SUMMER SCHOOL ALPBACH: MOTIVES, PERIODS AND TRANSCENDENCE

THE SPEAKERS

Abstract. These are notes from the Workshop on Motives, Periods and Transcendence in Alpbach, Austria organized by the ProDoc Arithmetic and Geometry module of the ETH and the Universität Zürich and meeting from July 13th – 19th, 2011. The talks aim to give a full exposition of the paper *On the relation between Nori Motives and Kontsevich Periods* by Annette Huber and Stefan Müller-Stach.

Contents

1. Periods from a classical point of view 1
2. Kontsevich-Zagier periods 5
3. Singular (co)homology 7
4. Sheaf cohomology 10
5. Algebraic de Rham cohomology and the period isomorphism 12
6. Relative de Rham cohomology 16
7. Nori's basic lemma 19
8. Tannakian categories 23
9. Nori's diagram category 27
10. Multiplicative structure of diagrams and their localization 31
11. Nori's rigidity criterion 34
12. Pairs of representations 36
13. Diagrams of pairs 38
14. Yoga of diagrams 42
15. Rigidifty for Nori motives 44

Note. Please attribute errors first to the scribe.

1. Periods from a classical point of view

Gisbert Wüstholz on the 13th of July, 2011.

The workshop organizers were G. Wüstholz (Chair), J. Ayoub, A. Huber-Klawitter, S. Müller-Stach with the help of C. Fuchs. The speakers were G. Wüstholz, J. Jermann, R. Paulin, B. Fazlija, R. von Känel, J. Skowera, J. Yu, K. Völkel, P. Wieland, M. Huicochea, H. Pham, C. Scheimbauer, A. Schmidt, J. von Wangenheim, M. Gallauer, T. Preu, L. Kühne, M. Wang, U. Choudhury, S. Rybakov, and S. Gorchinskiy. The notes were recorded by Jonathan Skowera.

1.1. **Periods.** Their roots descend to the nineteenth century, where periods arose from the study of elliptic and abelian integrals by Jacobi, Hurwitz and many others, and also from the problem of squaring the circle, or more generally, squaring ovals as studied by Leibniz in particular. These questions arise in celestial mechanics, in the study of orbits. If you calculate the area of sectors of an ellipse, you are lead to elliptic integrals, and so to periods.

And what are periods?

Definition 1.1. The periods of a smooth projective curve C are the integrals of global holomorphic forms over 1-cycles. They are characterized by the periods of a basis of the $k(C)$-vector space of global holomorphic differential forms integrated over a basis of the first singular homology over $\boldsymbol{C}$.

In the case of a circle, a basis is given by the single element

$$\frac{dx}{x}$$

In the case of an elliptic curve X over the complex numbers, which one might picture as a torus, or as the graph of its real points in $\boldsymbol{R}^2$. Assume the curve has been transformed to Weierstraß normal form

$$y^2 = x^3 - ax - b$$

Then the zeroeth cohomology $H^0(X, \Omega_X)$, is spanned by two holomorphic differential forms,

$$\begin{aligned} \omega &= \frac{dx}{y} \\ \eta &= x\omega \end{aligned}$$

Using the Hodge decomposition $H^1(X) \cong H^{1,0}(X) \oplus H^{0,1}(X)$[1], we distinguish ω as a form of the first kind, and η as a form of the second kind. Then we may define the periods by integrating.

$$\begin{aligned} \omega(\sigma) &= \int_\sigma \omega, \qquad \omega(f) = \int_f \omega \\ \eta(\sigma) &= \int_\sigma \eta, \qquad \eta(f) = \int_f \eta \end{aligned}$$

Rewriting the equation of the elliptic curve in Legendre form

$$y^2 = x(x-1)(x-\lambda)$$

allows easier generalization to the higher genus case, where we consider curves of the form

$$y^2 = x(x-1)(x-\lambda_1)\cdots(x-\lambda_{2g-1})$$

This presents the curve as a ramified cover of $\boldsymbol{P}^1$ by projecting onto the y-axis.

The study of periods leads to hypergeometric geometric functions and to variations of Hodge structures.

1.2. **Hilbert's seventh problem.** Much of transcendental theory in the twentieth century grew out of Hilbert's seventh problem. Prior to 1900, Riemann proved that π is transcendental. Hilbert's problem is as follows:

[1] Recall that Poincaré duality for the curve X says that $H^0(X, \Omega_X) \cong H^1(X, \mathcal{O}_X) =: H^1(X)$.

Conjecture 1.2 (Hilbert's seventh problem)**.** *Complex numbers α, β and α^β are in $\overline{\boldsymbol{Q}}$ if and only if $\beta \in \boldsymbol{Q}$ or $\alpha = 0$ or 1.*

Hilbert regarded this problem as more difficult than the Riemann Hypothesis in one of his lectures. The problem was solved in 1934 by Gel'fand and Schneider, while the Riemann hypothesis remains open. Note this does not necessarily mean Hilbert was wrong.

In 1936, Schneider extended the result to the case of elliptic curves, which relied on the theory of elliptic integrals of the first and second kind. For example, he considered the following

Question 1.3. When is a point on an elliptic curve over $\boldsymbol{C}$ is transcendental.

Gel'fand was more interested in extending in the direction of logarithmic forms in two variables. For example, he considered the following

Question 1.4. When are numbers of the form

$$\Lambda = \beta_1 log \alpha_1 + \beta_2 \log \alpha_2$$

transcendental?

The work of Gel'fand was then extended to case of at least three logarithms.

A famous problem originating with Gauß asks the following

Question 1.5. Which number fields have class number one?

Baker solved this problem using n-logarithms, which have the form,

$$\Lambda = \sum_i \beta_i \log \alpha_j,$$

where $\alpha_i, \beta_j \in \overline{\boldsymbol{Q}}$. He made the qualitative observation that

$$\dim_{\overline{\boldsymbol{Q}}} \langle \log \alpha_1, \dots, \log \alpha_n \rangle = \dim_{Q} \langle \log \alpha_1, \dots, \log \alpha_n \rangle.$$

He also proved the quantitative result that

$$|\Lambda| > B^{-c},$$

where B is the height of the linear form $\sum_i \beta_i x$, and c is a constant.

Siegel proved that if you have a plane algebraic curve with rational coefficients, you look at the integral points on this rational curve. In that case, there are only finitely many such integral points, except when the curve has genus 0 and the divisor at infinity has at most two components, in which case there are infinitely many points.

1.3. **Commutative algebraic groups.** Around 1970, Serge Lang reformulated everything I told you so far in terms of algebraic groups.

Theorem 1.6 (Rosenlicht)**.** *Let G be an algebraic group over $\overline{\boldsymbol{Q}}$. The G is an extension of the form*

$$0 \to L \to G \to A \to 0$$

where

$$L = \boldsymbol{G}_a^k \times \boldsymbol{G}_m^l,$$

and A is an abelian variety such that

$$A(\boldsymbol{C}) \cong \boldsymbol{C}^n / \Lambda,$$

for the Λ *the fundamental group of* A.

These group extensions may be studied using $H^q(A, \mathcal{O}_A)$ and $\mathrm{Pic}^0(A)$, spaces which relate closely to differential forms.

Recalling the discussion about the origin of periods, in the study of periods on circles, we arrived at rational functions, and in the study of elliptic curves, at the Weierstraß$\mathfrak{p}$-function. This may be generalized as follows.

Let G be a Lie group. We will abuse notation and also write G to mean $G(\boldsymbol{C})$. It has a Lie algebra $\mathfrak{g}$, which we consider as a $\overline{\boldsymbol{Q}}$-vector space. In general, given a subspace $\mathfrak{a}$, it corresponds to an analytic subgroup $A \leq G$.

Question 1.7. Does

$$A(\overline{\boldsymbol{Q}}) = G(\overline{\boldsymbol{Q}}) \cap A(\boldsymbol{Q})?$$

If A is *a fortiori* an algebraic subgroup, then the above equation holds.

In general, $A(\overline{\boldsymbol{Q}}) = \{1\}$. If A is not semi-stable, then there are cases when $A(\overline{\boldsymbol{Q}}) = H(\overline{\boldsymbol{Q}})$ for $H \leq A$ algebraic.

1.4. **Periods from a modern point of view.** There are two viewpoints.

- One may consider periods attached to an algebraic variety.
- One may consider all periods together, an approach due to Kontsevich.

We take the first viewpoint in this first talk. Let X be a projective algebraic variety over $\overline{\boldsymbol{Q}}$, and $\mathcal{M}_X^n$ be the sheaf of meromorphic functions[2] on X. Let $\xi \in \Gamma(X, \mathcal{M}_X^n)$ be a meromorphic function of degree 0. Given a polar divisor D, we define

$$U = X \setminus |D|$$

which is open in X. Fix a global holomorphic differential form $\xi \in H^0(U, \Omega_U^1)$.

Definition 1.8. The periods of X with respect to ξ are

$$\begin{aligned} H_1(U, \boldsymbol{Z}) &\to \boldsymbol{C} \\ (\gamma : [0,1] \to U(\boldsymbol{C})) &\mapsto \int_\gamma \xi \end{aligned}$$

Question 1.9. Is

$$\int_\gamma \xi$$

transcendental?

The problem may be transformed to a question about the path space

$$\mathcal{P}_U := \{\gamma : [0,1] \to U(\boldsymbol{C})\},$$

which forms an infinite-dimensional differential manifold.

$$\mathcal{P}_U := \left\{\gamma \in \mathcal{P}_n : \gamma(0), \gamma(1) \in X(\overline{\boldsymbol{Q}})\right\}$$

Question 1.10. How does $\mathcal{P}_U(\overline{\boldsymbol{Q}})$ look?

[2]The meromorphic functions are defined over $\overline{\boldsymbol{Q}}$

We will see a picture of what this looks like in terms of mixed Hodge structures. The periods are characterized by a function on the path space.

$$\begin{aligned} I(\xi) : \mathcal{P}_U &\to \boldsymbol{C} \\ \gamma &\mapsto \int_\gamma \xi \end{aligned}$$

One could replace this path space by the fundamental groupoid.

A significant problem lies in generalizing this story to higher cohomology.

2. Kontsevich-Zagier periods

Gisbert Wüstholz on the 14th of July, 2011.

See *Periods and algebraic de Rham cohomology* by B. Friedrich for reference

2.1. Let X be a smooth projective variety over $\overline{\boldsymbol{Q}}$, and D a smooth, normal crossings divisor[3] on X. Let $U = X \setminus D$ be the complement, and $\xi \in \Gamma(X, \mathcal{M}_X^1)$ a meromorphic form of degree 0.

If D is a polar divisor[4], then $\xi \in H^0(U, \Omega_U^1)$.

Consider again the path space $\mathcal{P}_U$. It contains a subset $\mathcal{E}_U$

$$\mathcal{P}_U(\overline{\boldsymbol{Q}}) = \{\gamma \in \mathcal{P}_U : \gamma(0), \gamma(1) \in U(\overline{\boldsymbol{Q}})\} \supset \mathcal{E}_U = \left\{\gamma \in \mathcal{P}_U(\overline{\boldsymbol{Q}}) : \int_\gamma \xi \in \overline{\boldsymbol{Q}} \cup \{\infty\}\right\}$$

Question 2.1 (Leibniz, Arnol'd). What is $\mathcal{E}_U$?

Consider an embedding of mixed Hodge structures $i : H \hookrightarrow H_1(U, \boldsymbol{Z})$. It induces an exact sequence

$$0 \longrightarrow H^\perp \longrightarrow H^1(U, \boldsymbol{Z}) \xrightarrow{i^\vee} H^\vee \longrightarrow 0$$

Theorem 2.2. *A path γ lies in $\mathcal{E}_U$ if and only if the following conditions hold*

(i) *There exists a mixed Hodge sub-structure $H \hookrightarrow H_1(U, \boldsymbol{Q})$ such that $\gamma \in H \times_{\boldsymbol{Q}} \boldsymbol{C}$.*

(ii) *There exists $\eta \in \Gamma(X, \Omega_X^1[\log D]^5)$ and $\phi \in \Gamma(U, \mathcal{O}_U)$ such that*

$$\xi = \eta + d\phi \quad \textit{and} \quad [\xi] \in H^\vee \otimes_{\overline{\boldsymbol{Q}}} \boldsymbol{C}.$$

The second property holds whenever

$$\int_\gamma \xi = \int_\gamma (\eta + d\phi) = \phi(\gamma(1)) - \phi(\gamma(0)) \in \overline{\boldsymbol{Q}}$$

In general, there are at most countably many mixed Hodge sub-structures.

Given a U and xi, one can define their associated generalized Albanese variety $\mathrm{Alb}(U)$ with the associated embedding $\phi : U \to \mathrm{Alb}(U)$. This is the variety which is universal for morphisms

[3]That is, a divisors whose support has at most nodal singularities

[4]A polar divisor the negative part of a divisor of a meromorphic form

[5]This means the meromorphic forms whose poles are all simple and lie along the divisor D

from U to Abelian varieties, i.e.,

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[dr, "\forall"'] & \mathrm{Alb}(U) \arrow[d, dashed, "\exists"] \\ & A \end{tikzcd}$$

Then there exists an $\omega \in \mathrm{LieAlb}(U)^\vee$ such that $\xi \in \phi^*\omega$, and

$$\mathcal{P}_U \xrightarrow{\phi_*} \mathcal{P}_{\mathrm{Alb}(U)} \longrightarrow \mathrm{LieAlb}(U) \cong H_1(\mathrm{Alb}(U), \boldsymbol{C})$$

This succeeds in transferring the problem from the variety U and form ξ to the algebraic group $\mathrm{Alb}(U)$ with a tangent vector ω.

Consider the exact sequence

$$0 \longrightarrow \boldsymbol{G}_a^k \times \boldsymbol{G}_m^l \longrightarrow G \longrightarrow A \longrightarrow 0\ .$$

There is another sequence

$$0 \longrightarrow V \longrightarrow G \longrightarrow S \longrightarrow 0\ ,$$

where S is a semiabelian variety, and corresponds to the logarithmic structure $\log D$ and gives a mixed Hodge structure H in the above setting.

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow G \longrightarrow A \longrightarrow 0$$

2.2. **Periods according to Kontsevich.** The formulation of Huber-Müller-Stach begins with a quadruple (X, D, ω, γ) such that

X is an algebraic variety over $\boldsymbol{Q}$
D is a (not necessarily closed) subvariety of X
$\omega \in H^d_{dR}(X, D)$ is a holomorphic d-form vanishing along D.
$\gamma \in H_d(X(\boldsymbol{C}), D(\boldsymbol{C}), \boldsymbol{Q})$ is a rational d-cycle avoiding D.

Define the set

$$\mathcal{R} = \boldsymbol{Q}\left[\ldots, \int_\gamma \omega, \ldots\right]$$

It is a ring by Frobenius' theorem.

Let $\mathcal{P}$ be the free $\boldsymbol{Q}$-algebra generated by the set of quadruples.

$$\begin{tikzcd} (X, D, \omega, \gamma) \arrow[dd, mapsto] & \mathcal{P} \arrow[dd] \arrow[dr] & \\ & & \mathcal{P}^+ \arrow[dl, "\tau"] \\ \int_\gamma \omega & \mathcal{R} & \end{tikzcd}$$

where $\mathcal{P}^+$ is $\mathcal{P}$ with the Tate element, $2\pi i$, inverted.

Conjecture 2.3. *The map τ is injective.*

This conjecture lies deep, and might take another century to solve.

Question 2.4. What is the kernel of the period map from $\mathcal{P}$ to $\mathcal{R}$?

The ring $\mathcal{R}$ is defined by relations.

(i) They are linear in ω, γ
(ii) $(X, D, f^*\omega', \gamma') = (X', D', \omega', f_*\gamma)$ for all $f : X \to X'$ such that $f(D) \subset D'$.
(iii) $(Y, Z, \omega, d\gamma) = (X, Y, \partial\omega, \gamma)$ for all $Z \subset Y \subset X$.

2.3. **The case of an elliptic curve.** Let E be an elliptic curve with complex multiplication, and let

$$\begin{aligned} H_{dR}(E) &= \overline{\boldsymbol{Q}}\omega \oplus \overline{\boldsymbol{Q}}\eta \\ H_1(E) &= \mathcal{O}e \end{aligned}$$

where $\mathcal{O}$ is an order in a imaginary quadratic field induced by the complex multiplication on E, making it a two dimensional algebra over $\boldsymbol{Q}$.

Examine the quadruple $(E, \emptyset, \xi, \gamma)$ where $\xi = \omega, \eta$ and $\gamma = e$. Then

$$\mathcal{P} = \left\langle (E, \emptyset, \eta, e), (E, \emptyset, \omega, e), (\boldsymbol{P}^1, \{0\} + \{\infty\}, \frac{dt}{t}, s') \right\rangle_{\boldsymbol{Q}}.$$

Furthermore,

$$\mathcal{P} \to \boldsymbol{Q}[\omega(e), \eta(e), 2\pi i] = \boldsymbol{Q}[\omega(e), \eta(e)].$$

The equality follows from the Legendre relations. This map is injective.

3. Singular (co)homology

Jonas Jermann and Roland Paulin on the 14th of July, 2011.

See *Algebraic Topology* by Allen Hatcher for reference.

3.1. A variety will be a reduced, separated scheme of finite type over $\boldsymbol{Q}$. After base change to $\boldsymbol{C}$, it carries both a Zariski and a Euclidean topology.

Definition 3.1 (Complex analytic space)**.** A complex analytic space is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to a space of the form $(U, \mathcal{O}_U)$, such that,

$$U = \{z \in D^n \mid f_1(z) = \cdots = f_n(z) = 0\}, \qquad f_1, \ldots, f_n \in \Gamma(D^n, \mathcal{O}_{D_n})\text{holomorphic},$$

and

$$\mathcal{O}_U = \mathcal{O}_{D^n}/(f_1, \ldots, f_n).$$

where $D^n = \{z \in \boldsymbol{C}^n \mid |z_i| < 1\}$.

Let X be a scheme of finite type over $\boldsymbol{C}$. X is locally of the form $X \subset Y = \operatorname{Spec} \boldsymbol{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We view $f_1, \ldots, f_m$ as holomorphic functions. Let $Y^n = \{z \mid f_1(z) = \cdots = f_m(z) = 0\}$ and $O_{Y^{an}} = O_{\boldsymbol{C}^n}/(f_1, \ldots, f_m)$

There is a functor a_n which maps

$$\begin{aligned} (\text{schemes of finite type over } \boldsymbol{C}) &\to \text{complex analytic spaces} \\ (\text{smooth of finite type over } \boldsymbol{C}) &\to \text{complex manifolds} \\ \boldsymbol{CP}^n &\to \boldsymbol{CP}^n_{an} \end{aligned}$$

where the "an" subscript will denote the associated analytic space.

3.2. **Singular homology.** Let X be a topological space and R a unital, commutative ring. Let Δ^n be the standard n-simplex.

Definition 3.2 (Singular n-simplex)**.** A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$.

Definition 3.3 (Singular n-chain)**.** The singular n-chains on X with coefficients in R are the R-module

$$\langle \sigma : \Delta^n \to X \mid \ \sigma \text{ is continuous } \rangle_R$$

Definition 3.4 (Face of a singular chain)**.** The ith face of a singular n-chain σ is

$$\sigma_i(t_0, \dots, t_{n-1}) = \sigma(t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_{n-1})$$

We define a complex with the boundary map by its action on simplices

$$\begin{array}{rcl} C_n(X;R) & \to & C_{n-1}(X;R) \\ \partial\sigma & \mapsto & \Sigma_{i=0}^n (-1)^i \sigma_i \end{array}$$

It extends to chains by linearity.

Lemma 3.5. *This is a chain complex, i.e.,*

$$\partial^2 = 0.$$

3.3. **Homology of a pair.** Let $Y \subset X$ be a subspace of X. Then define a chain complex by

$$C(X, Y; R) = C_n(X;R)/C_n(Y;R)$$

The boundary map ∂ descends to these R-modules, which form a chain complex.

Define the homology $H_n(X, Y; R)$ to be the homology of this complex. It is functorial in (X, Y), i.e.,

$$\begin{array}{rcl} (X,Y) & \mapsto & H_n(X,Y;R) \\ \begin{pmatrix} f : X \to X' \\ f(Y) \subset Y' \end{pmatrix} & \mapsto & (f_* : \phi \mapsto f \circ \phi) \end{array}$$

3.4. **Singular cohomology.**

Definition 3.6. A singular cochain on X is

$$C^n(X, Y; R) = \mathrm{Hom}_R(C_n(X, Y; R), R)$$

Since $\mathrm{Hom}_R(\cdot, R)$ is a contravariant functor, it transforms the singular chain complex into a singular cochain complex with R-modules $C^n(X, Y; R)$ and boundary map δ.

Proposition 3.7. *Let $Z \subset Y \subset X$ be topological spaces. There are long exact sequences of homology and cohomology*

$$\cdots \longrightarrow H_n(Y,Z;R) \longrightarrow H_n(X,Z;R) \longrightarrow H_n(X,Y;R) \longrightarrow H_{n-1}(Y,Z;R) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^n(X,Y;R) \longrightarrow H^n(X,Z;R) \longrightarrow H^n(Y,Z;R) \longrightarrow H^{n+1}(X,Y;R) \longrightarrow \cdots$$

Proof. Consider $(Y, Z) \hookrightarrow (X, Z) \hookrightarrow (X, Y)$. It induces a short exact sequence

$$0 \longrightarrow C_*(Y,Z;R) \longrightarrow C_*(X,Z;R) \longrightarrow C_*(X,Y;R) \longrightarrow 0$$

By the snake lemma, this induces a long exact sequence in homology.

For cohomology, apply the Hom_R functor to the short exact sequence, to obtain a left-exact sequence. It is in fact a short exact sequence, which can be seen be examining $C^n(X, Z; R) \to C^n(Y, Z; R)$. □

3.5. Properties of singular cohomology.

Proposition 3.8 (Homotopy invariance). *Let $f, g : (X, Y) \to (X', Y')$ be homotopic morphisms of pairs, i.e., continuous functions such that $f(Y), g(Y) \subset Y'$, and such that there is some $G : X \times [0, 1] \to X'$ such that $G_t(Y) \subset (Y')$ for all t. Then f and g induce the same R-module morphism on the relative homology.*

Define the cup product by

$$\begin{array}{rcl} C^l(X; R) \times C^k(X; R) & \to & C^{l+k}(X; R) \\ (\phi, \psi) & \mapsto & (\phi \cup \psi : \sigma \mapsto \phi(\sigma \circ \alpha)\psi(\sigma \circ \beta)) \end{array}$$

where

$$\begin{array}{rcl} \alpha : (t_0, \dots, t_k) & \mapsto & (t_0, \dots, t_k, 0, \dots, 0) \\ \beta : (t_0, \dots, t_l) & \mapsto & (0, \dots, 0, t_0, \dots, t_l) \end{array}$$

This satisfies

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^d \phi \cup (\delta\psi)$$

and induces a cup product on the cohomology, making $H^*(X; R) = \oplus_n H^n(X; R)$ into an R-algebra.

Proposition 3.9 (Künneth formula). *There is a cross-product on singular cohomology*

$$\begin{array}{rcl} \times : H^k(X; R) \otimes H^l(Y; R) & \to & H^{k+l}(X \times Y; R) \\ \alpha \times \beta & \mapsto & p_1^*(\alpha) \cup p_2^*(\beta). \end{array}$$

It interacts with the cup product according to the formula

$$(\alpha \otimes \beta) \cup (\alpha' \otimes \beta') = (-1)^{kl}(\alpha \cup \alpha') \otimes (\beta \cup \beta').$$

If $H^k(Y; R)$ is a finitely generated free R-module for all k, then the cross-product induces the isomorphism of graded rings

$$\begin{array}{rcl} H^*(X; R) \otimes_R H^*(Y; R) & \to & H^*(X \times Y; R) \\ \alpha \otimes \beta & \mapsto & \alpha \times \beta. \end{array}$$

3.6. Universal coefficients theorem for fields.

Proposition 3.10. *Let $X \supset Y$ be topological spaces and L/K a field extension. There are isomorphisms*

$$\begin{array}{rcl} H_n(X, Y; L) & \cong & H_n(X, Y; K) \otimes_K L \\ H^n(X, Y; L) & \cong & \mathrm{Hom}_K(H_n(X, Y; K), L) \end{array}$$

3.7. **Smooth chains.** Given a smooth manifold X, we can form a chain complex of C^∞*−chains*, which are R-linear sums of C^∞-functions from standard simplices to X. One can show that the resulting homology and cohomology are isomorphic to singular homology and cohomology respectively.

3.8. **Poincaré duality.** Let M be a compact, oriented smooth manifold without boundary. It corresponds to a fundamental class $[M]$ in its own homology $H_n(M;R)$.

$$\begin{array}{rcl} H^k(M;R) & \xrightarrow{\sim} & H_{n-k}(M;R) \\ \alpha & \mapsto & \alpha \cap [M] \end{array}$$

where $\cap$ denotes the cap product, induced by the cap product on singular chains,

$$\begin{array}{rcl} C_k(X;R) \times C^l(X;R) & \to & C_{k-l}(X;R) \\ (\sigma, \phi) & \mapsto & \sigma \cap \phi = \phi(\sigma \circ \alpha)(\sigma \circ \beta), \end{array}$$

where we leave α and β undefined.

It interacts with the boundary map according to

$$\partial(\sigma \cap \phi) = (-1)^l(\partial\sigma \cap \phi - \sigma \cap \delta\phi)$$

4. Sheaf cohomology

Bleder Fazlija and Rafael von Känel on the 14th of July, 2011.

See Chapter 5 of *Foundations of differentiable manifolds and Lie groups* by Warner for reference.

4.1. **First definitions.** Let K be a principal ideal domain.

Definition 4.1. A sheaf cohomology theory on a differentiable manifold M with coefficients in a sheaf of K-modules consists of

(i) For all sheaves $\mathcal{S}$ on M and all integer $q \in \boldsymbol{Z}$, a K-module $H^q(M, \mathcal{S})$.
(ii) For all morphisms $\mathcal{S} \to \mathcal{S}'$ and $q \in \boldsymbol{Z}$, a morphism $H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S}')$.
(iii) For all short exact sequences $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ and $q \in \boldsymbol{Z}$, a morphism $H^q(M, \mathcal{S}'') \to H^{q+1}(M, \mathcal{S}')$

such that

(a) all $q < 0$, $H^q(M, \mathcal{S}) = 0$. Also, $H^0(M, \mathcal{S}) = \Gamma(M, \mathcal{S})$ and for all morphisms $\mathcal{S} \to \mathcal{S}'$, the diagram

$$\begin{array}{ccc} H^0(M, \mathcal{S}) & \longrightarrow & \Gamma(M, \mathcal{S}) \\ \downarrow & & \downarrow \\ H^0(M, \mathcal{S}') & \longrightarrow & \Gamma(M, \mathcal{S}') \end{array}$$

commutes

(b) A fine sheaf is a sheaf such that for every locally finite cover $\{U_i\}$ of M, there exists a family of endomorphisms l_i such that
supp$l_i \subset U_i$
$\sum_i l_i = id$
We assume that for all fine sheaves $\mathcal{S}$ and all $q \in \boldsymbol{Z}$, $H^q(M, \mathcal{S}) = 0$.
(c) For all short exact sequences of sheaves, $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$, there is a long exact sequence $H^q(M, \mathcal{S}') \to H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S}'') \to H^{q+1}(M, \mathcal{S}') \to \cdots$.

(d) The identity $1 : \mathcal{S} \to \mathcal{S}$ induces the identity $1 : H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S})$.
(e) For all commuting triangles

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[dr] & \mathcal{S}' \arrow[d] \\ & \mathcal{S}'' \end{tikzcd} ,$$

the diagram

$$\begin{tikzcd} H^q(M, \mathcal{S}) \arrow[r] \arrow[dr] & H^q(M, \mathcal{S}') \arrow[d] \\ & H^q(M, \mathcal{S}'') \end{tikzcd}$$

commutes.
(f) For all morphisms of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S}' \arrow[r] \arrow[d] & \mathcal{S} \arrow[r] \arrow[d] & \mathcal{S}'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{T}' \arrow[r] & \mathcal{T} \arrow[r] & \mathcal{T}'' \arrow[r] & 0 \end{tikzcd}$$

the diagram

$$\begin{tikzcd} H^r(M, \mathcal{S}'') \arrow[r] \arrow[d] & H^{q+1}(M, \mathcal{S}') \arrow[d] \\ H^q(M, \mathcal{T}'') \arrow[r] & H^{q+1}(M, \mathcal{T}') \end{tikzcd}$$

commutes.

Proposition 4.2. *Let $\mathcal{S}_{pre}$ be a presheaf on M, where we will use the notation $\mathcal{S}_U$ for its open sests and $\rho_{U,V}$ for its restriction maps. Let cS be its associated sheaf.*

$$(\mathcal{S}_M)_0 = \{s \in \mathcal{S}_M \mid \rho_{m,M}(s) = 0 \text{ for all } m \in M\}$$

If $\mathcal{P}$ satisfies the glueing axiom, then

$$0 \longrightarrow (\mathcal{S}_M)_0 \longrightarrow \mathcal{S}_M \xrightarrow{r} \Gamma(M, \mathcal{S}) \longrightarrow 0$$

is exact, where $r(s) = (m \mapsto \rho_{m,M}(s))$.

Theorem 4.3. *Let H be a cohomology theory on M and*

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}_0 \longrightarrow \mathcal{S}_1 \longrightarrow \cdots$$

be a fine resolution of $\mathcal{S}$. Then

$$H^q(M, \mathcal{S}) = H^q(\Gamma(M, \mathcal{S}))$$

for all $q \in \boldsymbol{Z}$.

Proof. Let $q = 0$. Since the resolution is exact and Γ is left-exact,

$$0 \longrightarrow \Gamma(\mathcal{S}) \longrightarrow \Gamma(\mathcal{S}_0) \xrightarrow{d^0} \Gamma(\mathcal{S}_1)$$

is exact. Thus

$$H^0(M,\mathcal{S}) = \Gamma(M,\mathcal{S}) = \ker(d^0) = H^0(\Gamma(M,\mathcal{S}^*))$$

Now consider the case $q = 1$. Let K_1 be the kernel of $S_1 \to S_2$. Since 4.3 is exact,

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}_0 \longrightarrow \mathcal{K}_1 \longrightarrow 0$$

is exact. Property (c) of sheaf cohomology implies that

$$H^0(M,\mathcal{S}_0) \to H^0(M,\mathcal{K}_1) \to H^0(M,cS) \to H^1(M,cS_0)$$

Then

$$\Gamma(\mathcal{S}_0) \xrightarrow{d^0} \Gamma(\mathcal{K}_n) \longrightarrow H^1(M,\mathcal{S}) \longrightarrow 0$$

implying that

$$\begin{aligned} H^0(M,S) &= \Gamma(M,\mathcal{K}_1)/\Gamma(M,\mathcal{S}_0) \\ &= \Gamma(M,\mathcal{K}_1)/\mathrm{im}\, d^0 \\ &= H^1(\Gamma(M,\mathcal{S}^*)) \end{aligned}$$

□

Definition 4.4. Let $\mathcal{G}$ be a K-module. Then $S^p(U,\mathcal{G})$ is the K-module consisting of functions which assign to each singular p-simplex an element in $\mathcal{G}$.

Definition 4.5. The classical singular cohomology groups of M with coefficients in $\mathcal{G}$ are defined by

$$H^q_\Delta(M,\mathcal{G}) = H^q(\mathcal{S}^*(M,\mathcal{G})) \qquad H^q_{\Delta\infty}(M,\mathcal{G}) = H^q(\mathcal{S}^*_\infty(M,\mathcal{G}))$$

Theorem 4.6. *The classical cohomology groups are canonically isomorphics as K-modules to the sheaf cohomology groups $H^q(M,\mathcal{S})$.*

Proof. We have

$$H^q(\mathcal{S}^*(M,\mathcal{G})_{(\infty)}) \cong H^q(\Gamma(M,\mathcal{S}^*_{(\infty)}(M,\mathcal{G})))$$

This together with 4.5 imply that

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{S}^0(M,\mathcal{G}) \xrightarrow{d} \mathcal{S}^1(M,\mathcal{G}) \xrightarrow{d} \cdots$$

is a fine resolution and

$$H_\Delta(M,\mathcal{G}) \cong H^q(M,\mathcal{G}) \cong H^q(M,\mathcal{G})$$

□

Let X be a topological space, and $D \subset X$ a closed subspace with open complement $j : U \to X$. Then

$$H^i(X, j_! A) = H^i(X, D; A)$$

for any constant abelian group sheaf A, where the lower shriek is extension by zero, defined by taking the sheafification of

$$V \mapsto \begin{cases} A, & V \subset U \\ 0 & 0 \end{cases}$$

5. Algebraic de Rham cohomology and the period isomorphism

Konrad Völkel and Peter Wieland on the 15th of July, 2011

5.1. **Introduction.** Let X_0 be a smooth projective variety over $\boldsymbol{Q}$ and $X = X_0 \times_{\boldsymbol{Q}} \boldsymbol{C}$ the base extension to $\boldsymbol{C}$. We will prove the isomorphism

$$H^i_{dR}(X_0, \boldsymbol{Q}) \otimes_{\boldsymbol{Q}} \boldsymbol{C} \xrightarrow{\sim} H^i_{dR}(X, \boldsymbol{C})$$

as the composition of four different isomorphism.

5.2. **De Rham cohomology.** Let $\mathcal{F}$ e a sheaf on a topological space X with values in abelian groups. Take an injective resolution

$$0 \longrightarrow X \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$$

Apply the global sections functor and drop the $\Gamma(X, -)$ term to get

$$0 \longrightarrow \Gamma(X, I^0) \xrightarrow{d^0} \Gamma(X, I^1) \xrightarrow{d^1} \cdots$$

The homology of this complex is the sheaf cohomology of $\mathcal{F}$, i.e.,

$$H^k(X, \mathcal{F}) = \frac{\ker d^k}{\operatorname{im} d^{k-1}}$$

Let X be a variety over K and denote Ω_X the sheaf of Kähler differentials on X.

Consider

$$\mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \Omega^2_X \longrightarrow \cdots$$

where $\Omega^k_X = \wedge^k_{i=1} \Omega_X$ is the ith exterior power of Ω_X. Let $U \subset X$ be an open affine subvariety

$$U \cong \operatorname{Spec} A, \quad A := K[x_1, \ldots, x_n]$$

and

$$d_{:} f dx_{i_1} \wedge \cdots \wedge dx_{i_n} \mapsto df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}$$

where $df := \sum_{i=1}^n \frac{\partial f}{x_i} dx_i$.

Choose a double complex $I^{\bullet,\bullet}$ such that the kth column is an injective resolution of Ω^k_X.

$$H^i_{dR}(X, k) := H^i(\Omega^\bullet_X) = H^i(\Gamma(X, \operatorname{tot} I^{\bullet,\bullet}))$$

$$\begin{array}{ccccc}
I^{0,0} & \xrightarrow{d} & I^{1,0} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
I^{0,1} & \xrightarrow{e} & I^{1,1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & & \ddots
\end{array}$$

$$\operatorname{tot}(I^{\bullet,\bullet})_n := \bigoplus_{\substack{k,l \\ k+l=n}} I^{k,l}, \qquad \partial_n :=$$

For $U = \text{Spec } A$ where $A = K[x_1, \ldots, x_n](f_1, \ldots, f_r)$, the sections of Ω_X on U are

$$\Omega_U = \frac{\langle dx_1, \ldots, dx_n \rangle_A}{\langle df_1, \ldots, df_r \rangle}$$

5.3. **Examples.**

Example 5.1 ($\boldsymbol{A}^1$). Let $X = \boldsymbol{A}^1$. We examine the complex

$$0 \to \mathcal{O}_X \to \Omega_X^1 \to \Omega_X^2 \to \cdots$$

The differential is

$$d : k[x] \to \langle dx \rangle$$

So

$$\begin{aligned} H^0_{dR}(X, k) &= k \\ H^1_{dR}(\boldsymbol{G}_m, k) &= 0 \end{aligned}$$

Example 5.2 ($\boldsymbol{G}_m$). Let $X = \boldsymbol{G}_m = \text{Spec } k[x, x^{-1}]$ and

$$\mathcal{O}_X \xrightarrow{d} \Omega_X^1 \longrightarrow \Omega_X^2 \longrightarrow \cdots$$

where

$$d : k[x, x^{-1}] \to \langle dx \rangle_{k[x, x^{-1}]}$$

is the surjective differential. Here $\ker d = k$. Since $\frac{dx}{x} \notin \text{im } d$,

$$d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$$

So

$$\begin{aligned} H^0_{dR}(X, k) &= k \\ H^1_{dR}(\boldsymbol{G}_m, k) &= \left\langle \frac{dx}{x} \right\rangle_k \end{aligned}$$

5.4. **The isomorphism $H^i_{dR}(X_0, \boldsymbol{Q}) \otimes_{\boldsymbol{Q}} \boldsymbol{C} \cong H^i_{dR}(X, \boldsymbol{C})$.**

Let X_0 be a smooth variety over k and $k_0 \subset k$ a subfield. Then we have an isomorphism

$$H^i_{dR}(X_0, k_0) \otimes_{k_0} k \cong H^i_{dR}(X, k)$$

Since X_0 is an affine variety, $\pi^* \Omega_{X_0/k_0} = \Omega_{X/k}$ by Hartshorne II.8.

$$\begin{tikzcd} \pi^{-1}\Omega_{X_0,k_0} \arrow[rr] \arrow[dr] & & \pi^*\Omega_{X_0} = \Omega_X \\ & \pi^{-1}\Omega_{X_0} \otimes_{k_0} k \arrow[ur] & \end{tikzcd}$$

5.5. **The isomorphism $H^i_{dR}(X, \boldsymbol{C}) \cong H^i_{dR}(X^{an})$.**

Now let Y be a complex manifold.

$$\mathcal{A}_Y^{\bullet;\bullet} = \text{smooth } \boldsymbol{C}\text{-valued differential forms on } Y$$

Select the subcomplex of holomorphic differential forms

$$\Omega_Y^\bullet := \ker \overline{\partial} \subset \mathcal{A}_Y^{\bullet,0}.$$

Now the de Rham cohomology of Y is defined to be

$$H^i_{dR}(Y) := H^i(Y, \Omega^\bullet_Y).$$

Theorem 5.3 (GAGA[6])**.** *Let X be a smooth projective variety over $\boldsymbol{C}$ with structure sheaf $\mathcal{O}_X$. Consider for some coherent $\mathcal{O}_X$-module $\mathcal{F}$*

$$\mathcal{F}^{an} := j^*\mathcal{F} = j^{-1}\mathcal{F} \otimes_{j^{-1}\mathcal{O}_X} \mathcal{O}_{X^{an}}$$

where $j : X^{an} \to X$ is a morphism of locally ringed spaces. Then

$$H^i(X, \mathcal{F}) \cong H^i(X^{an}, \mathcal{F}^{an})$$

More generally, for any complex of $\mathcal{O}_X$-modules, there is an isomorphism of hypercohomology

$$\boldsymbol{H}^i(X, \mathcal{F}^\bullet) \cong \boldsymbol{H}^i(X^{an}, \mathcal{F}^{an}).$$

Furthermore,

$$\boldsymbol{H}^i(X, \Omega^\bullet_X) \cong \boldsymbol{H}^i(X^{an}, (\Omega^\bullet_X)^{an}) \cong \boldsymbol{H}^i(X^{an}, \Omega^\bullet_{X^{an}})$$

where the second isomorphism comes from the isomorphism $j^*\Omega^\bullet_X \to \Omega^\bullet_{X^{an}}$.

5.6. **The isomorphism $H^i_{dR}(X^{an}) \cong H^i_{sing}(X^{an}, \boldsymbol{C})$.** The following exact sequence induces $H^i_{dR}(X^{an}) \cong H^i_{sing}(X^{an}, \boldsymbol{C})$.

Lemma 5.4 (Poincaré lemma)**.** *There is an exact sequence*

$$0 \to \boldsymbol{C} \to \mathcal{O}_{X^{an}} \to \Omega^1_{X^{an}} \to \cdots .$$

$$\begin{tikzcd}
 & \Omega^\bullet_{X^{an}} \arrow[d, hook] \\
\underline{\boldsymbol{C}} \arrow[ur, "qis"] \arrow[r, "qis"] \arrow[dr, "qis"'] & \mathcal{A}^\bullet_{X^{an}} \arrow[d, "\int"] \\
 & \mathcal{C}^\bullet_{sing}(X^{an}, \boldsymbol{C})
\end{tikzcd}$$

The integration morphism is

$$\int : \begin{array}{ccc} \mathcal{A}^n_{X^{an}} & \to & \mathcal{C}^n_{sing}(X^{an}, \boldsymbol{C}) \\ \omega & \mapsto & (z \mapsto \int_Z \omega) \end{array}$$

Then

$$\begin{tikzcd}
\underline{C} \arrow[r] \arrow[dr, "ev"'] & \mathcal{O}_{X^{an}} \arrow[d] & & \mathcal{F} \arrow[d] \\
 & \mathcal{C}^0_{sing}(X^{an}, \boldsymbol{C}) & & \int_{[p]} f = f(p)
\end{tikzcd}$$

[6]The abbreviation GAGA stands for *Géometrie algébrique et géometrie analytique*, the title of the paper by J.-P. Serre.

and

$$\begin{tikzcd}
\mathcal{A}^n \arrow[r, "d"] \arrow[d, "\int"'] & \mathcal{A}^{n+1} \arrow[d, "\int"] \\
\mathcal{C}^n_{sing} \arrow[r, "d"] & \mathcal{C}^{n+1}_{sing}
\end{tikzcd}$$

follows from

$$\int_{\partial Z} \omega = \int_Z d\omega$$

So the integration morphism induces an isomorphism of hypercohomology

$$\boldsymbol{H}^i(X^{an}, \Omega_{X^{an}, \Omega^\bullet_{X^{an}}} \xrightarrow{\sim} \boldsymbol{H}^i(X^{an}, \mathcal{C}^\bullet_{sing}(X^{an}, \boldsymbol{C})$$

Hence

$$H^i_{dR}(X^{an}) \xrightarrow{\sim} H^i_{sing}(X^{an}, \boldsymbol{C}).$$

5.7. **The isomorphism** $H^i_{sing}(X^{an}, \boldsymbol{C}) \cong H^i_{sing}(X^{an}, \boldsymbol{Q}) \otimes_{\boldsymbol{Q}} \boldsymbol{C}$**.** The last isomorphism follows directly from the universal coefficient theorem for fields. Cf. the talk on singular cohomology.

6. Relative de Rham cohomology

By Mario Huicochea and Hiep Pham on the 15th of July, 2011.

Let X be a smooth algebraic vareity over a field k, and let D be a normal crossings divisor whose decomposition into prime divisors is

$$D = \sum_{i=1}^{r} D_i$$

Define also the intersections

$$D_{i_0 \cdots i_p} := \cap_{j=1}^{p} D_{i_j}$$

We will abbreviate $\{0, \dots, m\}$ as $[m]$. Let $f : [m] \to [n]$, i.e., a function such that $f(i) < f(j)$ for $i < j$. It induces

$$D^\bullet(f) : \coprod_{1 \leq i_0 < \cdots < i_n \leq r} D_{i_0 \cdots i_n} \to \coprod_{1 \leq j_0 < \cdots < j_m \leq r} D_{j_0 \cdots j_m}$$

which the natural inculsions

$$D_{i_0 \cdots i_n} \hookrightarrow D_{i_{f(0)} \cdots i_{f(n)}}$$

We also define an order preserving function for each pair of natural numbers l and m by

$$\delta^m_l : [m] \to [m+1]$$

where δ^m_l includes $[m]$ into $[m+1]$ by excluding l.

$$\begin{tikzcd}
\coprod_{a=1}^r D_a & \coprod_{1 \leq a < b \leq r} D_{ab} \arrow[l, shift right, "D^\bullet(\delta^0_1)"'] \arrow[l, shift left, "D^\bullet(\delta^0_0)"] & \coprod_{1 \leq a < b < c \leq r} D_{abc} \arrow[l, shift right=2, "D^\bullet(\delta^1_2)"'] \arrow[l, "D^\bullet(\delta^1_1)"] \arrow[l, shift left=2, "D^\bullet(\delta^1_0)"]
\end{tikzcd}$$

Then include

$$Z \hookrightarrow \mathrm{supp} D = \cup_{i=1}^{r} D_i$$

and

$$i_*\Omega^\bullet_{Z/k}$$

We define

$$(i_*\Omega^\bullet)(\coprod_a Z_a) = \bigoplus_a i_*\Omega_{Z_a/k}$$

So we get morphisms

$$\bigoplus_{a=1}^r i_*\Omega^\bullet_{D_a/k} \underset{d_1^0}{\overset{d_0^0}{\rightrightarrows}} \bigoplus_{1\le a<b\le r} i_*\Omega^\bullet_{D_{ab}/k} \overset{d_0^1, d_1^1, d_2^1}{\to} \bigoplus_{1\le a<b<c\le r} i_*\Omega^\bullet_{D_{abc}/k} \cdots$$

Define its associated complex by

$$\bigoplus_{a=1}^r i_*\Omega^\bullet_{D_a/k} \xrightarrow{d^1} \bigoplus_{1\le a<b\le r} i_*\Omega^\bullet_{D_{ab}/k} \xrightarrow{d^2} \bigoplus_{1\le a<b<c\le r} i_*\Omega^\bullet_{D_{abc}/k} \xrightarrow{d^3} \cdots$$

where the differential is given by

$$\begin{aligned} d^1 &:= d_0^0 - d_1^0 \\ &\vdots \\ d^{m+1} &:= \sum_{l=0}^{m+1} (-1)^l d_l^m \end{aligned}$$

It is indeed a complex, i.e., $d^{m+1}\circ d^m = 0$, because

$$\delta_l^{m+1}\circ\delta_k^m = \delta_{k+1}^{m+1}\circ\delta_l^m, \qquad 0\le l\le k\le m$$

This gives a double complex whose terms are

$$\Omega^{p,q}_{D^\bullet/k} := \bigoplus_{1\le a_0<\cdots<a_q\le p} i_*\Omega^p_{D_{a_1\cdots a_q}}$$

We will denote its associated total complex by $\widetilde{\Omega}^\bullet_{D/k}$.

Definition 6.1. The de Rham cohomology is defined by

$$H^\bullet_{dR}(D/k) := \boldsymbol{H}^\bullet(D, \widetilde{\Omega}^\bullet_{D/k})$$

6.1. **Relative de Rham cohomology.** The relative case, with a pair (X, D)

$$D_i \hookrightarrow X$$
$$\Omega^\bullet_{X/k} \to (D_i \hookrightarrow X)_*\Omega^\bullet_{D_i/k}$$

Let $i : \mathrm{supp}(D) \hookrightarrow X$. Then there is a morphism of complexes

$$F : \Omega^\bullet_{X/k}[0] \to i_*\Omega^{\bullet,\bullet}_{D/k}.$$

We take the total complexes to get a morphism

$$f : \Omega^\bullet_{X/k} \to i_*\widetilde{\Omega}^\bullet_{D/k}$$

Now define a new complex M_f by

$$M_f := i_* \widetilde{\Omega_{D/k}}^{\bullet}[-1]$$

whose differential is

$$\begin{pmatrix} -d_D & f \\ 0 & d_X \end{pmatrix}.$$

The picture you should have is

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
 & & 0 \arrow[d] & & \Omega_X^{p+2} \\
0 \arrow[d] & & \Omega_X^{p+1} \arrow[urr] \arrow[dr] & & \\
\Omega_X^p \arrow[urr] \arrow[d] \arrow[dr] & & & \bigoplus \Omega_X \arrow[d] & \\
0 & \bigoplus \Omega_X \arrow[urr] \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Definition 6.2. The relative de Rham cohomology is given by

$$H^{\bullet}_{dR}(X, D; k) := \boldsymbol{H}^{\bullet}(X, \widetilde{\Omega}_{X,D/k})$$

6.2.

Lemma 6.3. *If $i : Z \hookrightarrow X$ is a closed immersion of k-varieties, and $\mathcal{I}$ is an injective sheaf of abelian groups on Z, then $i_*\mathcal{I}$ is also injective.*

Proof. If $j : \mathcal{F} \to \mathcal{G}$ is an injective morphism of sheaves... $\square$

Lemma 6.4. *If $i : Z \hookrightarrow X$ is a closed immersion and $\mathcal{F}^{\bullet}$ a cochain complex of sheaves bounded below, then there is a natural isomorphism*

$$H^{\bullet}(X, i_*\mathcal{F}^{\bullet}) \xrightarrow{\sim} H^{\bullet}(Z, \mathcal{F}^{\bullet})$$

Proof. Taking a quasi-isomorphism[7] $\mathcal{F}^{\bullet} \to \mathcal{I}^{\bullet}$ with $\mathcal{I}^{\bullet}$ a cochain complex of injective sheaves. Then

$$\begin{array}{rcl} H^{\bullet}(X, i_*\mathcal{F}^{\bullet}) & = & H^{\bullet}(\Gamma(X, i_*\mathcal{F}^{\bullet})) \\ \cdot & = & \cdot \end{array}$$

$\square$

[7]A quasi-isomorphism is a morphism which induces isomorphisms on the hypercohomology

Theorem 6.5. *If X is a smooth variety over k and D a normal crossing divisor on X, then there is a long exact sequence*

$$\cdots \to H^{p-1}_{dR}(D;k) \to H^p_{dR}(X,D;k) \to H^{p-1}_{dR}(X;k) \to H^p_{dR}(D;k) \to \cdots$$

Proof. Recall that

$$\widetilde{\Omega}^\bullet_{X,D/k} := i_*\widetilde{\Omega}^\bullet_{D/k}[-1] \oplus \Omega^\bullet_{X/k}$$

where the -1 denotes shift to the right, i.e., $A^p[-1] = A^{p-1}$ and examine the trivial sequence

$$0 \to i_*\Omega^\bullet_{D/k}[-1] \to \widetilde{\Omega}^\bullet_{X,D/k} \to \Omega^\bullet_{X/k} \to 0.$$

It induces the long exact sequence

$$\cdots \to H^p(X, i_*\widetilde{\Omega}^\bullet_{D/k}[-1]) \to H^p(X, \widetilde{\Omega}^\bullet_{X,D/k}) \to H^p(X, \Omega^\bullet_{X/k}) \to \cdots$$

This is isomorphic to

$$H^{p-1}(X, i_*\widetilde{\Omega}^\bullet_{D/k} \cong H^{p-1}_{dR}(D;k) \to H^p_{dR}(X,D;k) \to H^p_{dR}(X;k) \to \cdots .$$

□

7. Nori's basic lemma

Jonathan Skowera and Jun Yu on the 16th of July, 2011.

7.1. **Motivation.** Fix a subfield $k \leq \boldsymbol{C}$.

We will prove Nori's basic lemma. In analogy with the cellular complex for CW-complexes, Nori's lemma allows one to create a complex for affine k-varieties. Recall that the cellular complex associated to a CW-complex X is

$$\cdots \longrightarrow H_2^{sing}(X^2, X^1) \xrightarrow{d_2} H_1^{sing}(X^1, X^0) \xrightarrow{d_1} H_0^{sing}(X^0, \emptyset) \xrightarrow{d_0} 0$$

where X^k denotes the k-skeleton of X. Adding the long exact sequences for the pairs (X^k, X^{k-1}) to the above diagram shows from where the d_k come. The homology of the complex recovers the singular homology of X.

$$H_k^{sing}(X) = \frac{\ker d_k}{\mathrm{im}\ d_{k+1}}$$

The intuition is that $H_k(X^k/X^{k-1}) \cong H_k(\bigvee_i S^k) \cong \sum_{k\text{-cells}} e_i \boldsymbol{Z}$.

7.2. **Sheaves.**

Definition 7.1. A sheaf of sets $\mathcal{F}$ on a topological space is locally constant if the fibers of its étale space are all discrete. In other words, if every point $x \in X$ lies in a neighborhood U such that there exists a set S_U such that

$$\mathcal{F}(V) \cong S, \qquad \text{for all connected, open } V \subset U$$

For example, any constant sheaf, such as

$$\underline{\boldsymbol{Z}} : U \mapsto \boldsymbol{Z}, \qquad U \text{ connected.}$$

is locally constant. Another example is the sheaf of sections of a non-trivial $\boldsymbol{Z}/2$-torsor on a non-orientable manifold. Locally free sheaves are a non-example, except in trivial cases, since their sets of sections always grow upon restriction to subsets.

Definition 7.2 (Weakly constructible sheaf). Let X be a variety over k. Then a sheaf of sets $\mathcal{F}$ on $X(\boldsymbol{C})$ with the Euclidean topology is weakly constructible if there exists a stratification of X

$$X = \coprod_i X_i$$

by locally closed subvarieties $X_i \subset X$ such that every restriction to a cell, $\mathcal{F}_{X_i(\boldsymbol{C})}$, is locally constant. A stratification is decomposition into locally closed subvarieties called cells such that the closure of every cell is a union of cells, i.e., $\overline{X}_i = \cup_{X_j \subset \overline{X}_i} X_j$.

For example, let $X = \boldsymbol{A}^1$ with the stratification $X_0 = \{0\}$, $X_1 = \{y^2 = x^3 - x - 1\} \setminus X_0$ and $X_2 = \boldsymbol{A}^1 \setminus (X_1 \cup X_0)$. Then

$$\mathcal{F}|_{X_i} \cong \begin{cases} \underline{0}, & i = 0 \\ \underline{\boldsymbol{Z}/2}, & i = 1 \\ \underline{\boldsymbol{Z}/3}, & i = 2 \end{cases}$$

Definition 7.3 (Canonical exact sequence). Given a topological space X, an open subset $j : U \hookrightarrow X$, and its closed compliment $i : Y \hookrightarrow X$, then there is a functorial association of sheaves of abelian groups $\mathcal{F}$ on X to their corresponding canonical exact sequences,

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

where the first non-trivial morphism is the counit of the adjunction $j_! \dashv j^*$ and the second non-trivial morphism is the unit of the adjunction $i^* \dashv i_*$.

Remark 7.4. We will sometimes use the notation

$$\begin{aligned} \mathcal{F}_U &:= j_! j^* \mathcal{F} \\ \mathcal{F}_Y &:= i_* i^* \mathcal{F} \end{aligned}$$

7.3. **Basic lemmas.**

Lemma 7.5. *Let X be an affine variety over k and $Z \subset X$ a closed subvariety such that* $\dim Z < \dim X =: n$*. Then there exists an intermediary closed subvariety Y such that $X \supset Y \supset Z$,* $\dim Y < \dim X$ *and*

$$H^i_{sing}(X(\boldsymbol{C}), Y(\boldsymbol{C}), \boldsymbol{Q}) \cong \begin{cases} \boldsymbol{Q}^k, & i = n \\ 0, & i \neq n \end{cases}.$$

Example 7.6. If k is algebraically closed, the result follows from the Lefschetz hyperplane theorem. Let X be an affine variety over k, realized as a quasiprojective variety by

$$X \hookrightarrow \boldsymbol{A}^m_k \hookrightarrow \boldsymbol{P}^m_k.$$

Let $\overline{X}$ be the closure of X in $\boldsymbol{P}^n$ and $H := \boldsymbol{P}^m_k \setminus \boldsymbol{A}^m_k$. Let $H' \subset \boldsymbol{P}^m$ be the a hyperplane which intersects both $\overline{X}$ and $\overline{X} \setminus X$ transversely. Define Y to be the hyperplane section,

$$Y := X \cap H'.$$

Then the Lefschetz hyperplane theorem implies that

$$H_i(X, Y) = 0, \quad i \neq n.$$

Inductively repeating this construction, one can form a complex for X.

The proof of 7.5 follows immediately from a second lemma:

Lemma 7.7. *Let X be an affine variety over k of dimension n, and $\mathcal{F}$ be a weakly constructible sheaf on X. Then there exists a closed subvariety $i : Y \hookrightarrow X$ with open compliment $j : U \hookrightarrow X$ such that* $\dim Y < \dim X$ *and*

$$H^i(X, \mathcal{F}_U) \cong \begin{cases} \bigoplus_{i=1}^r \mathcal{F}_{X_i}, & i = n \\ 0, & i \neq n \end{cases}$$

Proof. Proof of the basic lemma. Assume we have proved 7.7. We have a closed $Z \subset X$, and we need a Y with the properties stated above. Define the sheaf $\mathcal{F} = \underline{Z}_{(X\ Z)}$ on X. It is weakly constructible

$$\begin{array}{cccc} & \underline{\boldsymbol{Q}} & & \underline{0} \quad . \\ & | & & | \\ X = & X \setminus Z & & \coprod Z \end{array}$$

The lemma 7.7 supplies a closed subvariety $V \subset X$. Define $Y := V \cup Z$. Then $Y \supset Z$ and $\dim Y < n$. But

$$(\underline{\boldsymbol{Q}}_{(X\setminus Z)})_{(X\setminus V)} = \underline{\boldsymbol{Q}}_{(X\setminus Y)}.$$

where the notation uses the convention 7.4. Thus

$$H^i_{sing}(X(\boldsymbol{C}), Y(\boldsymbol{C}), \boldsymbol{Q}) \cong H^i(X, \underline{Z}_{(X\setminus Y)}) \cong \begin{cases} \underline{Z}^r, & i = n \\ 0, & i \neq n \end{cases}.$$

using the isomorphism between singular cohomology and sheaf cohomology stated at the end of the lecture on sheaf cohomology. □

Before proving the second lemma, we need two new concepts.

Definition 7.8 (Leray spectral sequence)**.** For any morphism of ringed spaces $\pi : X \to Y$ and sheaf $\mathcal{F}$ on X, there are higher direct image sheaves $\{R^q\pi_*\mathcal{F} \mid q \geq 0\}$ defined by pushing forward an injective resolution of $\mathcal{F}$ by π_* and then taking homology sheaves of the resulting inexact complex. They serve as coefficients for the Leray spectral sequence

$$H^p(Y, R^q\pi_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}), \qquad p, q \geq 0.$$

Definition 7.9 (Variation on proper base change)**.** If $\pi : X \to Y$ is a proper morphism between ringed spaces, then for any sheaf $\mathcal{F}$ on X, the natural maps,

$$(R^q\pi_*\mathcal{F})_y \to H^q(\pi^*(y), \mathcal{F}|_{\pi^{-1}(y)}),$$

are isomorphisms for any $y \in Y$ and $q \geq 0$.

Proposition 7.10 (Variation)**.** *Let $f : P \to Q$ be a continuous map between locally compact, locally contractible topological spaces which is a fiber bundle and let $\mathcal{F}$ be a sheaf on P. Assume $\Delta \subset P$ is a closed subset such that*

(a) $\mathcal{F}|_{P\setminus\Delta}$ *is locally constant, and*
(b) $f|_\Delta : \Delta \to Q$ *is proper.*

Then

$$(R^q\pi_*\mathcal{F})_x \xrightarrow{\sim} H^q(\pi^{-1}(x), \mathcal{F}|_{\pi^{-1}(x)})$$

for all $x \in Q$ and all $q \geq 0$.

Proof. The statement is local, so we may assume that $P = T \times Q$, $p = pr_2 : T \times Q \to Q$ and $\Delta \subset K \times Q$ for some compact subset $K \subset T$. We may replace Δ by $K \times Q$. Shrinking Q further, we may assume $\mathcal{F}|_{(T \setminus K)} \times Q$ is a pull-back of a local system on $T \setminus K$. Now consider

$$0 \to \mathcal{F}|_{(T\setminus K)\times Q} \to \mathcal{F} \to \mathcal{F}_{K\times Q} \to 0$$

The result hold for $\mathcal{F}|_{(T\setminus K)\times Q}$ and $\mathcal{F}|_{K\times Q}$ by the Künneth formula and proper base change respectively, so it also holds for $\mathcal{F}$. ☐

Proof. Proof of second lemma.

(1) First, using Noether normalization, we let $\pi : X \to \boldsymbol{A}^n$ be a finite morphism. We may reduce to the case $X = \boldsymbol{A}^n$ by the following claim.

Claim 7.11. The sheaf $\pi_*\mathcal{F}$ on $\boldsymbol{A}^n$ is weakly constructible.

Generally, for any finite surjective morphisms $\pi : X \to X'$, $\pi \times \mathcal{F}$ is weakly constructibel if $\mathcal{F}$ is. By induction on dim X', there exists a closed subset $Y' \subset X'$ with dim $Y' <$ dim X' such that

$$\pi|_{X\setminus\pi'(Y')} : X' \setminus \pi^{-1}(Y) \to X' \setminus Y'$$

is a covering. Then $(\pi_*\mathcal{F})_{X\setminus Y}$ and $(\pi_*\mathcal{F})_Y$ are both weakly constructible, so

$$0 \to (\pi_*\mathcal{F})_{X\setminus Y} \to \pi_*\mathcal{F} \to (\pi_*\mathcal{F})_Y$$

is exact.

(2) Now induct on n. Let $\mathcal{F}$ be weakly constructible on $\boldsymbol{A}^n$. By replacing $\mathcal{F}$ by some $\mathcal{F}_V$, we can assume that there exist $f \in k[x_1, \ldots, x_n]$ such that $\mathcal{F}|_{D(f)}$ is locally constant. We may therefore assume that $\mathcal{F}|_{D(f)}$ is a local system and $\mathcal{F}|_{V(f)} = 0$. After a change of coordinates, we may assume f has no multiple factrs. We may write f in the form

$$f = x_n^k + x_n^{k-1} f_1 + \cdots + x_n f_{k-1} + f_k, \quad f_i \in k[x_1, \ldots, x_{n-1}]$$

Let $p : \boldsymbol{A}^n \to A^{n-1}$ be the projection $p : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$.

(3) By variation of proper base change,

$$(R^q p_*\mathcal{F})_x \to H^q(\pi^*(x), \mathcal{F}|_{\pi^{-1}(x)})$$

is an isomorphism for any x. It follows that $R^p p_*\mathcal{F} = 0$ if $q \neq 0, 1$. Also

$$(p_*^i \mathcal{F})_x \cong H^0(\boldsymbol{A}^1, \mathcal{H})$$

for some local system $\mathcal{H}$ extended by 0 on a proper open subset of $\boldsymbol{A}^1$. Hence $H^0(\boldsymbol{A}, \mathcal{H}) = 0$. So $R^0 p_*\mathcal{F} = 0$. The Leray spectral sequence implies that

$$H^i(\boldsymbol{A}^n, \mathcal{F}) \cong H^{i-1}(\boldsymbol{A}^n, R^1\pi_*\mathcal{F}).$$

(4)

Claim 7.12. The sheaf $R^1 p_*\mathcal{F}$ is weakly constructible.

Recall that $\mathcal{F}|_{D(f)}$ is locally constant. By 3, the stalk of $R^1 p_*\mathcal{F}$ at a point is $H^q(\boldsymbol{A}^1, \mathcal{F}|_{\boldsymbol{A}^1})$, where $\mathcal{F}_{\boldsymbol{A}^1}$ is locally constant outside a finite set of points $S \subset \boldsymbol{A}^1(\boldsymbol{C})$. In the complex analytic space $\boldsymbol{A}^1(\boldsymbol{C})$, let T be a tree[8] with vertices exactly at the points of S. Note that

[8]A tree is a graph with no cycles. In our case, we assume the tree is embedded such that the edges are diffeomorphic to the unit interval $[0, 1]$.

$\boldsymbol{A}^1\backslash S$ deformation retracts onto T. Then the isomorphism of sheaf cohomology of locally constant sheaves with singular cohomology, together with the homotopy invariance of singular cohomology, gives

$$\begin{aligned} H^i(\boldsymbol{A}^1, \mathcal{F}) &\cong H^i_{sing}(\boldsymbol{A}^1(\boldsymbol{C}), S, \boldsymbol{Q}) \\ &\cong H^i_{sing}(T, S, \boldsymbol{Q}) \\ &\cong H^i(T, \mathcal{F}|_T). \end{aligned}$$

Covering T with the open cover formed by the stars of each vertex,

$$\{U_v \mid \forall v \in vertices(T), U_v := v \cup (\cup_{e \ni v} \mathring{e})\},$$

where the union joins the interiors $\mathring{e}$ of edges containing the vertex v. The spectral sequence for this covering implies that

$$H^1(T, \mathcal{F}|T) \xrightarrow{\sim} \oplus_e \mathcal{F}_{\mu(e)}$$

where $\mu(e)$ is the center point of the edge e, and we have used the fact that an edge deformation retracts onto its center point, so $H^0(e, \mathcal{F}|e) \cong \mathcal{F}_{\mu(e)}$. This proves the claim.

(5) Applying the induction hypothesis, we assume the lemma holds for $R^1 p_* \mathcal{F}$ on $\boldsymbol{A}^{n-1}$. We get a closed subvariety $Z \hookrightarrow A^{n-1}$ with open compliment $V \hookrightarrow A^{n-1}$ such that

$$H^j(\boldsymbol{A}^{n-1}, (R^1 p_* \mathcal{F})_V) = 0, \quad \forall j \neq 0, 1$$

Claim 7.13. There is an isomorphism of sheaves on $\boldsymbol{A}^{n-1}$,

$$R^q p_*(\mathcal{F}_{p^{-1}(V)}) \cong (R^q p_* \mathcal{F})_V$$

Consider the exact sequence

$$0 \to \mathcal{F}_{p^{-1}(V)} \to \mathcal{F} \to \mathcal{F}|_{\boldsymbol{A}^n \backslash p^{-1}(V)}$$

The last two terms of the sequence satisfy the hypothesis of the variation of proper base change, so they satisfy its conclusion. For that reason, $\mathcal{F}|_{p^{-1}(V)}$ also satisfies the conclusions of the lemma, i.e.,

$$\begin{aligned} (R^q p_* \mathcal{F}_{p^{-1}(V)})_x &\cong H^q(p^{-1}(x), \mathcal{F}_{p^{-1}(V)}|_{p^{-1}(x)}) \\ &= \begin{cases} H^q(p^{-1}(x), \mathcal{F}|_{p^{-1}(x)}), & x \in V \\ 0, & \text{otherwise} \end{cases} \\ &= (R^q \pi_* \mathcal{F})_V|_x. \end{aligned}$$

This proves the claim.

Finally, from the claim, it follows that

$$H^i(\boldsymbol{A}^n, \mathcal{F}_{p^{-1}(V)}) = H^{i-1}(\boldsymbol{A}^{n-1}, R^1 p_*(\mathcal{F}_{p^{-1}(V)})) = H^{i-1}(\boldsymbol{A}^{n-1}, (R^1 p_* \mathcal{F})_V) = 0.$$

□

8. Tannakian categories

Claudia Scheimbauer and Andrin Schmidt on the 16th of July, 2011.

Let the field k be fixed.

8.1. **Affine group schemes and the category of representations.**

Definition 8.1 (Affine group scheme)**.** Affine group scheme over k is a scheme $G = \mathrm{Spec}\ A$ for a k-algebra A, and three morphisms

$$\begin{aligned} m : G \times G &\rightarrow G \\ \epsilon : \mathrm{Spec}\ k &\rightarrow G \\ i : G &\rightarrow G \end{aligned}$$

which satisfy diagrams of associativity, identity, inverse, and their compatibility. Alternatively, an affine group scheme over k is a functor

$$F : \mathrm{Alg}_k \rightarrow \mathrm{Gp}$$

such that the composition with forgetful functor to sets is representable.

Recall the equivalence of categories

$$\mathrm{Alg}_k \rightarrow \mathrm{Aff}_k$$

Definition 8.2 (Hopf algebra)**.** A commutative Hopf algebra A is the dual of an affine group scheme over k, i.e., a k-algebra A

$$\begin{aligned} \Delta : A &\rightarrow A \otimes A \\ \epsilon : A &\rightarrow k \\ i : A &\rightarrow A \end{aligned}$$

Let R be a k-algebra and $G(R) = \mathrm{Hom}_k(A, R)$. Then the functor

$$F_G : R \rightarrow \mathrm{Hom}(A, R)$$

is a group with the operation

$$A \longrightarrow A \otimes A \xrightarrow{\phi \otimes \psi} R \otimes R \longrightarrow R\ , \quad \phi, \psi \in \mathrm{Hom}(A, R)$$

Definition 8.3. An algebraic group over k is the dual under Spec of a finitely generated commutative Hopf algebra.

Theorem 8.4. *Every commutative Hopf algebra is a direct limit of finitely generated Hopf algebras. Every affine group scheme is an inverse limit of algebraic groups.*

Example 8.5. Let $A = k[T, T^{-1}]$ be the Hopf algebra with operations

$$\begin{aligned} T &\mapsto T \otimes T \\ T &\mapsto 1 \\ T &\mapsto T^{-1} \end{aligned}$$

Then

$$F_G(R) = \mathrm{Hom}_k(k[T, T^{-1}], R) \cong R^*$$

Definition 8.6 (Representation of affine group scheme)**.** Let $G = \mathrm{Spec}\ A$ be an affine group scheme. A representation of G is a pair (V, τ), where V is a k-vector space and $\tau : V \rightarrow A \otimes V$

is a k-linear morphism such that

$$\begin{tikzcd} V \arrow[r, "\tau"] \arrow[dr, "1"'] & A \otimes V \arrow[d, "End"'] \\ & k \otimes V \cong V \end{tikzcd}$$

and

$$\begin{tikzcd} V \arrow[r, "\tau"] \arrow[d, "\tau"] & A \otimes V \arrow[d, "1\otimes\tau"] \\ A \otimes V \arrow[r, "A\otimes 1"'] & A \otimes A \otimes V \end{tikzcd}$$

Definition 8.7. A morphism between representations $f : (V_1, \tau_1) \to (V_2, \tau_2)$

$$\begin{tikzcd} V_1 \arrow[r, "\tau_1"] \arrow[d, "f"'] & A \otimes V_1 \arrow[d, "1\otimes f"] \\ V_2 \arrow[r, "\tau_2"'] & A \otimes V_2 \end{tikzcd}$$

Denote the category of *finite dimensional* representations by Rep_G

Proposition 8.8. *The category* Rep_G *satisfies the following properties*

(i) *It is an abelian category.*
(ii) *It is monoidal with identity* $\mathbf{1} : k \to k \otimes k$*.*
(iii) *The functor*

$$\begin{array}{rcl} \mathrm{Rep}_G & \to & \mathrm{Hom}(\mathrm{Rep}_G \otimes X, Y) \\ X^\vee \otimes Y & \mapsto & \underline{\mathrm{Hom}}(X, Y) \end{array}$$

where the dual is $X^\vee := \underline{\mathrm{Hom}}(X, \mathbf{1})$*.*

(iv)

$$\mathrm{End}(\mathbf{1}) \cong k$$

(v)

$$\omega : \mathrm{Rep}_k \to \mathrm{Vect}_k$$

8.2. Neutral Tannakian categories.

Definition 8.9 (Tensor category)**.** A tensor category $(C, \otimes)$ is a functor

$$\otimes : C \times C \to C$$

together with an isomorphism

$$\phi_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

natural in X, Y, and Z, and an identity functor

$$\mathbf{1} : \begin{array}{rcl} U & \to & U \otimes U \\ T & \mapsto & U \otimes T \end{array}$$

Definition 8.10 (Internal hom)**.** The internal hom of a tensor category is

$$T \mapsto \mathrm{Hom}(T \otimes X, Y)$$

Definition 8.11 (Dual object)**.** An object X in a tensor category C with internal hom's is dualizable if there exists an object $X^\vee$ such that

$$X^\vee \otimes Y \xrightarrow{\sim} \underline{\mathrm{Hom}}(X, Y)$$

Definition 8.12 (Rigid category)**.** A category is rigid if every object is dualizable.

Definition 8.13 (Netural Tannakian category)**.** A neutral Tannakian category is a category C which is

(i) an abelian category
(ii) a tensor category
(iii) a rigid category
(iv)
$$\mathrm{End}(\mathbf{1}) \cong k$$
(v) The functor
$$\omega : C \to \mathrm{Vect}_k$$
is k-linear, faithful, exact and monoidal, i.e., compatible with the tensor product $\otimes$.

Example 8.14. Examples of neutral Tannakian categories include Rep_G, Vect_k, and $\mathrm{Vect}_k^{\boldsymbol{Z}}$[9]

8.3. **Tannaka duality.**

Definition 8.15. Let $(C, \otimes)$ be a tensor category, and $\omega, \eta : \boldsymbol{C} \to \mathrm{Vect}_k$, two functors. Then

$$\mathrm{Hom}^{\otimes}(\omega, \eta) : \begin{array}{rcl} \mathrm{Alg}_k & \to & \mathrm{Set} \\ R & \mapsto & \mathrm{Hom}^{\otimes}(\omega, \eta)(R) \end{array}$$

where $\mathrm{Hom}^{\otimes}$ denotes natural transformations which respect the tensor structure, as described below.

Definition 8.16. An element of the set $\mathrm{Hom}^{\otimes}(\omega, \eta)(R)$ is of the form

$$\sigma_X : R \otimes \omega(X) \to R \otimes \eta(X), \qquad X \in C$$

such that for all $f : X \to Y$, the diagram

$$\begin{tikzcd} R \otimes \omega(X) \arrow[r, "1\otimes\omega(f)"] \arrow[d, "\sigma_X"'] & R \otimes \omega(Y) \arrow[d] \\ R \otimes \eta(X) \arrow[r] & R \otimes \eta(Y) \end{tikzcd}$$

commutes,

$$\begin{tikzcd} R \otimes \omega(\mathbf{1}) \arrow[r, "\sigma_{\mathbf{1}}"] \arrow[d, equal] & R \otimes \eta(\mathbf{1}) \arrow[d, equal] \\ R \arrow[r, "1"'] & R \end{tikzcd}$$

and for all $X, Y \in C$,

$$\sigma_{X \otimes Y} = \sigma_X \otimes \sigma_Y$$

[9]We denote by $\mathrm{Vect}_k^{\boldsymbol{Z}}$ the category of $\boldsymbol{Z}$-graded k-vector spaces.

Theorem 8.17 (Tannaka's Theorem)**.** *Let G be an affine group scheme over k, and $\omega : \mathrm{Rep}_G \to \mathrm{Vect}_k$. Then there is an isomorphism of functors*

$$F_G \to \mathrm{Aut}^{\otimes}(\omega)$$

Let $G = \mathrm{Spec}\ A$, and R be a k-algebra,

$$\begin{array}{rcl} F_G(R) = \mathrm{Hom}(A, R) & \to & \mathrm{Aut}^{\otimes}(\omega)(R) \\ \xi & \mapsto & \sigma \end{array}$$

Take $(V, \tau) = X$. Then

$$V \xrightarrow{\tau} A \otimes V \xrightarrow{\xi \otimes 1} R \otimes V$$

implies

$$R \otimes V \xrightarrow{\sigma_X} R \otimes V$$

Theorem 8.18. *Let C be a netural Tannakian category over k with fiber functor $C \cong \mathrm{Rep}_G$ and G represents the functor $\mathrm{Aut}^{\otimes}(\omega)$. This induces an equivalence of categories*

$$\textit{(Category of algebraic group schemes)} \xrightarrow{\sim} \textit{(Category of Tannakian categories)}$$

Example 8.19. The category Vect_k corresponds to the trivial group scheme Spec k.

Example 8.20. The category $\mathrm{Vect}_k^{\boldsymbol{Z}}$ corresponds to the multiplicative group $\boldsymbol{G}_m$.

$$\mathrm{Aut}^{\otimes}(\omega)(R) \cong \boldsymbol{G}_m(R) \cong R^*$$

and

$$\begin{tikzcd} k \arrow[r, "f"] \arrow[d, "\omega"'] & \omega_0 \arrow[d, "\omega"] \\ k(f) \arrow[r, "\omega"] & \omega_0 \end{tikzcd},$$

where

$$\sigma_\omega|_{\omega[0]} = 1.$$

Also,

$$\begin{tikzcd} k[1] \arrow[r] \arrow[d, "\omega"'] & \omega_1 \arrow[d, "\omega"] \\ k \arrow[r] & \omega_1 \end{tikzcd},$$

where

$$\sigma_\omega|_{\omega_1} = \alpha \in k^*$$

for some α fixed for σ.

9. Nori's diagram category

Jonas von Wangenheim on the 16th of July, 2011.

Definition 9.1 (Diagram)**.** A diagram D is an oriented graph, i.e., it consists of a set of objects (vertices) and for all $p, q \in D$, a set of morphisms (edges) $D(p, q)$.

Example 9.2. Any small category is a diagram, i.e., a category whose objects form a set and whose Hom's always form sets. It has a rule for composing morphisms which is superfluous to the diagram structure.

Definition 9.3 (Morphism of diagrams)**.** A morphism $F : D \to D'$ between diagrams D and D' consists of a function $F : D \to D$ on objects and, for all $p, q \in D$, a function on morphisms $D(p,q) \to D'(F(p), F(q))$.

If D' is a category, we call this map a representation of D.

Theorem 9.4. *Let D be a diagram and*

$$T : D \to \mathrm{Vect}_k$$

a representation of D in finite dimensional k-vector spaces. Then there is a k-linear abelian category $C(T)$, called the diagram category, such that T factors through a representation $\widetilde{T}$:

$$\begin{tikzcd}
D \arrow[rr, "T"] \arrow[dr, "\widetilde{T}"'] & & \mathrm{Vect}_k \\
& C(T) \arrow[ur, "ff_T"'] &
\end{tikzcd}$$

where ff_T is a faithful k-linear functor.

Furthermore, for every k-linear abelian category $\mathcal{A}$ with such factorizations, there exists a k-linear faithful functor unique up to isomorphism.

$$\begin{tikzcd}
& A \arrow[dr] & \\
D \arrow[ur] \arrow[rr] \arrow[dr] & & \mathrm{Vect}_k \\
& C(T) \arrow[ur] \arrow[uu, dashed, "\exists! L(f)"] &
\end{tikzcd}$$

Then $C(T)$ arises as the category of finitely generated comodules over a certain coalgebra $A(D,T)$.

Proof. (Sketch) **Step one.** Assume D has finitely many objects. Then every $p \in D$ corresponds to a k-vector space T_p and

$$\prod_{p \in D} \mathrm{End}(T_p)$$

is a k-algebra with subalgebra

$$\mathrm{End}(T) := \left\{ c \in \prod_{p \in D} \mathrm{End}(T_p) \;\middle|\; \forall p, q \in D, a \in D(p,q), \quad \begin{tikzcd} T_p \arrow[r, "T_a"] \arrow[d, "e_p"'] & T_q \arrow[d, "T_a"] \\ T_p \arrow[r, "e_p"'] & T_q \end{tikzcd} \right\}$$

Form $C(T) = \mathrm{End}(T) - \text{modules}$. Then

$$\begin{array}{rcl} \mathrm{End}(T) \times T_p & \to & T_p \\ ((C_p)_{p\in D}, m) & \mapsto & e_p(m) \end{array}$$

is well-defined left-action on T_p, making T_p into an $\mathrm{End}(T)$-module. All T_a become $\mathrm{End}(T)$-linear, and

$$\begin{array}{ccccc} D & \to & C(T) & \to^{ff_T} & \mathrm{Vect}_k \\ p & \mapsto & T_p & \mapsto & T_p \end{array}$$

Let V be a k-vector space, and A a k-algebra. Define $A^\vee = \mathrm{Hom}(A, R)$. There is a canonical bijection between A-module structures and $A^\vee$-coalgebra structures on the dual coalgebra $A^\vee$.

$$\mathrm{Hom}(A \otimes V, V) \cong \mathrm{Hom}(V, \mathrm{Hom}(A, V)) \cong \mathrm{Hom}(V, V \otimes A^\vee)$$

Thus

$$(\text{Category of } \mathrm{End}(T)\text{-modules}) \cong (\text{Category of } \mathrm{End}(T)^\vee\text{-modules}).$$

Define

$$A(D,T) := \mathrm{End}(T)^\vee$$

We obtain the factorization

$$D \xrightarrow{\widetilde{T}} \mathrm{End}(T)^\vee\text{-Comod} \xrightarrow{ff_T} \mathrm{Vect}_k$$

Step two. Now let D be arbitrary.

$$\begin{array}{rcl} T : D & \to & \mathrm{Vect}_k \\ T_f : F & \to & \mathrm{Vect}_k \end{array}$$

Consider finite subsets $F \subset F' \subset D$.

$$\begin{tikzcd} \prod_{F'\in D} \mathrm{End}(T_p) \arrow[r, two heads] & \prod_{F\in D} \mathrm{End}(T_p) \\ \mathrm{End}(T_{F'}) \arrow[r] \arrow[u] & \mathrm{End}(T_F) \arrow[u] \end{tikzcd}$$

induces a functor from $\mathrm{End}(T_F)$-modules to $\mathrm{End}(T_{F'})$-modules. Define

$$C(T) := \lim_{F\subset D \text{ finite}} \mathrm{End}(T_F)\text{-Mod} = \lim(\mathrm{End}(T_F)^\vee\text{-Comod}) = (\lim \mathrm{End}(T_F)^\vee)\text{-Comod}.$$

This is an abelian category. $\square$

Theorem 9.5. *Let $\mathcal{A}$ be an abelian k-linear category, and $T : \mathcal{A} \to \mathrm{Vect}_k$, a faithful, k-linear, exact representation. Then the diagram category $C(T)$ satisfies*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "T"] \arrow[dr, "\widetilde{T}"'] & & \mathrm{Vect}_k \\ & C(T) \arrow[ur, "ff_t"'] & \end{tikzcd}$$

and $\widetilde{T}$ is an equivalence of category.

Proof. Let's consider $\mathcal{A}$ in the case there is a k-linear, exact, faithful functor such that

$$\begin{tikzcd}
 & \mathcal{A} \arrow[r] \arrow[dr, "S"] & C(S) \arrow[d] \\
D \arrow[ur, "F"] \arrow[rr, "T"] \arrow[dr, "\widetilde{T}"'] & & \mathrm{Vect}_k \\
 & C(T) \arrow[ur, "ff_T"'] &
\end{tikzcd}$$

We will construct a functor $C(T) \to C(S)$ that makes the above diagram commutes. Then

$$\begin{tikzcd}
D \arrow[rr, "F"] \arrow[d, "\widetilde{T}"] & & \mathcal{A} \arrow[d, "\widetilde{S}"] \\
\mathrm{End}(T)\text{-Mod} \arrow[rr] \arrow[dr, "ff_T"'] & & \mathrm{End}(S)\text{-Mod} \arrow[dl, "ff_S"'] \\
 & \mathrm{Vect}_k &
\end{tikzcd},$$

and also,

$$\prod_{q\in\mathcal{A}} \mathrm{End}(S_q) \twoheadrightarrow \prod_{q\in\mathrm{im}\ F} \mathrm{End}(S_q) \longrightarrow \prod_{p\in D} \mathrm{End}(S\circ F)$$

which restricts to

$$\mathrm{End}(S) \to End(S\circ F) \cong \mathrm{End}(T)$$

and induces an equivalence between $\mathrm{End}(T)$-Mod and $\mathrm{End}(S)$-Mod.

Note 9.6. The most important part for further talks is probably the method of computing $C(T)$.

Consider

$$\widetilde{T}: \begin{array}{ccc} \mathcal{A} & \to & C(T) \\ \langle p\rangle \cong \langle F\rangle & \to & \mathrm{End}(T_F)\text{-Mod} \cong \mathrm{End}(T_{\{p\}})\text{-Mod} \end{array}$$

where $p = \oplus_{p_i\in F} p_i$.

Without loss of generality, assume $\mathcal{A} = \langle p\rangle$. We want to show that $\mathcal{A} \to End(T_{\{p\}})$-Mod is an equivalence of categories.

$$\begin{array}{ccccc} \mathrm{End}(T)\text{-Mod} & \to & \mathcal{A} & \to^{\Gamma} & \mathrm{Vect}_k \\ M \cong K^n & \mapsto & \mathrm{Hom}(M,p) \cong p^n & \mapsto & \mathrm{Hom}(M, T_p) \end{array}$$

The second map comes from the following consideration.

$$\begin{tikzcd}
MN \arrow[r, "f"] \arrow[d] & N \arrow[d] \\
K^m \arrow[r, "A"] & K^n
\end{tikzcd} \mapsto (A^T : p^n \to p^m) \mapsto (A^T : T_p^n \to T_p^n)$$

Then

$$\circ f : \mathrm{Hom}(N, T_p) \to \mathrm{Hom}(M, T_p)$$

and

$$\circ A : \mathrm{Hom}(K^n, T_p) \to \mathrm{Hom}(K^m, T_p)$$

This allows us to define Γ. But Γ may be defined more generally for R-modules.

$$T : \begin{array}{rcl} \mathcal{A} & \to & R - Mod \\ \mathrm{Hom}(T_p, p) & \mapsto & \mathrm{Hom}(T_p, T_p) \end{array}$$

Then

$$g : \begin{array}{rcl} \mathrm{Hom}(T_p, T_p) & \to & \bigoplus_{i=1}^n \mathrm{Hom}_k(T_p, T_p) \\ \phi & \mapsto & (T_{a_i} \circ \phi - \phi \circ T_{a_i})_{i=1}^n \end{array}$$

Then

$$\ker g = \mathrm{End}(T)$$

Furthermore,

$$\tilde{g} : \begin{array}{rcl} \mathrm{Hom}(T_p, p) & \to & \bigoplus \mathrm{Hom}(T_p, p) \\ u & \mapsto & (a_i \circ u - u \circ T_{a_i}) \end{array}$$

Then $\ker \tilde{g}$ is a preimage of $\mathrm{End}(T)$ under T. □

10. Multiplicative structure of diagrams and their localization

Martin Gallauer on the 16th of July, 2011.

Recall the situation of the previous talk

$$\begin{tikzcd} & C(T) \arrow[dr, "ff_t"] & \\ D \arrow[ur, "\widetilde{T}"] \arrow[rr, "T"'] & & R\text{-Mod} \end{tikzcd}$$

We would like to put a tensor structure on the category $C(T)$, the category of finitely generated R-modules, which has an $A(T)$-comodule structure, where $A(T) = \mathrm{End}(T)^\vee$.

10.1. Let D be a diagram, $p \in D$ an object, and $\mathbf{1}_p$ a unit.

Definition 10.1. A grading on D given by $|\cdot| : D \to \boldsymbol{Z}, p \mapsto |p|$. If D_1 and D_2 are graded, then their product $D_1 \times D_2$ is graded by $(f, g) \mapsto |f| + |g|$.

Definition 10.2.

$$E(D_1 \times D_2) = \{\mathbf{1} \times \alpha | \alpha \in E(D_2)\} \cup \{\alpha \times \mathbf{1} | \alpha \in E(D_1)\}$$

Definition 10.3. Let D be a graded diagram. Then a commutative product structure on D is a function

$$\times : D \times D \to D$$

be a graded (degree 0) diagram map together with edges

$$\begin{array}{rcl} \alpha_{f,g} : f \times g & \to & g \times f \\ \beta_{f,g,h} : f \times (g \times h) & \to & (f \times g) \times h \end{array}$$

Since these are edges, there is no compatibility diagrams. Note this is slightly more precise than the preprint, where they assume there are strict equalities $f \times g = g \times f$ and $f \times (g \times h) = (f \times g) \times h$.

Let D be a graded diagram with a commutative product structure, and A, a graded commutative representation of D in R-free[10],

$$T : D \to R\text{-free}$$

together with isomorphisms

$$\tau_{f,g} : T(f \times g) \xrightarrow{\sim} T(f) \otimes T(g), \qquad \forall f, g \in D$$

such that

(1)

$$\begin{tikzcd} T(f \times g) \arrow[d, "T(\alpha_{f,g})"] & T(f) \otimes T(g) \arrow[l, "\tau^{-1}_{f,g}"'] \arrow[d, "(-1)^{|f||g|}"] \\ T(g \times f) \arrow[r, "\tau_{g,f}"'] & T(g) \otimes T(f) \end{tikzcd}$$

(2) For all $(\gamma : f \to f') \in D(f, f')$ and for all vertices $g \in D$, we require that the following diagram commutes

$$\begin{tikzcd} T(f \times g) \arrow[r, "T(\alpha\times\mathbf{1})"] \arrow[d, "\tau_{f,g}"'] & T(f' \times g) \arrow[d, "\tau_{f'}"] \\ T(f) \otimes T(g) \arrow[r, "T(\alpha)\otimes\mathbf{1}"'] & T(f') \otimes T(g) \end{tikzcd}$$

and that a similar diagram for $1 \times \alpha$ commutes.

(3) The diagram

$$\begin{tikzcd} T(f \times (g \times h)) \arrow[r, "T(\beta_{f,g,h})"] \arrow[d, "\tau"'] & T((f \times g) \times h) \arrow[d, "\tau"] \\ T(f) \otimes T(g \times h) \arrow[d, "\tau"'] & T(f \times) \otimes T(h) \arrow[d, "\tau"] \\ T(f) \otimes (T(g) \otimes T(h)) \arrow[r] & (T(f) \otimes T(g)) \otimes T(h) \end{tikzcd}$$

commutes.

Proposition 10.4. *Let D be a diagram and T a representation of it in R-Mod. Then $A(T)$ is naturally a*

Proof. There is an R-linear morphism

$$A(T) \times A(T) \to A(T)$$

But $\lim_F A(T|F) = A(T)$, so $\lim_F (A(T|F) \times A(T|F)) = A(T) \times A(T)$, so it suffices to show

$$A(T|F) \otimes A(T|F) \to A(T|F')$$

is R-linear. Then

$$\mathrm{End}(T|F') \to \mathrm{End}(T|F) \otimes \mathrm{End}(T|F)$$

[10]We denote the category of free R-modules by R-free.

with $F' \supset \{f \times g | f, g \in F\}$. But

$$\begin{aligned}
\mathrm{End}(T|F) \otimes \mathrm{End}(T|F) \subset \prod_f \mathrm{End}(T(f)) \otimes \prod_g \mathrm{End}(T(g)) &\cong \prod_{f,g} \mathrm{End}(T(f)) \otimes \mathrm{End}(T(g)) \\
&\cong \prod_{f,g} \mathrm{End}(T(f) \otimes T(g)) \\
&\cong \prod_{f,g} \mathrm{End}(T(f \times g)) \ni (\alpha_{f\times g})_{f,g}
\end{aligned}$$

The rest of the verification is ommitted. □

Corollary 10.5. *Let D be a diagram, and T a representation of it in R-Mod. Then $C(T)$ is a tensor category with unit R, and $ff_T : C(T) \to R$-Mod is a tensor functor.*

Proof. Let $X, Y \in C(T)$. Then

$$X \otimes Y \to (A(T) \otimes X) \otimes (A(T) \otimes Y) \cong (A(T) \otimes A(T)) \otimes (X \otimes Y) \to A(T) \otimes (X \otimes Y).$$

□

10.2. **Localization of Nori's diagram category.** We begin with a few definitions, and then discuss the main theorem.

Definition 10.6 (Localized diagram)**.** Let D^{eff} be a graded diagram, $f_0 \in D$ and $n_0 = |f_0| \in \boldsymbol{Z}$. Then

$$\begin{tikzcd}
& C(T)^{eff} \arrow[dr] & \\
D^{eff} \arrow[ur, "\widetilde{T}^{eff}"] \arrow[rr, "T^{eff}"'] & & R\text{-free}
\end{tikzcd}$$

The localized diagram D of D^{eff} with respect to f_0 is defined by

$$\begin{aligned}
D &= \{\text{symbols } f(n) \mid f \in D, n \in \boldsymbol{Z}\} \\
D(f,g) &= \{\text{symbols } \alpha(n) \mid \alpha \in D^{eff}(f,g), n \in \boldsymbol{Z}\} \cup \{(f \times f_0)(n) \to f(n+1)\}
\end{aligned}$$

It is a graded diagram with the grading

$$|f(n)| := |f|^{eff} + n \cdot n_0$$

and has product structure

$$f(n) \times g(m) := (f \times g)(n+m).$$

We assume that

- $T^{eff}(f_0)$ is a rank one module.
- $2|n_0$.

We call this localization, since the endofunctor

$$- \otimes \widetilde{T}^{eff}(f_0) : C(T^{eff}) \to C(T^{eff})$$

satisfies a universal property?

Lemma 10.7. *The functor T^{eff} extends uniquely to $T : D \to R$-free such that*
$$T(f(n)) = T^{eff}(f) \otimes T^{eff}(f_0)^{\otimes n}$$

Proof. Ommitted, because it is clear enough. □

Proposition 10.8. *Let D^{eff}, T, and f_0 be as above. Then*
1. *The category $C(T)$ is the localization of $C(T^{eff})$ with respect to $\widetilde{T}^{eff}(f_0)$.*
2. *The category $A(T)$ is the localization of $A(T^{eff})$ with respect to $\chi \in A(T^{eff})$, where*
$$\begin{array}{rcl} A(T^{eff}|_{f_0}) \cong \mathrm{End}(T(f_0)) & \longleftrightarrow & \mathrm{End}(T(f_0)) \\ \chi & \longleftarrow & \mathbf{1} \end{array}$$

11. Nori's rigidity criterion

Thomas Preu on the 18th of July, 2011.

11.1. Let $R = k$ be a field of characteristic 0.

Theorem 11.1 (Tannaka duality). *Let (C, ω) be a neutral k-Tannakian category, then there is an affine group scheme G such that (C, ω) is equivalent to (Rep_G, F). Conversely, if G is an affine k-group shceme, then (Rep_G, F) is a neutral k-Tannakian category. [Szamuely, 6.5]*

Remark 11.2. In the theorem above, G is algebraic if and only if $C \cong \langle X \rangle_{\otimes,\mathrm{rigid}}$[11] for some $X \in C$. [Milne, Deligne, LNM 900, Prop 2.20]

Remark 11.3. If one drops rigidity from the definition of a neutral Tannakina category, then the above proposition holds, with affine group scheme replaced with k-monoid scheme M in place of G.

Lemma 11.4. *Let G be an affine algebraic group over k and $M \subset G$ a closed, affine algebraic submonoid. Then M is a fortiori an affine algebraic group over k.*

Proof. We would like to show that
$$\begin{tikzcd} G \arrow[r, "inv"] & G \\ M \arrow[u, hook, "inv|M"] \arrow[r, dashed] & M \arrow[u, hook] \end{tikzcd},$$
where inv is the inverse morphism. So we base change to the algebraic closure $\overline{k}$, because this a purely topological property.

Over $\overline{k}$, we may work wiht $\overline{k}$-rational points instead of the group scheme, since we are proving a topological property. We must show that for any $g \in M(\overline{k}) \cong \overline{M}^{top}$, $g^{-1} \in M(\overline{k})$. Since $M(\overline{k})$ is a monoid,
$$g \cdot M(\overline{k}) \subset M(\overline{k})$$
Inductively applying $g^{-1} \in G(\overline{k})$ to both sides, we arrive at a sequence
$$\cdots \subset g^n M(\overline{k}) \subset g^{k-1} M(\overline{k}) \subset \cdots \subset M(\overline{k})$$

[11] This means the minimal rigid, tensor, abelian category generated by X.

Then Z-closed subsets of a noetherian affine scheme, so it stabilizers, i.e., there exists an $n \in \boldsymbol{N}$ such that

$$g^{n+1}M(\overline{k}) = g^k M(\overline{k})$$

Multiplying by $(g^{-1})^{(n+1)}$ gives

$$M(\overline{k}) = g^{-1}M(\overline{k}) \ni g^{-1}$$

□

Definition 11.5 (Neutral Tannakian category except for rigidity)**.** Let $V \in C$. Then we say that V admits a perfect duality if there is a morphism $q : V \otimes V \to \mathbf{1}$ such that $T(q)$ is a non-degenerate bilinear pairing over k.

Lemma 11.6. *Let $V \in C$ be a generator, i.e., $C = \langle V \rangle_{\otimes}$[12]. If V admits a perfect duality pairing, then C is rigid.*

Proof. Tannakian duality says that the dual of V is an affine monoid scheme M over k. Translating this to Hopf algebras, we have a bi-algebra A such that Spec $A = M$. By 11.4, it suffices to give a closed immersion of M into an affine algebraic group, because then M must be an affine algebraic group over k and by Tannaka duality, C must be rigid.

Equivalently, we may find a finitely generated Hopf algebra $\widetilde{A}$ with a surjection $\widetilde{A} \twoheadrightarrow A$.

We know that $A = \lim A_n$, where A_n is the associated coalgebra to the abelian category $C_n = \langle \mathbf{1}, V, V^{\otimes 2}, \dots, V^{\otimes n} \rangle$[13]. Thus

$$\textstyle\bigoplus_{i=0}^{n} \mathrm{End}_k(T(V)^{\otimes i})^{\vee} \twoheadrightarrow A_n \ .$$

Taking union and the colimit of both sides transforms this to

$$s : \textstyle\bigoplus_{i=0}^{\infty} \mathrm{End}_k(T(V)^{\otimes i})^{\vee} \twoheadrightarrow A \ .$$

Then A has a commutative algebra structure via

$$s' : s^{*}\mathrm{End}_k(T(V)^{\vee}) \twoheadrightarrow A \ .$$

Now we use the perfect pairing. Since $T(q)$ is represented as the invertible matrix Φ, it commutes with the elements of A, i.e., if X in the image of $\mathrm{End}_k(T(V))^{\vee}$ under s', then representing X as a matrix,

$$\Phi = X^T \Phi X.$$

Since Φ is invertible, so is X. This means s' factors via

$$(\text{Hopf algebra associated to } \boldsymbol{GL}(V)) \twoheadrightarrow A \ .$$

□

Proposition 11.7. *Let C be a neutral Tannakian category except for rigidity, and $T : C \to$ Vect_k the fiber functor. Assume the set $\boldsymbol{V} = \{V_i \mid i \in I\} \subset C$ satisfies the following conditions.*

(1) *C is generated as a tensor abelian category by $\boldsymbol{V}$.*
(2) *For every $V \in \boldsymbol{V}$, there is a $W_i \in C$ such that there exists a $q_i : V_i \otimes W_i \to \mathbf{1}$ such that*

$$T(q_i) : T(V_i) \otimes T(W_i) \to k.$$

[12]This notation means the minimal tensor abelian category containing V.

[13]This notation means the smallest abliean subcategory of C containing the listed objects.

Then C is rigid.

Proof. Step 1. Replace V_i and W_i by $V_i' := V_i \oplus W_i$ then V_i' has the property that this is a perfect duality pairing

$$q_i' : V_i' \otimes V_i' \to \mathbf{1}$$

Explicitly, q_i' is T applied to multiplication by the matrix

$$\begin{pmatrix} 0 & q_i \\ \tilde{q}_i & 0 \end{pmatrix},$$

where $\tilde{q}_i = W_i \otimes V_i \cong V_i \otimes W_i \to \mathbf{1}$. (Generation property obtained by replacing $\boldsymbol{V}$ by $(\boldsymbol{V}_1, \dots, \boldsymbol{V}_n)$.

Step 2. Write $C = \cup_{J \subset I, |J| < \infty} \langle V_j \mid j \in J \rangle_{\otimes}$ and use the construction with lim. Then apply 11.6. □

Remark 11.8. This works also with only a dual pairing,

$$\begin{aligned} \mathbf{1} &\to V \otimes V \\ \mathbf{1} &\to V_i \otimes W_i. \end{aligned}$$

Question 11.9. Why does $\langle X \rangle_{\otimes}$ exist? In particular, we probably need to know that the intersection of two tensor abelian categories is a tensor abelian category.

Question 11.10. If there are two abelian subcategories C_1, C_2 of C, is the intersection $C_1 \cap C_2$ an abelian category? It's not, for trivial reasons, i.e., the intersection of two disjoint zero categories is the empty category, which is not abelian. But if the inclusions are exact and full functors, is their intersection an abelian category?

Section 8.6 of *Categories and Sheaves* by Kashiwara and Schapira might be a useful reference for answering this question.

12. Pairs of representations

Minxi Wang and Lars Kühne on the 18th of July, 2011.

12.1. Let D be a commutative, graded diagram with a product structure $\times : D \times D \to D$, and

$$T_1, T_2 : D \to \mathrm{Vect}_{\boldsymbol{Q}}$$

two vector spaces.

Example 12.1. For example, $D = C(T)$, Nori's diagram category, and $T_1 := H^*_{dR}$ and $T_2 := H^*_{sing}$.

Definition 12.2 ($A_{1,2}$)**.** In the above situation, $A_{1,2}$, let $F \subset D$ be a finite subdiagram. Then

$$\mathrm{Hom}(T_1|_F, T_2|_F) = \left\{ \prod_{f \in F} (a_f) \in \prod_{f \in F} \mathrm{Hom}(T_1(f), T_2(f)) \;\middle|\; \begin{array}{ccc} T_1(f)T_1(m) & \xrightarrow{a_f} & T_2(f) \\ \downarrow & & \downarrow{\scriptstyle T_2(m)} \\ T_1(f') & \xrightarrow[a_{f'}]{} & T_2(f') \end{array} \quad \forall (f \to f') \in D|_F \right\}$$

Then $F_1 \subset F_2$ gives a natural map

$$\mathrm{Hom}(T_1|_{F_1}, T_2|_{F_2}) \to \mathrm{Hom}(T_1|_{F_1}, T_2|_{F_2})$$

and hence a map

$$\mathrm{Hom}(T_1|_{F_1}, T_2|_{F_2})^\vee \to \mathrm{Hom}(T_1|_{F_1}, T_2|_{F_2})^\vee$$

Define

$$A_{1,2} = \mathrm{colim}_F \mathrm{Hom}(T_1|_F, T_2|_F)^\vee.$$

There is a natural map

$$\mathrm{End}(T_1|_F) \times \mathrm{Hom}(T_1|_F, T_2|_F) \to \mathrm{Hom}(T_1|_F, T_2|_F)$$

giving a natural map

$$\mathrm{Hom}(T_1|_F, T_2|_F))^\vee \to \mathrm{End}(T_1|_F) \times \mathrm{Hom}(T_1|_F, T_2|_F)$$

and inducing

$$A_{1,2} \to A_1 \otimes A_{1,2}.$$

Similarly, there is a map

$$\begin{array}{rcl} \mathrm{Hom}(T_1|_{F'}, T_2|_{F'}) & \to & \mathrm{Hom}(T_1|_F, T_2|_F) \otimes \mathrm{Hom}(T_1|_F, T_2|_F) \\ a = \prod_{f' \in F'} (a_{f'}) \in \prod_{f' \in F'} \mathrm{Hom}(T_1(f'), T_2(f')) & \mapsto & \mu|_F(a) \end{array}$$

giving a map

$$\mathrm{Hom}(T_1|_F, T_2|_F)^\vee \otimes \mathrm{Hom}(T_1|_F, T_2|_F)^\vee \to \mathrm{Hom}(T_1|_{F'}, T_2|_{F'})^\vee.$$

and inducing

$$\mu : A_{1,2} \times A_{1,2} \to A_{1,2},$$

as follows: Choose F' such that $F \times F \subset F'$. Then for all $f, g \in F$,

$$\begin{tikzcd} T_1(f \times g) \arrow[r, "a_{f\times g}"] \arrow[d, "\tau_1"] & T_2(f \times g) \arrow[d, "\tau_2"] \\ T_1(f) \otimes T_1(g) \arrow[r, "\mu^*(a)|_{(f,g)}"'] & T_2(f) \otimes T_2(g) \end{tikzcd}$$

So $A_{1,2}$ is an algebra.

Definition 12.3 ($P_{1,2}$)**.** Let $P_{1,2}$ be the $\boldsymbol{Q}$-vector space

$$\langle (p, w, \gamma) \in D \times T_1(p) \times T_2(p)^\vee \rangle_{\boldsymbol{Q}} / \text{relations}$$

where the relations are given by

$$\begin{array}{rcl} (p, \alpha \cdot w, \beta \cdot \gamma) + (p, \alpha' \cdot w, \beta'\gamma) & = & (p, (\alpha + \alpha')w, (\beta + \beta')\gamma) \\ (p, w, \gamma) + (p, w', \gamma') & = & (p, w + w', \gamma + \gamma') \\ (p, w, T_2(m)^\vee(\gamma) & = & (p', T_1(m)(w), \gamma), \qquad \forall (m : p \to p') \in D, w \in T_1(p), \gamma \in T_2(p')^\vee \\ (p, w, \gamma) \times (p', w', \gamma') & = & (p \times p', w \times w', \gamma \times \gamma') \end{array}$$

Then

$$P_{1,2} = \mathrm{colim}_F P_{1,2}(p|_F)$$

There is a morphism

$$\begin{aligned} \psi : P_{1,2} &\to A_{1,2} \\ \psi|_F : P_{1,2}|_F &\to A_{1,2}|_F, \qquad F \subset D \text{ finite} \end{aligned}$$

Consider

$$\begin{tikzcd} \sum_{p\in F} T_1(p) \otimes T_2(p)^\vee \arrow[r, two heads] \arrow[d, "\cong"'] & P_{1,2}|_F \arrow[d, "\psi|_F"] \\ \sum_{p\in F} \mathrm{Hom}(T_1(p), T_2(p))^\vee \arrow[r, two heads] & A_{1,2}|_F \end{tikzcd}$$

For all $\omega \in T_1(p)$ and for all $\gamma \in T_2(\gamma')^\vee$. Then

$$\phi(p')T_1(f) - T_2(f)\phi(p) = 0$$

implies, through an ommitted chain of reasoning, that

$$(\pi \circ \widetilde{\psi})(T_1(f)(\omega) \otimes \gamma - \omega \otimes T_2(f)^\vee(\gamma)) = 0.$$

Theorem 12.4. *Let $k/\boldsymbol{Q}$ be a field extension such that*

$$(T_1 \otimes k) \xrightarrow{\sim} (T_2 \otimes k), \qquad (\textit{as } \boldsymbol{Q}\textit{-vector spaces}).$$

Then $\psi : P_{1,2} \xrightarrow{\sim} A_{1,2}$.

For example, the hypotheses are satisfied when T_2 is singular cohomology, T_1 is de Rham cohomology, and $k \cong \boldsymbol{C}$.

Proof. Spec $K \to$ Spec $\boldsymbol{Q}$ is faithfully flat. Then

$$\psi \otimes K : (P_{1,2} \otimes K) \to (A_{1,2} \otimes K)$$

Since $T_1 \otimes K \cong T_2 \otimes K$, we may assume $T := T_1 = T_2$, so that $A := A_{1,2} = A_1 = A_2$ and $P := P_{1,2} = P_1 = P_2$. As before, there is a commutative diagram

$$\begin{tikzcd} \sum_{p\in F} T(p) \otimes T(p)^\vee \arrow[r, two heads] \arrow[d, "\cong"] & P|_F \arrow[d, "\psi|_F"] \\ \sum_{p\in F} \mathrm{Hom}(T(p), T(p))^\vee \arrow[r, two heads, "\pi"'] & A|_F \end{tikzcd}$$

The remainder of the proof is ommitted. ◻

Remark 12.5. Sergey Gorchinskey completed the proof as follows. Observe that

$$\mathrm{End}(V) \cong \mathrm{End}(V)^\vee \cong (V^\vee \otimes V^\vee)^\vee \cong V \otimes V^\vee,$$

and consider

$$\begin{aligned} \mathrm{End}(V) \otimes \mathrm{End}(V) &\to k \\ A \otimes B &\mapsto \mathrm{tr}(A \cdot B). \end{aligned}$$

13. Diagrams of pairs

Utsav Choudhury on the 18th of July, 2011.

Our goal will be a diagram of the form

$$\begin{tikzcd} D \arrow[rr, "T"] \arrow[dr] & & \mathrm{Vect}_Q \\ & C(D,T) \arrow[ur] & \end{tikzcd}$$

13.1. **Three diagrams.** We define three diagrams, D^{eff}, D^{eff}_{Nori}[14] and $\widetilde{D}^{eff}$.

Definition 13.1 (D^{eff})**.** The diagram D^{eff} has vertices

$$\{(X,Y,i) \mid X, Y\ \boldsymbol{Q}\text{-varieties}, Y \subset X, i \in \boldsymbol{Z}\},$$

and edges

$$f : (X', Y', i) \to (X, Y, i), \qquad f : X \to X', \qquad f(Y) \subset Y'.$$

Definition 13.2 (D^{eff}_{Nori})**.** The diagram D^{eff}_{Nori} is the full sub-diagram of D^{eff} with vertices

$$\left\{(X,Y,i) \mid \forall j \neq i, H^j(X(\boldsymbol{C}), Y(\boldsymbol{C}), \boldsymbol{Q}) = 0\right\}$$

Definition 13.3 ($\widetilde{D}^{eff}$)**.** The diagram $\widetilde{D}^{eff}$ is the full sub-diagram of D^{eff}_{Nori} with vertices

$$\left\{(X,Y,i) \mid X \text{ is affine and } \left\{ \begin{array}{c} X \text{ is smooth over } Y \quad \dim X = i \\ X = Y, \quad \dim X < i \end{array} \right\} \right\}$$

It admits a norm

$$\begin{array}{rcl} |\cdot| : D^{eff}_{Nori} & \to & \boldsymbol{Z} \\ (X,Y,i) & \mapsto & i \end{array}$$

and a product

$$(X,Y,i) \times (X',Y',i') := (X \times X', X \times Y' \cup Y \times X', i + i')$$

By the Künneth formula,

$$H^j(X \times X', X \times Y' \cup Y \times X') = 0, \qquad j \neq i + i'.$$

13.2. **Localizing D^{eff}_{Nori}.** Define $f_0 := (\boldsymbol{G}_m, \{1\}, 1) \in D^{eff}_{Nori}$. Then $|f_0| = 1$. For every $f \in D^{eff}_{Nori}((X,Y,i),(X',Y',i'))$,

$$f(n) : (X,Y,i)(n) \xrightarrow{f(n)} (X',Y',i')(n)$$

Then

$$(X \times \boldsymbol{G}_m, X \times \{1\} \cup Y \times \boldsymbol{G}_m, i+1)(n) \to (X,Y,i)(n+1)$$

Then

$$\begin{array}{rcl} H^*_{sing} : D^{eff}_{Nori} & \to & \mathrm{Vect}_{\boldsymbol{Q}} \\ (X,Y,i) & \mapsto & H^i_{sing}(X(\boldsymbol{C}), Y(\boldsymbol{C}), \boldsymbol{Q}) \end{array}$$

By the Künneth formula, H^*_{sing} is a graded representation,

$$H^i(X(\boldsymbol{C}), Y(\boldsymbol{C}), \boldsymbol{Q}) \otimes H^{i'}(X'(\boldsymbol{C}), Y'(\boldsymbol{C}), \boldsymbol{Q}) \cong H^{i+i'}((X \times X')(\boldsymbol{C}), X \times Y' \cup Y \times X', \boldsymbol{Q})$$

[14]Apologies to Nori for this notation.

We can extend it to

$$H^*_{sing} : D_{Nori} \to \mathrm{Vect}_{\boldsymbol{Q}}$$

by noticing that

$$H^*_{sing}(X, Y, i)(n) = H^i(X, Y, \boldsymbol{Q} \otimes H^1(\boldsymbol{G}_m\{1\})^{\otimes n}$$

Secondly, we can define

$$\begin{array}{rcl} H^*_{dR} : D^{eff}_{Nori} & \to & \mathrm{Vect}_{\boldsymbol{Q}} \\ (X, Y, i) & \mapsto & H^i_{dR}(X, Y0 \end{array}$$

We can show this can be extended in the same way by the Künneth formula, giving a graded representation.

Using the same construction as the lecture on Saturday, we can extend this to

$$\begin{tikzcd} D_{Nori} \arrow[r, "H^*_{sing}"] \arrow[dr, "\widetilde{H}^*_{sing}"'] & \mathrm{Vect}_{\boldsymbol{Q}} \\ & C(D_{Nori}, H_{sing}) =: MM_N \arrow[u, "ff_{H^*_{sing}}"'] \end{tikzcd}$$

where MM_N is the tensor category of Nori's mixed motives. Furthermore,

$$\begin{tikzcd} D^{eff}_{Nori} \arrow[rr, "H^*_{sing}"] \arrow[dr] & & \mathrm{Vect}_{\boldsymbol{Q}} \ . \\ & C(D^{eff}_{Nori}, H^*sing) \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} D_{Nori} \arrow[r, shift left, "H^*_{sing}"] \arrow[r, shift right, "H^*_{dR}"'] & \mathrm{Vect}_{\boldsymbol{Q}} \end{tikzcd}$$

$$H^*_{sing}(X(\boldsymbol{C}), Y(\boldsymbol{C}), \boldsymbol{Q}) \otimes \boldsymbol{C} \cong H^*_{dR}(X, Y) \otimes \boldsymbol{C}$$

Then H^*_{sing} gives $ff_{H^*_{sing}} : MM_N \to \mathrm{Vect}_{\boldsymbol{Q}}$, which induces in turn

$$\widetilde{H}^*_{dR} : MM_N \to \mathrm{Vect}_{\boldsymbol{Q}}.$$

Lemma 13.4. *If*

$$\begin{tikzcd} D \arrow[r, shift left, "T_1"] \arrow[r, shift right, "T_2"'] \arrow[d, "\widetilde{T}_1"'] & \mathrm{Vect}_{\boldsymbol{Q}} \\ C(D, T_1) \arrow[ur, "ff_{T_1}"'] & \end{tikzcd}$$

and

$$T_1 \otimes \boldsymbol{C} \cong T_2 \otimes \boldsymbol{C},$$

then one can extend T_2 to $C(D, T_1)$.

Proof.

$$\begin{tikzcd}
& C(D,T_1) \arrow[rd, "ff_{T_1}"] \arrow[dd, "\exists!\widetilde{f}"] & \\
D \arrow[ru, "\widetilde{T}_1"] \arrow[rd, "f"'] & & \mathrm{Vect}_{\boldsymbol{Q}} \\
& \mathcal{A} \arrow[ru, "g"'] &
\end{tikzcd}$$

where g is faithful and exact and $\mathcal{A}$ the abelian $\boldsymbol{Q}$-linear category with objects

$$\mathcal{A} := \{(V_1, V_2, \psi) \mid V_1, V_2\ \boldsymbol{Q}\text{-vector spaces}, \psi : V_1 \otimes \boldsymbol{C} \to V_2 \otimes \boldsymbol{C}\}$$

and morphisms $(\phi_1, \phi_2) : (V_1, V_2, \psi) \to (V_1', V_2', \psi')$ given by $\phi : V_1 \to V_1'$ and $\phi_2 : V_2 \to V_2'$ such that

$$\begin{tikzcd}
V_1 \otimes \boldsymbol{C} \arrow[r, "\psi"] \arrow[d, "\phi_1 \otimes \boldsymbol{C}"'] & V_2 \otimes \boldsymbol{C} \arrow[d, "\phi_2 \otimes \boldsymbol{C}"] \\
V_1 \otimes \boldsymbol{C} \arrow[r, "\psi"'] & V_2' \otimes \boldsymbol{C}
\end{tikzcd} .$$

□

There exists a functor

$$f : \begin{array}{ccc} D & \to & A \\ p & \mapsto & (T_1(p), T_2(p), \boldsymbol{P}(p)) \end{array},$$

where $\boldsymbol{P}$ is the period isomorphism, and a projection

$$pr_1 : \mathcal{A} \to \mathrm{Vect}_{\boldsymbol{Q}}$$

Define

$$\tilde{f} = (C(D, T_1) \to \mathcal{A} \to^{pr_1} \mathrm{Vect}_{\boldsymbol{Q}}$$

13.3. **Formal periods and period numbers.**

$$H^*_{sing}, H^*_{dR} : D^{eff} \to \mathrm{Vect}_{\boldsymbol{Q}}$$

Define the $\boldsymbol{Q}$-vector space

$P_{1,2} := \left\{(p, w, \gamma) : p \in D^{eff}, w \in D^*_{dR}(p) = H^i_{dR}(X, Y), \gamma \in (H^*_{sing}(p))^\vee \cong (H^i_{sing}(X(\boldsymbol{C}), Y(\boldsymbol{C}), \boldsymbol{Q})\right\} /$ (relation

where the relations are given by

(1) linearity in w and γ
(2) for all $f : p \to p'$, $p, p' \in D^{eff}$ and all $\gamma \in H^*_{sing}(p')^\vee$ and all $w \in H^*_{dR}(p)$

$$(p', H^*_{dR}(f)(w), \gamma) = (p, w, H^*_{sing}(f)^\vee(\gamma))$$

Definition 13.5 (Evaluation morphism)**.** The evaluation morphism is

$$ev : \begin{array}{ccc} P_{1,2} & \to & \boldsymbol{C} \\ (p, w, \gamma) & \mapsto & \int_\gamma w \end{array}$$

Definition 13.6 (Period numbers)**.** The period numbers are the image of the map

$$H^i_{dR} \times H_i \to \boldsymbol{C}$$

14. Yoga of diagrams

Sergey Rybakov on the 19th of July, 2011.

Definition 14.1. The diagram D^{eff} has

vertices: (X, Y, i), where $X \supset Y$ closed subvariety of $\boldsymbol{Q}$-variety X, $i \in \boldsymbol{Z}$.

edges: $f : (X', Y', i) \to (X, Y, i)$, where $f : X \to Y'$ and $f(Y) \subset Y'$. And for all chains $X \supset Y \supset Z$ of closed subvarieties, there is an edge

$$(Y, Z, i) \to (X, Y, i+1)$$

Definition 14.2. The diagram D^{eff}_{Nori} is the full subdiagram of D^{eff} with vertices (X, Y, i) such that

$$H^j(X(\boldsymbol{C}), Y(\boldsymbol{C}), \boldsymbol{Q}) = 0, \qquad j \neq i$$

Definition 14.3. A subdiagram $\widetilde{D}^{eff}$ with vertices (X, Y, i) such that X is affine and $X \setminus Y$ is smooth.

There is a multiplicative structure on D^{eff}_{Nori} which induces a tensor structure on

$$C(D^{eff}_{Nori}, T) = MM_N$$

Theorem 14.4. *The natural functors*

$$C(\widetilde{D}^{eff}, T) \to C(D^{eff}_{Nori}, T) \to C(D^{eff}, T)$$

are equivalences.

Then $C(D^{eff}, T) = C(D^{eff})$, and we define

$$\mathcal{A} = C(D^{eff}_{Nori}, T)$$

Proposition 14.5. *Let* $\widetilde{T} : \widetilde{D}^{eff} \to \mathcal{A}$*. Then there is a contravariant, triangulated functor,*

$$R : C^b(\boldsymbol{Z}[\mathrm{Var}]) \to D^b(\mathcal{A})$$

where the morphisms of $\boldsymbol{Z}[\mathrm{Var}]$ *are of the form* $\sum \alpha_i f_i$ *for morphisms of varieties* $X_i \to Y_i$*. It is an additive category with disjoint union (properly defined–be careful about multiple connected components) as the sum.*

For all good pairs (X, Y, i),

$$H^j(R(\mathrm{Cone}(Y \to X)) = \begin{cases} 0, & j \neq i \\ \widetilde{T}(X, Y, i), & j = i \end{cases}$$

where Cone sets Y in degree -1 and X in degree 0.

Proof. Proof of theorem. The composition of functors

$$D^{eff} \to \mathcal{A} \to C(D^{eff}) \to \mathrm{Vect}_{\boldsymbol{Q}}$$

implies that $\mathcal{A} \to C(D^{eff})$ is an equivalence. The first functor has the form,

$$(X, Y, i) \mapsto H^i(R(X, Y)) \in \mathcal{A},$$

where

$$R(X, Y) = R(\mathrm{Cone}(Y \to X)).$$

For each $X \supset Y \supset Z$, take a triangle in the derived category $R(X,Y) \to R(X,Z) \to R(Y,Z)$. That it is a triangle follows by general reasoning from the definition of R using cones. It has connecting morphism,

$$\delta : H^i(R(Y,Z)) \to H^{i+1}(R(X,Y)).$$

□

In the remained of the talk, we will prove the proposition.

Definition 14.6 (Rigidified affine cover)**.** A rigidified affine cover of X is an affine cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X and for all closed points $x \in X$, an index $i(x)$ such that $x \in U_{i(x)}$.

Definition 14.7 (Morphism of rigidified affine covers)**.** Given rigidified affine covers $\mathcal{V} = \{V_j\}_{j\in J}$ of Y and $\mathcal{U} = \{U_i\}_{i\in I}$ of X is given by a morphism of varieties

$$f : X \to Y$$

and a map

$$\phi : I \to J$$

such that $f(U_i) \subset V_{\phi(i)}$, and for all $x \in X$, $\phi(i(x)) = j(f(x))$.

Lemma 14.8. (1) *Filtered system. Rigidified affine covers form a filtered system*
(2) *Functorialityof system. If $f : X \to Y$ and $\mathcal{V}$ is a rigidified affine cover of Y, then there exists a rigidified affine cover $\mathcal{U}$ of X and a morphism $\mathcal{U} \to \mathcal{V}$.*

Definition 14.9 (Affine cover)**.** Let $X \in C^b(\boldsymbol{Z}[\mathrm{Var}])$. And affine cover of X is a chain

$$X_1 \xrightarrow{d_1} X_2 \xrightarrow{d_2} \cdots \longrightarrow X_n$$

together with rigidified affine covers $\mathcal{U}_m$ of X_m, and morphisms $\mathcal{U}_m \to \mathcal{U}_{m+1}$ for all $f : X_m \to X_{m+1}$, where $d_m = \sum a_i f_i$.

Lemma 14.10. (1) *Filtered System. The affine covers of any $X \in C^b(\boldsymbol{Z}[\mathrm{Var}])$ form a non-empty filtered system.*
(2) *Functoriality of system. If $f : X \to Y$ is a morphism and $\mathcal{V}$ is an affine cover of Y, then there exists an affine cover $\mathcal{U}$ of X and a morphism $\mathcal{U} \to \mathcal{V}$.*

Definition 14.11 (Čech complex)**.** The Čech complex of a rigidified affine cover $\mathcal{U} = \{U_i\}_{i\in I}$ has chain groups of the form

$$C^{-n}(\mathcal{U}) = \coprod_{i_0 < \cdots < i_n} \cap_{j=0}^{n} U_{ij}$$

with differential, whose definition is ommitted.

Definition 14.12 (Čech complex)**.** Given $X \in C^b(\boldsymbol{Z}[\mathrm{Var}])$, fix an affine cover $U_\bullet$ of X. The Čech complex of X is the total complex of the bicomplex $C^{-i}(U_j)$

$$C^\bullet(\mathcal{U}) \in C^b(\boldsymbol{Z}[\mathrm{Aff}])$$

In a previous lecture, we proved the following theorem.

Lemma 14.13 (Nori's basic lemma)**.** *Let $X \subset Z$ be affine varieties suc that Z is closed in X,* $\dim X = n$ *and* $\dim Z < n$*. Then there exists a closed subvariety $Y \subset X$ such that $X \supset Y \supset Z$ and (X, Y, n) is a very good pair.*

Corollary 14.14. (1) *Existence. If X is affine, then X has a very good filtration*
$$\emptyset = F_{-1}X \subset F_0 X \subset F_1 X \subset \cdots \subset F_n X = X$$
such that $(F_i X, F_{i-1}X, i)$ is a very good pair
(2) *Filtered system. The very good filtrations on X form a filtered system.*
(3) *Functoriality of system. Given $f : X \to Y'$ and a very good filtration $F_\bullet X$, there exists a very good filtration $G_\bullet Y$ such that $f(F_i X) \subset G_i(Y)$.*

Proof. (1) Let $Z = \mathrm{Sing}X$. Then dim $Z < n$. By Nori's lemma, there is a $Y \supset Z$ which forms a very good pair with X. Define $F_{n-1}X = Y$ and proceed by induction to define a very good filtration of X.
(2) Let $F_\bullet X$ and $F'_\bullet X$ be very good filtrations. One can define a third very good filtration to which the first two map. Its nth term is $G_n X = X$. The rest of the terms are formed inductively, beginning with $Z = F_{n-1}X \cup F'_{n-1}X$, applying Nori's lemma, and defining $G_{n-1} = Y$.
(3) Let $Z = f(F_{n-1}X)$. If dim $Z <$ dim Y', then define $G_{n+1}Y' = Y$ and $G_n Y' = Y'$. In the case dim $Z =$ dim Y', use (Y', Y', n).

□

Lemma 14.15. *If $X_\bullet \in C^b(\boldsymbol{Z}[Aff])$, then there exists a very good filtration of X, and such filtrations form a functorial filtered system.*

Proof. Let $\widetilde{R}(F_\bullet X_\bullet) \in C^b(\mathcal{A})$ be of the form
$$\cdots \to \widetilde{T}(F_j X_\bullet, F_{j-1}X) \to \widetilde{T}(F_{j+1}X, F_j X_j) \to \cdots$$
and $X_\bullet \in C^b(\boldsymbol{Z}[\mathrm{Var}])$. Take an affine cover $\mathcal{U}$ of $X_\bullet$. Then define
$$R(X_\bullet) = \widetilde{R}(F_\bullet C^\bullet(\mathcal{U})).$$

□

Proposition 14.16.
$$H^i(X(\boldsymbol{C}), \boldsymbol{Q}) = ff_T(R(X))$$

Corollary 14.17.
$$\begin{tikzcd}
\widetilde{R}(F'_\bullet C^\bullet(\mathcal{U}')) \arrow[dr] & \\
& ? \\
\widetilde{R}(F_\bullet C^\bullet(\mathcal{U})) \arrow[ur] &
\end{tikzcd}$$

This gives a map $f : X_\bullet Y_\bullet$ and $\mathcal{V} \to \mathcal{U}$. Then $F_\bullet \mathcal{V}_\bullet$ gives $G_\bullet \mathcal{U}_\bullet$, giving a map $R(Y_\bullet) \to R(X_\bullet)$.

15. Rigidifty for Nori motives

Sergey Gorchinskiy on the 19th of July, 2011.

15.1. Reminder.

Periods [15]

$$\begin{aligned} H_B : D^{eff} = \{\text{effective pairs}\} &\to \mathrm{Vect}_{\boldsymbol{Q}} \\ (X, Y, i) &\mapsto H^i_B(X, Y) \end{aligned}$$

Tensor product The diagram D^{eff}_{Nori} is composed of effective good pairs, i.e., where $H^j(X, Y) = 0$ for $j \neq i$.

Rigidity The diagram $\widetilde{D}^{eff}$ is composed of very good pairs, i.e., where $X \setminus Y$ is smooth affine.

If you want to create some new world, it's natural to create some new objects in the new world. But category theory does it differently. We do not create no objects in the new category, but use the original ones, and do not even create new morphisms, but restrict morphisms and create a non-full subcategory.

(i) All three universal abelian categories corresponding to the above three diagrams coincide in $\mathrm{Vect}_{\boldsymbol{Q}}$, giving MM^{eff}_N.
(ii) D^{eff}_{Nori} has a commutative group structure, implying that MM^{eff}_N is tensor.
(iii) $\mathbf{1}(-1) := H_B(\boldsymbol{G}_m, \{1\}, 1)$. Localize MM^{eff}_N by $\mathbf{1}(-1)$ to get MM_N.
(iv) If $f : X \to Y$ is a mixed motive in MM_N such that $H_B(f)$ is an isomorphism, then f is an isomorphism.

There exists a covariant functor

$$R : C^b(\boldsymbol{Z}[\mathrm{Var}_k]) \to D^b(MM_N)$$

such that for every good pair (X, Y, i),

$$R(Y \to X) = (X, Y, i)[-i]$$

where Y is in degree -1, X in degree 0.

Recall that the construction of R was in two steps. First take affine cover, then take the filtration. If there are two varieties, there are two coverings, and we can take their product. One can also take the product of the filtrations. This gives

$$R(X \times Y) \cong R(X) \otimes R(Y)$$

There is also a morphism,

$$\Delta^* : R(X) \otimes R(X) \to R(X),$$

where Δ is the diagonal morphism.

Definition 15.1. Define the Nori cohomology of X to be

$$H^i_N(X) := H^i(R(X)) \in MM_N.$$

Perhaps this should be called the Nori motive, in analogy with other motivic theories.

15.2. Nori motivic cycle classes.

[15]We will denote Betti cohomology by H_B.

Lemma 15.2. *There exists canonical isomorphisms*

$$\begin{array}{rcll} H^0_N(V) & \cong & \mathbf{1}, & V \text{ connected} \\ H^{2i}_N(\boldsymbol{P}^d) & \cong & \mathbf{1}(-i), & 0 \le i \le d \\ H^i_N(\boldsymbol{P}^d) & = & 0, & \text{else} \\ H^{2d}_N(X) & \cong & H^{2d}_N(X) \cong \mathbf{1}(-d) & X^d \text{ smooth projective} \end{array}$$

Proof. All the time, use the property (iv) listed above. Construct a morphism in the mixed category of Nori motives, and check this is an isomorphism of cohomology.

For the first isomorphism, consider $V \to pt$ which induces

$$H^0_N(pt) \xrightarrow{\sim} H^0_N(V)$$

For the second isomorphism, consider the case $d = 1$. Then $\boldsymbol{P}^1 = U_1 \cup U_2$, where $U_1 := \boldsymbol{P}^1 \setminus \{0\}$ and $U_2 = \boldsymbol{P}^1 \setminus \{\infty\}$, so $U_i \cong \boldsymbol{A}^1$ for $i = 1, 2$, and $H^\bullet_N(\boldsymbol{A}^1) \cong \mathbf{1}$. Furthermore, $U_{12} \cong \boldsymbol{G}_m$. Thus the Čech complex is

$$R(U_1) \oplus R(U_2) \longrightarrow R(U_1 \cap U_2)$$

$$\begin{array}{cccc} 1 & & & \mathbf{1}(-1) \\ & & & \\ 0 & \mathbf{1} \oplus \mathbf{1} & \longrightarrow & \mathbf{1} \end{array}$$

which gives $H^0_N(\boldsymbol{P}^1) \cong \mathbf{1}$ and $H^1_N(\boldsymbol{P}^1) \cong \mathbf{1}(-1)$.

In the case $d \ge 2$, consider the finite morphism of degree m

$$\pi : \underbrace{\boldsymbol{P}^1 \times \cdots \times \boldsymbol{P}^1}_{d} \to \boldsymbol{P}^d$$

This induces

$$\pi^*/m : H^{2d}_N(\boldsymbol{P}^d) \xrightarrow{\sim} H^{2d}_N((\boldsymbol{P}^1)^d) \cong \mathbf{1}(-d)$$

Then the linear map $\boldsymbol{P}^i \hookrightarrow \boldsymbol{P}^d$ induces

$$H^{2i}_N(\boldsymbol{P}^d) \xrightarrow{\sim} H^{2i}_N(\boldsymbol{P}^i)$$

For the last isomorphism, consider the finite morphism of degree m

$$\pi : X^d \to \boldsymbol{P}^d$$

It induces π^*/m on top degree cohomology. □

Proposition 15.3. *Let $Y \subset X$ be smooth projective variety, with Y closed in X. Let $c :=$* $\mathrm{codim}_X Y$*. If $0 \ne [Y] \in H^{2c}_B(X)$, then there exists a canonical mixed motive $\mathbf{1}(-c)[-2c] \to R(X)$ which induces $[Y] \cdot \boldsymbol{Q} \hookrightarrow H^{2c}_B(X)$ after applying H_B.*

Proof. Consider the pairing:

$$H^{2c}_N(X) \otimes H^{2e}_N(X) \to H^{2d}_N(X) \cong \mathbf{1}(-d)$$

where $c := \dim Y, d := \dim X$. By the projection formula for $Y \hookrightarrow X$ in Betti cohomology, we get

$$K := \ker(H_N^{2c}(X) \to H_N^{2c}(X \setminus Y)) \otimes \underbrace{\text{im } (H_N^{2e}(X))}\, \mathbf{1}(-e) \to H_N^{2e}(Y)) \to \mathbf{1}(-d)$$

This pairing is perfect after taking Betti cohomology:

$$K \cong \mathbf{1}(-d) \otimes \mathbf{1}(-e)^{-1} \cong \mathbf{1}(-c) \xrightarrow{\alpha} H_N^{2c}(X)$$

Since the first nontrivial cohomology of $\mathrm{Cone}(R(X) \to R(X \setminus Y))[-1]$ is in degree $2c$, α lifts canonically to $\mathbf{1}(-c)[2c] \to R(X)$. □

We want to show that all objects are dualizable. Then general plan is to show that $R(X)$, for X smooth and projective, generates the category of mixed motives. Any category generated by dualizable objects, then every object is dualizable.

Definition 15.4. Let $(C, \otimes)$ be a commutative tensor category. Then an object $X \in C$ is dualizable if there exists an object $X^\vee \in C$ and dualizing morphisms $\mathbf{1} \to X \otimes X^\vee \to \mathbf{1}$ such that

$$1_X = (X \to X \otimes X^\vee \otimes X \to X)$$

and

$$1_{X^\vee} = (X^\vee \to X^\vee \otimes X \otimes X^\vee \to X^\vee)$$

where the first equation comes from tensoring the dualizing morphism on the right by X, and the second from tensoring on the left by $X^\vee$.

Remark 15.5. Let X, Y be dualizable, and $f : X \to Y$. Then f has a dual $f^\vee : Y^\vee \to X^\vee$ defined by

$$Y^\vee \longrightarrow Y^\vee \otimes X \otimes X^\vee \xrightarrow{f} Y^\vee \otimes Y \otimes X^\vee \longrightarrow X^\vee .$$

Proposition 15.6. *Let $(\mathcal{A}, \otimes)$ be an abelian category such that the tensor product $\otimes$ is exact. Let $(f : X \to Y) \in D^b(\mathcal{A})$ be dualizable objects. Then $C := cone(f)$ is dualizable, and $C^\vee := cone(f^\vee)[-1]$ such that*

$$X \xrightarrow{f} Y \longrightarrow C \longrightarrow X[1]$$

$$C^\vee \longrightarrow X^\vee \xrightarrow{f^\vee} Y^\vee \longrightarrow C^\vee[1]$$

15.3. **Rigidity.**

Remark 15.7. Let X^d be a smooth projective variety. Then $R(X)$ is dualizable in $D^b(MM_N)$,

$$\mathbf{1}(-d)[-2d] \to R(X) \otimes R(X) \to R(X) \to \mathbf{1}(-d)[-2d]$$

where the first term is the motivic cycle classes of $\Delta : X \to X \times X$.

If we show the following proposition,

Proposition 15.8. *$R(MM_N)$ is triangulated by $R(X)$ for X smooth projective.*

then as a corollary, we will get rigidity, because you will get dualizing maps for the pure motives. Note that

$$R(X)^\vee \cong R(X)(d)[2d]$$

Let $D \subset X$ be a smooth divisor. Then duality from $R(X)$ to $R(D)$ induces

$$R(D)(-1)[-2] \to R(X)$$

Let $D = \cup_i D_i$ be the decomposition into prime divisors. Then

$$0 \to \cdots \to \oplus_{i,j} R(D_i \cap D_j)(-2)[-4] \to \oplus R(D_i)(-1)[-2] \to R(X)$$
$$T \xrightarrow{\sim} R(X \setminus D)$$

Lemma 15.9. *For all $V \subset U$, very good pair,*

$$R(V \to U) \cong R(X \setminus (D_1 \cap D_2) \to X \setminus D_2)$$

where X is smooth projective and $D = D_1 \cup D_2 \subset X$ is a logarithmic divisor.

Proof. Take the projective closure of V and U, desingularizing the boundary and V using Hironaka's lemma . Then

$$\begin{tikzcd} V' \arrow[r] \arrow[d, "\text{proper}"'] & U' \supset & U' \setminus V' \\ V \arrow[r] & U \supset & U \setminus V \arrow[from=1-2, to=2-2, "\text{proper}"] \end{tikzcd}$$

implies

$$R(V \to U) \to R(V' \to U')$$

$\square$

Corollary 15.10. *Nori's category of mixed motives MM_N with $\otimes$ is rigid.*

15.4. **Corollaries of rigidity.**

Corollary 15.11.
(i) *Given $Y \hookrightarrow X$ of codimension c, there is a morphism $R(Y)(-c)[-2c] \to R(X)$.*
(ii) *Let $R_c(U) := R(Y \to X)$, where X, Y are smooth projective and $U = X \setminus Y$ is smooth.*
(iii) *$R(U^d)^\vee \cong R_c(U)(d)[2d]$*

The morphism $\mathbf{1} \to R(U) \otimes R(U)$ can be constructed geometrically.

Question 15.12. Can one construct a mixed motive in (i) and (ii) geometrically, i.e., in terms of morphisms between pairs?

Remark 15.13. Let $T : D \to \mathrm{Vect}_k$. The mixed motives between $p, q \in D$ in $C(D,T)$ can be larger than in the additive closure of $T(D)$.

Example 15.14. Let $D = \{p\}$, and A be composed of the arrows of D. A forms a subalgebra of $Mat_{n\times n}(k)$. Let

$$\begin{array}{rcl} T : D & \to & \mathrm{Vect}_k \\ p & \mapsto & k^n \end{array}$$

Let

$$C(D,T) \cong Mod(Z(A))$$

where $Z(A)$ is the centralizer of Z in $Mat_{n\times n}(k)$. Then

$$\mathrm{End}_{C(D,T)}(p) = Z(Z(A)) \supsetneqq A.$$

Corollary 15.15.
- *Proper morphisms $f : Y \to X$ between smooth varieties induce,*
$$f_* : R(Y)(c)[2c] \to R(X)$$
where $c = \dim X - \dim Y$.
- *Cycles on the product $\alpha \in CH^{\dim (Y)}(X \times Y)$ of degree* $\dim Y$ *induce*
$$\alpha_* : R(Y) \to R(X)$$
- *There is a functor*
$$\{\textit{Chow motives}\} \to D^b(MM_N)$$
- *There is a functor*
$$\begin{array}{rcl} \{\textit{homological motives}\} & \to & MM_N \\ X^d & \mapsto & \oplus_{i=0}^{2d} H^i_N(X) \end{array}$$
- *The functor*
$$DM_{Sm}(k) \to D^b(MM_N)$$
is A^1-equivariant, fiber, push-forward.

15.5. **Torsors.**

Corollary 15.16. *Mixed motives with Betti cohomology*
$$H_B : MM_N \to \mathrm{Vect}_{\boldsymbol{Q}}$$
form a Tannakian category, and the correspondence $G_M := \underline{\mathrm{Isom}}(H_B, H_B)$ is the Nori motivic Galois group.

Remark 15.17. The functor H_B factors through $\mathrm{Vect}^{\boldsymbol{Z}}(\boldsymbol{Q})$, inducing $\boldsymbol{G}_m \to G_M$.

Recall that $\underline{\mathrm{Isom}}^{\otimes}(H_B, H_{dR}) = \mathrm{Spec}\ (\boldsymbol{P})$, for $\boldsymbol{P}$ the formal period algebra.
$$H_{dR} : MM_N \to \mathrm{Vect}_{\boldsymbol{Q}}$$
Then Spec $\boldsymbol{P}$ is a torsor under G_M. The isomorphism
$$\boldsymbol{C} \otimes H_{dR} \cong \boldsymbol{C} \otimes H_B$$
induces Spec $\boldsymbol{C} \to$ Spec $\boldsymbol{P}$.

Corollary 15.18. *K-Z conjecture is equivalent to the conjecture that the image of the unique point on* Spec $\boldsymbol{C}$ *is the generic point of* Spec $\boldsymbol{P}$.

15.6. **Conjecture.** Now we will consider the three conjectures, and the form of their mutual implications.

Conjecture 15.19. *For all i, X, consider*
$$C := \langle H^i_N(X) \rangle_{\otimes} \hookrightarrow MM_N$$
The evaluation morphism
$$ev : \boldsymbol{P}_{\boldsymbol{P}^r} \to \boldsymbol{C}$$
is injective, where $\boldsymbol{P}_{\boldsymbol{P}^r}$ is the formal periods of smooth projective varieties such that Spec $\boldsymbol{P}_{\boldsymbol{P}^r} = \underline{\mathrm{Isom}}(H_{dR}|_C, H_B|_C)$

This splits into two conjectures, as discussed with Joseph Ayoub.

Conjecture 15.20. *For all smooth projective varieties X over $\overline{\boldsymbol{Q}}$,*

$$(2\pi\sqrt{(-1)})^{\pm i} H^{2i}_{dR}(X) \cap H^{2i}_B(X) \subset \boldsymbol{C} \otimes H^{2i}_B(X)$$

are algebraic classes.

Conjecture 15.21.

$$\overline{\text{Spec } \boldsymbol{C}} \subset \text{Spec } \boldsymbol{P}_{\boldsymbol{P}^r}$$

is a torsor under a subgroup of $\underline{\text{Isom}}(H_B|_C, H_B|_C)$.

Proposition 15.22. *Conjecture 1 is equivalent to the conjunction of conjecture 2 and conjecture 3.*

Remark 15.23. Fix X and i. Then $\langle H^i_N(X)\rangle_{\otimes} = C(X,i) \subset MM_k$. So $\underline{\text{Isom}}(H_B|_{C(X,i)}) = G_M(X,i)$.

The Hodge conjecture implies

$$G_M(X^i_1) \cong \{\text{Mumford-Tate group of } H^i(X)\}$$

The Tate conjecture implies that

$$\boldsymbol{Q}_l \times G_M(X,i) \text{“}=\text{''} \{ \text{ Zariski closure of } G_k \qquad H^j_{et}(\overline{X}, i) \ \}$$

15.7. **Why motives?** Realizations!

$$\text{Var}_k \to \begin{cases} H^\bullet_{dR}(X) \\ H^\bullet_B(X) \\ H^\bullet_l(X) \\ H^\bullet_{sing}(X) \end{cases}$$

The cohomology theories map to rigid tensor abelian categories.

In all cohomological theories,

$$H^{2i}(\boldsymbol{P}^n) = \begin{cases} \boldsymbol{Q}(-i), & 0 \le i \le n \\ 0, & \text{else} \end{cases}$$

Realizations should be exact and faithful functors from MM_k to Hodge or Galois.

Consider the Chow groups. Let X be a smooth projective surface. Then

$$CH^2(X)_0 \longrightarrow CH^2(X) \xrightarrow{\deg} \boldsymbol{Z}$$

and

$$T(X) \longrightarrow CH^2(X_0) \xrightarrow{AJ} \text{Alb}(X)$$

A slogan: homological equivalence between cycles reconstructs rational equivalence. This is not strictly true, but an example illustrates its meaning.

Example 15.24. Let X be a smooth projective curve. Then there is an exact sequence,

$$0 \longrightarrow J(X) \longrightarrow CH^1(X) = \text{Pic}(X) \xrightarrow{\deg} \boldsymbol{Z} \longrightarrow 0$$

which is isomorphic to

$$0 \to H^1(X,\boldsymbol{C})/(F^1 + H^1(X,\boldsymbol{Z})) \to \text{Hom}(\mathbf{1}(-1), H^\bullet(X)) \to \text{Hom}(\mathbf{1}(-1), H^2(X)) \to 0$$

Conjecture 15.25 (Beilinson)**.**

$$\mathrm{Hom}(\mathbf{1}(-i)[-2i], R(X)) = CH^i(X)$$

for $R(X) \in D^b(MM_k)$. This holds in $DM_{Sm}(k)$, "implying" that there exists a filtration $F^\nu CHI^i(X)$ which is multiplicative, respects pull-back and push-forward, and such that $F^1CH^i(X) = CH^i(X)_{hom}$.

This implies

(i) Bloch's conjecture: For all smooth projective surfaces X,
$$H^0(X, \Omega_X^2) = 0 \Rightarrow T(X) = 0.$$

(ii) If $\alpha \in CH^2(X \times X)$ such that $\alpha \sim 0$, then the action of the kernel of Veronese is 0:
$$0 = (\alpha_* : T(X) \to T(X))$$

(iii) If $H^1(X) = 0$, then there exists an $N \in \boldsymbol{N}$ such that
$$\sum_{\sigma \in S_N} (-1)^{sgn(\sigma)} \Gamma_\sigma \in CH^{2N}(X^N \times X^N)$$