MOTIVIC DECOMPOSITION OF CELLULAR DELIGNE-MUMFORD STACKS

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1. Introduction

Let a relative cellular space be a smooth, complete variety over a field with an increasing sequence of closed subvarieties whose successive differences ("cells") are vector bundles over complete bases varieties. The cohomology of a relative cellular space decomposes into twisted sums of the cohomology of the bases, as shown by N. A. Karpenko [Kar, Corollary 6.11]. In fact, this decomposition is on the level of Chow motives.

We introduce a more general notion of cellularity for Deligne-Mumford stacks and generalise the decomposition of Karpenko. We prove that the $\mathbf{Q}$-motives of relative cellular Deligne-Mumford stacks decompose as a shifted, twisted sum of motives of the cell bases. The proof here does not rely on properties of Chow groups, as Karpenko's does. It relies instead on Voevodsky's vanishing theorem [Voe2, Corollary 4.2.6] which says that there are no nontrivial morphisms between the motives $M(X)$ and $M(Y)[i]$ for $i > 0$, if X and Y are smooth, proper schemes over k.

Smooth, proper Deligne-Mumford stacks with torus actions are often relative cellular (cf. Example 4.9), in which case the cells are unions of orbits and the cell bases are fixed point loci, as in the case of schemes [Bro, Theorem 3.5].

In this paper, we work over field k of characteristic zero. All our results is true over any perfect base field k admitting functorial resolution of singularities.

2. Notation and Terminology

Let R be a commutative ring. Let the tensor triangulated categories $\mathbf{DM}^{\text{eff}}_{\text{ét}}(k,R)$ and $\mathbf{DM}^{\text{eff}}(k,R)$ be constructed as in [MVW, Definition 9.2, 14.1]. There exists a canonical equivalence of tensor triangulated categories $\mathbf{DM}^{\text{eff}}(k,R) \cong \mathbf{DM}^{\text{eff}}_{\text{ét}}(k,R)$, for $\mathbf{Q} \subset R$ [MVW, Theorem 19.30].

The objects of $\mathbf{DM}^{\text{eff}}(k,\mathbf{Q})$ are complexes of presheaves with transfers. Let $SmCor(k)$ be the category of finite correspondences whose objects are smooth k-schemes X. The morphisms between X, $Y \in Sm/k$, are given by the free abelian group $Cor(X,Y)$ generated by integral closed subschemes $W \subset X \times Y$ which are finite and surjective on a connected component of X. For $X = \coprod_i X_i$, this implies $Cor(X \times Y) = \bigoplus_i Cor(X_i \times Y)$. Then the category, $PST(k,\mathbf{Q})$, of presheaves with transfers is the category of contravariant, additive functors $SmCor(k) \to \mathbf{Q}{-}Vect$. For each $X \in Sm/k$, the functor $\mathbf{Q}_{tr}(X) : SmCor(k) \to \mathbf{Q}{-}Vect$ is given by $Y \in Sm/k \mapsto Hom_{SmCor(k)}(Y,X) \otimes \mathbf{Q}$.

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A morphism $f : K' \to K$ between complexes of presheaves with transfers is invertible in $\mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$ if it is a quasi-isomorphism on (étale) stalks. If a morphism of complexes of presheaves with transfers has the form $\mathbb{Z}_{tr}(\mathbf{A}^1 \times X)[n] \to \mathbb{Z}_{tr}(X)[n]$ for $X \in Sm/k$ and $n \in \mathbb{Z}$, then it is invertible in $\mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$. The *motive* of a smooth scheme X over k, denoted $M(X)$, is the sheaf $\mathbf{Q}_{tr}(X)$ viewed as an object of $\mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$.

Since k is perfect and admits resolution of singularities, there exists a fully faithful functor $\iota : Ch^{\mathrm{eff}}_{\mathbf{Q}}(k) \to \mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$ [MVW, Proposition 20.1], where $Ch^{\mathrm{eff}}_{\mathbf{Q}}(k)$ is the category of effective rational *Chow motives*. A motive $M \in \mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$ is called a *Chow motive* if it is in the essential image of ι. It is called a *geometric motive* if it lies in the full subcategory $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$ (cf. [Voe2, Definition 2.1.1]).

Let F be a smooth and separated Deligne-Mumford stack (in the étale topology) over a field k. By [Ch, Definition 2.7], it has a motive $M(F) \in \mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$. It can be described as follows. Let $U \to F$ be an étale atlas. One can associate to F a simplicial scheme $U_\bullet$ in Sm/k, the category of smooth schemes over a field k, formed from the $(i+1)$-fold products $U_i = U \times_F \cdots \times_F U$. Relative diagonals and partial projections serve as the face and degeneracy maps. The simplicial scheme $U_\bullet$ also has an associated motive $M(U_\bullet) \in \mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$ such that

$$M(U_\bullet) := N(\mathbf{Q}_{tr}(U_\bullet)),$$

where $N(-)$ gives the normalized chain complex (cf. [GJ, p. 145]). The canonical morphism, $M(U_\bullet) \to M(F)$, is an isomorphism in $\mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$ [Ch, Corollary 2.14]. Since k admits functorial resolution of singularities, $M(F)$ is a geometric motive [Ch, Corollary 4.7].

3. Affine fibrations

The results on cellular stacks follow in part from the observation that the motive of the total space of an affine bundle is isomorphic to the motive of the base space. The proof proceeds first through the case of schemes.

Lemma 3.1 (Homotopy invariance)**.** *Let $p : Y \to X$ be a smooth morphism of smooth k-schemes such that the geometric fibers are affine spaces. Then $M(Y) \cong M(X)$ in $\mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$.*

Proof. Examining the restrictions of the morphism to each connected component, the smooth scheme X may be assumed to be irreducible. Let n be the relative dimension of p. The scheme X can be filtered by smooth, open subschemes,

$$\emptyset = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = X,$$

such that for all i, $W_i := V_i \setminus V_{i-1}$ is smooth, and there is a pullback square of the form,

$$\begin{array}{ccc} \mathbf{A}^n_{U_i} & \longrightarrow & U_i \\ {\scriptstyle \text{ét}}\downarrow & & \downarrow{\scriptstyle \text{ét}} \\ p^{-1}(W_i) & \longrightarrow & W_i \end{array}$$

such that the vertical morphisms are étale and surjective. Indeed, the hypotheses applied to the generic point of X show that p is generically an affine bundle, that is, there exists an étale morphism $u_1 : U_1 \to X$ such that $p^{-1}(U_1) \cong \mathbf{A}^n_{U_1}$. If u_1 is not surjective, this process may be continued inductively as follows.

Let $Z_1 := X \setminus V_1$ be the complement of the image V_1 of u_1. The field k is perfect, so Z_1 is generically smooth. It has a dense, smooth subscheme $W_2 \subset Z_1$ which is the image of an étale morphism $u_2 : U_2 \to Z_1$ trivializing the affine bundle, i.e., $p^*U_2 \cong \mathbf{A}^n_{U_2}$. Let $V_2 := U_1 \cup W_2$. Since $\operatorname{codim}_X V_2 < \operatorname{codim}_X V_1$, this process terminates in the promised filtration of X.

Induction on i shows that $M(V_i) \cong M(p^{-1}(V_i))$ for all i. The case $i = 1$ follows from the observation that $M(V_1) \cong M((U_1)_\bullet)$ and $M(p^{-1}(V_1)) \cong M((p^{-1}(U_1))_\bullet)$. In each simplicial degree, $(p^{-1}(U_1))_\bullet$ is isomorphic to $(\mathbf{A}^n_{U_1})_\bullet$. Hence the canonical morphism, $\mathbf{Q}_{tr}((p^{-1}(U_1))_\bullet) \to \mathbf{Q}_{tr}((U_1)_\bullet)$, induces an $\mathbf{A}^1$-weak equivalence in each simplicial degree, and an isomorphism $M(V_1) \cong M(p^{-1}(V_1))$.

For the general case, the Gysin triangle exists [Voe2], since k admits resolution of singularities. The required isomorphism then follows from the morphism of triangles,

$$\begin{array}{ccccc} M(p^{-1}(V_{i-1})) & \longrightarrow & M(p^{-1}(V_i)) & \longrightarrow & M(p^{-1}(W_i))(c_i)[2c_i] \\ \downarrow\wr & & \downarrow & & \downarrow\wr \\ M(V_{i-1}) & \longrightarrow & M(V_i) & \longrightarrow & M(W_i)(c_i)[2c_i] \end{array}$$

where $c_i := \operatorname{codim}_{V_i} W_i$. □

Remark 3.2. *The morphism $p : Y \to X$ of the previous lemma also induces an isomorphism in $\mathbf{DM}^{\mathrm{eff}}_{\acute{e}t}(k, \mathbb{Z})$. If moreover the fibers of p are affine spaces, then the isomorphism additionally holds in $\mathbf{DM}^{\mathrm{eff}}(k, \mathbb{Z})$.*

Corollary 3.3. *Let $p : F' \to F$ be a smooth, representable morphism of smooth Deligne-Mumford stacks such that the geometric fibers are affine spaces. Then $M(F') \cong M(F)$ in $\mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$.*

Proof. Let $U \to F$ (resp . $V := U \times_F F' \to F'$) be an atlas, and let $U_\bullet$ (resp. $V_\bullet$) be the associated Čech simplicial scheme. For each simplicial degree i, the smooth morphism $f_i : V_i \to U_i$ induced by $f' : V_\bullet \to U_\bullet$ has affine geometric fiber, so the morphism $\mathbf{Q}_{tr}(f) : \mathbf{Q}_{tr}(V_\bullet) \to \mathbf{Q}_{tr}(V_\bullet)$ is an $\mathbf{A}^1$-weak equivalence in each simplicial degree. This gives an isomorphism $M(f) : M(F') \cong M(F)$. □

4. Relative cellular Deligne-Mumford stacks and cellular decompositions of motives

Lemma 4.1. *Let $X, Y \in Sm/k$, such that X is proper. Then*

$$Hom_{\mathbf{DM}^{\mathrm{eff}}(k,\mathbf{Q})}(M(Y)(c)[2c], M(X)[1]) = 0.$$

Proof. When Y is proper, this is [Voe2, Corollary. 4.2.6]. We follow the same arguement here. Let $\dim(X) = d$. Since X is proper, by [MVW, Example 20.11]

$$\underline{Hom}(M(X), \mathbf{Q}(d)[2d]) \cong M(X).$$

Hence, by [MVW, Theorem 19.3, Proposition 14.16]

$$\begin{aligned} Hom_{\mathbf{DM}^{\mathrm{eff}}(k,\mathbf{Q})}(M(Y)(c)[2c], M(X)[1]) &= H^{2(d-c)+1}_M(Y \times X, d-c) \\ &= 0, \end{aligned}$$

where $H^p_M(-, q)$ is the motivic cohomology functor (cf. [MVW, Definition 3.4]). □

Proposition 4.2. *Let $Z \subset F$ be a smooth, closed, codimension c substack of a smooth Deligne-Mumford stack of finite type. Assume that $M(F \setminus Z)$ is a Chow motive. Then there is an isomorphism $M(F) \cong M(Z)(c)[2c] \oplus M(F \setminus Z)$.*

Proof. By [Ch, Lemma 3.9], there is an exact triangle,

$$M(F \setminus Z) \to M(F) \to M(Z)(c)[2c] \to M(F \setminus Z)[1],$$

so the proof will be done if $Hom_{\mathbf{DM}^{eff}(k,\mathbf{Q})}(M(Z)(c)[2c], M(F \setminus Z)[1]) = 0$. Since $M(F \setminus Z)$ is a Chow motive, and $M(Z)$ is a direct factor of $M(Y)$ (cf.[Ch, Corollary 4.7]) for some smooth k-scheme Y, it suffices to show that for any smooth scheme Y and any smooth projective scheme X,

$$Hom_{\mathbf{DM}^{\mathrm{eff}}(k,\mathbf{Q})}(M(Y)(c)[2c], M(X)[1]) = 0.$$

This follows from Lemma 4.1. □

Remark 4.3. *Suppose $\{X_i\}_{i=1}^n$ are proper smooth Deligne-Mumford stacks. Let $dim(X_i) = d_i$ and $\sigma_i \in Ch^{d'_i}((F \setminus Z) \times X_i)$. Since X_i's are proper, the cycles σ_i induce morphisms*

$$\sigma_i : M(F \setminus Z) \to M(X_i)(d'_i - d_i)[2(d'_i - d_i)]$$

in $\mathbf{DM}^{eff}(k, \mathbf{Q})$ by [Ch, Lemma 4.4] *and* [J]. *Assume that the induced morphism $\oplus_i \sigma_i : M(F \setminus Z) \to \oplus_i M(X_i)(d'_i - d_i)[2(d'_i - d_i)]$ is an isomorphism in $\mathbf{DM}^{eff}(k, \mathbf{Q})$. Then to give the isomorphism of proposition 4.2, it is enough to give a splitting of the morphism $\iota : M(F \setminus Z) \to M(F)$. This will follow if we can construct cycles $\sigma'_i \in Ch^{d'_i}(F \times X_i)$, such that $(\oplus_i \sigma'_i) \circ \iota = \oplus_i \sigma_i$. Let $\sigma'_i := \overline{\sigma_i} \in Ch^{d'_i}(F \times X_i)$, i.e the closure of σ_i in $F \times X_i$. The cycle σ'_i restricted to $(F \setminus Z) \times X$ gives the cycle σ_i. Hence $(\oplus_i \sigma'_i) \circ \iota = \oplus_i \sigma_i$.*

Definition 4.4. *A* Chow cellular *Deligne-Mumford stack is a smooth Deligne-Mumford stack F satisfying the following properties.*

(1) There exists a finite increasing filtration by closed (not necessarily smooth) substacks,

$$\phi = F_{-1} \subset F_0 \subset \cdots \subset F_n = F;$$

(2) The successive differences, $F_{i \setminus i-1} := F_i \setminus F_{i-1}$, are smooth of pure codimension in F and the $M(F_{i \setminus i-1})$ are Chow motives.

The successive differences are called cells. *If the successive differences are affine quasi-fibrations, i.e., there exist morphisms, $p_i : F_{i \setminus i-1} \to Y_i$, to smooth, proper Deligne-Mumford stacks, Y_i, whose geometric fibers are affine spaces, then the stack is* geometrically relative cellular *(cf.* [Kar, Definition 6.1]*).*

Remark 4.5. *Geometrically relative cellular Deligne-Mumford stacks are also Chow cellular. By corollary 3.3, $M(F_{i \setminus i-1}) \cong M(Y_i)$ for all i. But $M(Y_i)$ is a Chow motive, because Y_i is proper* [Ch, Theoerm 3.9].

Proposition 4.6. *Let F be a Chow cellular Deligne-Mumford stack. Then there is an isomorphism in $\mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Q})$,*

$$M(F) \cong \oplus_{i=0}^n M(F_{i \setminus i-1})(d_i)[2d_i],$$

where $d_i := \mathrm{codim}(F_{i \setminus (i-1)}, F)$.

Proof. The proof proceeds by descending induction on i applied to hypothesis that each $M(F \setminus F_i)$ is a Chow motive and moreover, that

$$M(F \setminus F_i) \cong \oplus_{j=i+1}^{n} M(F_{j\setminus j-1})(d_j)[2d_j].$$

The result will then follow for $i = -1$.

For $i = n-1$, $M(F \setminus F_{n-1})$ is a Chow motive by definition.

Now assume that

$$M(F \setminus F_k) \cong \oplus_{j=k+1}^{n} M(F_{j\setminus j-1})(d_j)[2d_j],$$

for all $n-2 \geq k \geq i$. We have $F \setminus F_{i-1} = F \setminus F_i \cup F_{i\setminus i-1}$. The motive $M(F \setminus F_i)$ is Chow by the induction hypothesis, and $M(F_{i\setminus i-1})$ is Chow by assumption. Then

$$M(F \setminus F_{i-1}) = \oplus_{j=i}^{n} M(F_{j\setminus j-1})(d_j)[2d_j].$$

follows from Proposition 4.2 and the induction hypothesis. □

Corollary 4.7. *Let F be a geometrically relative cellular stack, retaining the above notation. Then there is an isomorphism,*

$$M(F) \cong \oplus_{i=0}^{n} M(Y_n)(d_i)[2d_i],$$

where $d_i := \operatorname{codim}(F_{i\setminus(i-1)}, F)$.

Proof. The proof follows from Remark 4.5 and Proposition 4.6. □

Remark 4.8. *The morphism $M(F) \to M(F \setminus F_0) \cong \oplus_{j=1}^{n} M(Y_j)(d_j)[2d_j]$ is given by the correspondence $\oplus_{j=1}^{n} \overline{\Gamma}_{p_j}$, where $\overline{\Gamma}_{p_j} \subset F \times Y_j$ is the closure of the cycle $\Gamma_{p_j} \subset F_{j\setminus j-1} \times Y_j$. This follows from Remark 4.3 and the proof of Proposition 4.6 which imply that, for all i, the morphism $M(F \setminus F_{i-1}) \to M(F \setminus F_i) \cong \oplus_{j=i+1}^{n} M(Y_j)(d_j)[2d_j]$ is given by the correspondence $\oplus_{j=i+1}^{n} \overline{\Gamma}_{p_j}$.*

Example 4.9. *If a smooth, proper Deligne-Mumford stack over an algebraically closed field with a scheme for a coarse moduli space has a $\mathbf{G}_m$-action, then it admits a Białinycki-Birula decomposition [SK] and hence a decomposition of its motive. The assumption on the coarse moduli space may be dropped if the base field has characteristic zero.*

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