2\ln(a) - \ln(3c) = \ln(a^2) - \ln(3c) = \ln\left(\frac{a^2}{3c}\right)$

2) $3\ln(y) + \frac{1}{3}\ln(y) = \frac{10}{3}\ln(y) = \ln\left(y^{\frac{10}{3}}\right) = \ln\left(\sqrt[3]{y^{10}}\right)$

3) $4\ln(m) - \frac{1}{6}\ln(n) = \ln\left(m^4\right) - \ln\left(n^{\frac{1}{6}}\right) = \ln\left(m^4\right) - \ln\left(\sqrt[6]{n}\right) = \ln\left(\frac{m^4}{\sqrt[6]{n}}\right)$

b) (1 point)

$$e^{5a} \cdot e^{x^3} \cdot \left(e^{1-t}\right)^t = e^{5a} \cdot e^{x^3} \cdot e^{(1-t)\cdot t} = e^{5a+x^3+(1-t)\cdot t}$$

c) ($\frac{1}{2}$ point for each subtask)

1) $e^{-\ln(2x)} = \left(e^{\ln(2x)}\right)^{-1} = (2x)^{-1} = \frac{1}{2x}$

2) $e^{3\ln(w)} = \left(e^{\ln(w)}\right)^3 = w^3$

3) $e^{\frac{1}{2}\ln(2y)} = \left(e^{\ln(2y)}\right)^{\frac{1}{2}} = (2y)^{\frac{1}{2}} = \sqrt{2y}$

4) $e^{-\frac{1}{3}\ln(d)} = \left(e^{\ln(d)}\right)^{-\frac{1}{3}} = (d)^{-\frac{1}{3}} = \left(d^{-1}\right)^{\frac{1}{3}} = \left(\frac{1}{d}\right)^{\frac{1}{3}} = \frac{1}{(d)^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{d}}$

d) (Subtasks 1) and 2) $\frac{1}{4}$ point each, subtask 3) $\frac{1}{2}$ point)

1) $\ln(a^9 b) = \ln(a^9) + \ln(b) = 9\ln(a) + \ln(b)$

2) $\ln(\mathrm{e}^3 + 5) \quad \Rightarrow$ cannot be simplified since $\ln(a+b) \neq \ln(a) + \ln(b)$

3) $\ln\left(\dfrac{\mathrm{e}^{-x}}{b}\right) = \ln(\mathrm{e}^{-x}) - \ln(b) = -x\ln(\mathrm{e}) - \ln(b) = -x - \ln(b)$

e) (1 point)

$$\begin{aligned} 2^x - 2^{x-1} &= 2 && \Big|\cdot 2 \\ 2\cdot 2^x - 2^x &= 4 && \Big| \\ 2^x &= 4 && \Big| \\ x &= 2 && \Big| \end{aligned}$$