

Supplementary Material for ‘Errata and addendum to “Covariance Tapering for Interpolation of Large Spatial Datasets” published in the *Journal of Computational and Graphical Statistics*, 15, 502–523’

Reinhard FURRER, Marc G. GENTON and Douglas NYCHKA

We follow the notation of FGN as closely as possible. The proof of Proposition 1 is as follows.

*Proof.* Without loss of generality, we suppose that  $\alpha = 1$  and so  $f_{1,\nu}(\|\boldsymbol{\omega}\|) = M_1/(1+\|\boldsymbol{\omega}\|^2)^{\nu+d/2}$ ,  $\nu > 0$ . We need to prove that the limit

$$\lim_{\|\boldsymbol{\omega}\| \rightarrow \infty} \frac{\int_{\mathbb{R}^d} f_{1,\nu}(\|\mathbf{x}\|) f_{\theta}(\|\mathbf{x} - \boldsymbol{\omega}\|) \, d\mathbf{x}}{f_{1,\nu}(\|\boldsymbol{\omega}\|)} \quad (1)$$

exists and is not zero. As the spectral densities are radially symmetric, we choose an arbitrary direction for  $\boldsymbol{\omega}$  and we set  $\|\mathbf{x}\| = r\|\mathbf{u}\|$  and  $\|\boldsymbol{\omega}\| = \rho\|\mathbf{v}\|$ , with  $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ . The convolution reduces to

$$\int_{\mathbb{R}^d} f_{1,\nu}(\|\mathbf{x}\|) f_{\theta}(\|\mathbf{x} - \boldsymbol{\omega}\|) \, d\mathbf{x} = \int_{\partial B_d} \int_0^{\infty} f_{1,\nu}(r) f_{\theta}(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} \, dr \, dU(\mathbf{u}), \quad (2)$$

where  $\partial B_d$  is the surface of the unit sphere in  $\mathbb{R}^d$  and  $U$  is the uniform probability measure on  $\partial B_d$ . In what follows let  $\rho > 0$ . Recall that  $f_{1,\nu}(\rho)$  is monotonically decreasing in  $\rho$  and for all  $0 \leq r \leq b < \rho$ ,  $\rho - b \leq \|r\mathbf{u} - \rho\mathbf{v}\| \leq \rho + b$ .

We separate the inner integral of (2) into five integrals whose bounds are given by  $[0, \sqrt{\rho}]$ ,  $[\sqrt{\rho}, \rho/2]$ ,  $[\rho/2, \rho - \sqrt{\rho}]$ ,  $[\rho - \sqrt{\rho}, \rho + \sqrt{\rho}]$ ,  $[\rho + \sqrt{\rho}, \infty)$ . We consider now these five cases for sufficiently large  $\rho$ .

Case  $r \in [0, \sqrt{\rho}]$ : We have

$$\begin{aligned} \int_0^{\sqrt{\rho}} f_{1,\nu}(r) f_{\theta}(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} \, dr &\leq \sup_{s \in [0, \sqrt{\rho}]} f_{\theta}(\|s\mathbf{u} - \rho\mathbf{v}\|) \int_0^{\sqrt{\rho}} f_{1,\nu}(r) r^{d-1} \, dr \\ &\leq \sup_{s \in [\rho \pm \sqrt{\rho}]} f_{\theta}(s) \int_0^{\sqrt{\rho}} f_{1,\nu}(r) r^{d-1} \, dr \\ \int_0^{\sqrt{\rho}} f_{1,\nu}(r) f_{\theta}(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} \, dr &\geq \inf_{s \in [0, \sqrt{\rho}]} f_{\theta}(\|s\mathbf{u} - \rho\mathbf{v}\|) \int_0^{\sqrt{\rho}} f_{1,\nu}(r) r^{d-1} \, dr \\ &\geq \inf_{s \in [\rho \pm \sqrt{\rho}]} f_{\theta}(s) \int_0^{\sqrt{\rho}} f_{1,\nu}(r) r^{d-1} \, dr. \end{aligned}$$

Thus

$$\begin{aligned} \lim_{\rho \rightarrow \infty} f_{1,\nu}(\rho)^{-1} \int_{\partial B_d} \int_0^{\sqrt{\rho}} f_{1,\nu}(r) f_{\theta}(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} \, dr \, dU(\mathbf{u}) &\quad (3) \\ &\leq \lim_{\rho \rightarrow \infty} \frac{\sup_{s \in [\rho \pm \sqrt{\rho}]} f_{\theta}(s)}{f_{1,\nu}(\rho)} \int_{\partial B_d} \int_0^{\sqrt{\rho}} f_{1,\nu}(r) r^{d-1} \, dr \, dU(\mathbf{u}) \\ &= \lim_{\rho \rightarrow \infty} \frac{\sup_{s \in [\rho \pm \sqrt{\rho}]} f_{\theta}(s)}{f_{1,\nu}(\rho)} \times \lim_{\rho \rightarrow \infty} \int_{\partial B_d} \int_0^{\sqrt{\rho}} f_{1,\nu}(r) r^{d-1} \, dr \, dU(\mathbf{u}) \\ &\leq \frac{1}{M_1} \lim_{\rho \rightarrow \infty} \sup_{s \geq \rho - \sqrt{\rho}} f_{\theta}(s) (1 + \rho^2)^{\nu+d/2+\epsilon} \end{aligned}$$

and by a similar argument, (3) is bounded below by

$$\lim_{\rho \rightarrow \infty} \frac{\inf_{s \in [\rho \pm \sqrt{\rho}]} f_{\theta}(s)}{f_{1,\nu}(\rho)} \leq \frac{1}{M_1} \lim_{\rho \rightarrow \infty} \inf_{s \geq \rho - \sqrt{\rho}} f_{\theta}(s) (1 + \rho^2)^{\nu+d/2}.$$

By the Modified Taper Condition, the limit of  $f_\theta(\rho)\rho^{2k}$  exists and this implies that (3) is equal to zero if  $k > \nu + d/2$  and equal to  $M/M_1$  if  $k = \nu + d/2$ .

Case  $r \in [\sqrt{\rho}, \rho/2]$ : Given that  $\|r\mathbf{u} - \rho\mathbf{v}\| \geq \rho/2$ , we have

$$\begin{aligned} & \lim_{\rho \rightarrow \infty} f_{1,\nu}(\rho)^{-1} \int_{\partial B_d} \int_{\sqrt{\rho}}^{\rho/2} f_{1,\nu}(r) f_\theta(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} dr dU(\mathbf{u}) \\ & \leq \lim_{\rho \rightarrow \infty} f_{1,\nu}(\rho)^{-1} \int_{\partial B_d} \sup_{s \in [\sqrt{\rho}, \rho/2]} f_\theta(\|s\mathbf{u} - \rho\mathbf{v}\|) \int_{\sqrt{\rho}}^{\rho/2} f_{1,\nu}(r) r^{d-1} dr dU(\mathbf{u}) \\ & \leq \lim_{\rho \rightarrow \infty} \frac{\sup_{s \geq \rho/2} f_\theta(s)}{f_{1,\nu}(\rho)} \times \lim_{\rho \rightarrow \infty} \int_{\partial B_d} \int_{\sqrt{\rho}}^{\infty} f_{1,\nu}(r) r^{d-1} dr dU(\mathbf{u}). \end{aligned}$$

The first limit is finite by the Modified Taper Condition and the second vanishes for  $\rho$  tending to infinity.

Case  $r \in [\rho/2, \rho - \sqrt{\rho}]$ : We have

$$\begin{aligned} & \lim_{\rho \rightarrow \infty} f_{1,\nu}(\rho)^{-1} \int_{\partial B_d} \int_{\rho/2}^{\rho - \sqrt{\rho}} f_{1,\nu}(r) f_\theta(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} dr dU(\mathbf{u}) \\ & \leq \lim_{\rho \rightarrow \infty} f_{1,\nu}(\rho)^{-1} \int_{\partial B_d} f_{1,\nu}(\rho/2) \int_{\rho/2}^{\rho - \sqrt{\rho}} f_\theta(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} dr dU(\mathbf{u}) \\ & \leq \lim_{\rho \rightarrow \infty} \frac{f_{1,\nu}(\rho/2)}{f_{1,\nu}(\rho)} \times \lim_{\rho \rightarrow \infty} \int_{\partial B_d} \int_{\sqrt{\rho}}^{\infty} f_\theta(r) r^{d-1} dr dU(\mathbf{u}) \end{aligned}$$

The first limit is finite by the Modified Taper Condition and the second vanishes for  $\rho$  tending to infinity.

Case  $r \in [\rho - \sqrt{\rho}, \rho + \sqrt{\rho}]$ : Again using the monotonicity of  $f_{1,\nu}$ , we have

$$\begin{aligned} & \lim_{\rho \rightarrow \infty} f_{1,\nu}(\rho)^{-1} \int_{\partial B_d} \int_{\rho - \sqrt{\rho}}^{\rho + \sqrt{\rho}} f_{1,\nu}(r) f_\theta(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} dr dU(\mathbf{u}) \tag{4} \\ & \leq \lim_{\rho \rightarrow \infty} \frac{f_{1,\nu}(\rho - \sqrt{\rho})}{f_{1,\nu}(\rho)} \int_{\partial B_d} \int_{\rho - \sqrt{\rho}}^{\rho + \sqrt{\rho}} f_\theta(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} dr dU(\mathbf{u}) \\ & = \lim_{\rho \rightarrow \infty} \frac{f_{1,\nu}(\rho - \sqrt{\rho})}{f_{1,\nu}(\rho)} \times \lim_{\rho \rightarrow \infty} \int_{\partial B_d} \int_{\rho - \sqrt{\rho}}^{\rho + \sqrt{\rho}} f_\theta(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} dr dU(\mathbf{u}) = 1 \times 1, \end{aligned}$$

because  $f_\theta$  is a density in  $\mathbb{R}^d$ . Similarly, (4) has a lower bound

$$\begin{aligned} & \geq \lim_{\rho \rightarrow \infty} \frac{f_{1,\nu}(\rho + \sqrt{\rho})}{f_{1,\nu}(\rho)} \int_{\partial B_d} \int_{\rho - \sqrt{\rho}}^{\rho + \sqrt{\rho}} f_\theta(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} dr dU(\mathbf{u}) \\ & \geq 1 \times \lim_{\rho \rightarrow \infty} \int_{\partial B_d} \int_0^{\sqrt{\rho}} f_\theta(r) r^{d-1} dr dU(\mathbf{u}) = 1 \times 1. \end{aligned}$$

Case  $r \in [\rho + \sqrt{\rho}, \infty)$ : We have

$$\int_{\rho + \sqrt{\rho}}^{\infty} f_{1,\nu}(r) f_\theta(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} dr \leq f_{1,\nu}(\rho) \int_{\rho + \sqrt{\rho}}^{\infty} f_\theta(\|r\mathbf{u} - \rho\mathbf{v}\|) r^{d-1} dr.$$

As  $\rho$  tends to infinity, the integral will tend to zero.

Thus, collecting all five cases, equation (1) is equal to 1 if  $k > \nu + d/2$  and equal to  $M/M_1 + 1$  if  $k = \nu + d/2$ .  $\square$

*Remark:* In the proof of FGN, only three annuli were used, which resulted in limits of the form  $\lim_{\rho \rightarrow \infty} \rho^{2\nu+d}/(\rho/2)^{2\nu+d} = 2^{2\nu+d}$ . In order to avoid the fraction in the denominator, two additional annuli are required.