# Serie 3

Please hand in your solutions until march 19th, 2012.

**Exercise 1**. A $\mathbb{C}^n$-valued random variable can be written as $Z = X + iY$, where $X$ and $Y$ are $\mathbb{R}^n$-valued random variables. We assume that the $\mathbb{R}^{2n}$-valued random variable $(X, Y)$ is Gaussian with mean 0. We define the covariance matrix $\Sigma$ and the relation matrix $D$ of $Z$ by $\Sigma = \mathbb{E}Z\overline{Z}^T$ and $D = \mathbb{E}ZZ^T$ (i.e. $\Sigma_{i,j} = \mathbb{E}Z_i\overline{Z}_j$). Show:

a) The covariance matrix of $(X, Y)$ is given by

$$\frac{1}{2}\begin{pmatrix} \mathrm{Re}(\Sigma + D) & \mathrm{Im}(D - \Sigma) \\ \mathrm{Im}(\Sigma + D) & \mathrm{Re}(\Sigma - D) \end{pmatrix}.$$

b) Let $A \in \mathbb{C}^{n\times n}$. Then $AZ$ has covariance matrix $A\Sigma\overline{A}^T$ and relation matrix $ADA^T$.

c) Let $Z \in \mathbb{C}^n$ such that all involved real random variables are independent and standard Gaussian ($Z$ is called a n-dimensional standard complex Gaussian random variable). Determine the matrices $\Sigma$ and $D$. Show that $Z$ and $UZ$ have the same distribution if and only if $U \in U(n)$. Show also that $Z/\|Z\|$ and $UZ/\|Z\|$ have the same distribution if $U \in U(n)$.

**Exercise 2** (Complex Gaussian random variables). A random variable $Z$ described in Exercise 1 is called complex Gaussian if it has the additional property $D = 0$.

a) Let $Z = X + iY \in \mathbb{C}$ be standard complex Gaussian. Let $u, v \in \mathbb{C}$ be complex numbers. Show:

$$\mathbb{E}e^{uZ+v\overline{Z}} = e^{2uv}.$$

b) A random variable $Z'$ is complex Gaussian if and only if it can be written as $Z' = AZ + b$, where $Z$ is n-dimensional standard complex Gaussian, $A \in \mathbb{C}^{n\times n}$, and $b \in \mathbb{C}^n$. Conclude that $\mathbb{E}e^{u^TZ'+v^T\overline{Z'}} = e^{u^T\Sigma v}$, $u, v \in \mathbb{C}^n$, if $Z'$ has mean 0.

**Exercise 3** (Random variables on $S^1_{\mathbb{C}}$). Let $U_1, \dots, U_n$ be independent random variables uniformly distributed on $[0, 2\pi]$ and let $Z_j = \exp(iU_j)$, $j = 1, \dots, n$. Let $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ and $a = (b_1, \dots, b_n) \in \mathbb{N}^n$. Show:

$$\mathbb{E}\prod_{j=1}^{n} Z_j^{a_j}\overline{Z}_j^{b_j} = \delta_{ab}.$$

Show also that the distribution of $Z$ is uniquely determined by its moments.

**Exercise 4** (Weyl's integration formula). Let $U \in U(n)$ be Haar distributed and let $k \geq n$ be an integer. Let $Z(U) \in \mathbb{C}^n$ be the vector of the eigenvalues of $U^k$, where we choose uniformly one of the $n!$ possible orders. Show:

$$\begin{aligned}
&\mathbb{E} \prod_{j=1}^{n} Z(U)_j^{a_j} \overline{Z(U)}_j^{b_j} \\
&\quad = \frac{1}{(2\pi)^n n!} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{\sum_{j=1}^{n} ik\theta_j (a_j - b_j)} \prod_{1 \leq k < l \leq n} |e^{i\theta_k} - e^{i\theta_l}|^2 d\theta_1 \dots \theta_n = \delta_{ab}.
\end{aligned}$$

(Hint: Use Weyl's integration formula and the identity $|e^{i\theta_k} - e^{i\theta_l}|^2 = (e^{i\theta_k} - e^{i\theta_l})(e^{-i\theta_k} - e^{-i\theta_l})$.)
Conclude that the eigenvalues of $U^k$ are uniformly and independent distributed on $S^1_{\mathbb{C}}$.