

# A GENERALIZATION OF DOOB'S MAXIMAL IDENTITY

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ABSTRACT. In this paper, using martingale techniques, we prove a generalization of Doob's maximal identity in the setting of continuous nonnegative local submartingales  $(X_t)$  of the form:  $X_t = N_t + A_t$ , where the measure  $(dA_t)$  is carried by the set  $\{t : X_t = 0\}$ . In particular, we give a multiplicative decomposition for the Azéma supermartingale associated with some last passage times related to such processes and we prove that these non-stopping times contain very useful information. As a consequence, we obtain the law of the maximum of a continuous nonnegative local martingale  $(M_t)$  which satisfies  $M_\infty = \psi(\sup_{t \geq 0} M_t)$  for some measurable function  $\psi$  as well as the law of the last time this maximum is reached.

## 1. INTRODUCTION

Throughout this paper,  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  will be a filtered probability space, satisfying the usual assumptions. If  $(X_t)$  denotes a stochastic process, we note  $\bar{X}_t = \sup_{s \leq t} X_s$ . We also recall the definition of local submartingales of the class  $(\Sigma)$  which will play a crucial role in this paper. These processes were first introduced by Yor ([27]) and further studied in [15, 17]. For simplicity, we restrict ourselves to the case of continuous processes:

**Definition 1.1.** Let  $(X_t)$  be a positive local submartingale, which decomposes as:

$$X_t = N_t + A_t.$$

We say that  $(X_t)$  is of class  $(\Sigma)$  if:

- (1)  $(N_t)$  is a continuous local martingale, with  $N_0 = 0$ ;
- (2)  $(A_t)$  is a continuous increasing process, with  $A_0 = 0$ ;
- (3) the measure  $(dA_t)$  is carried by the set  $\{t : X_t = 0\}$ .

If additionally,  $(X_t)$  is of class  $(D)$ , we shall say that  $(X_t)$  is of class  $(\Sigma D)$ .

Well known examples of such stochastic processes are  $(|M_t|)$ , the absolute value of some local martingale  $M$ ,  $M^+$  or  $(\bar{M}_t - M_t)$ . Many other processes or simple transforms of diffusions fall in the class  $(\Sigma)$  (see [17] for more details).

*Remark 1.2.* In [17], the local martingale part  $N_t$  is allowed to be right continuous with left limit. Here, for simplicity, we restrict our attention to the case of continuous processes. But some of the results in this paper would hold in the more general setting under some extra assumptions (such as conditions on the sign of the jumps).

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In [16], nonnegative local martingales  $(M_t)$  which satisfy  $\lim_{t \rightarrow \infty} M_t = 0$  are studied in depth: in particular, the law of  $\sup_{t \geq 0} M_t$  (which is distributed as  $\frac{M_0}{U}$ , where  $U$  is a uniform random variable, and known as Doob's maximal identity<sup>1</sup>) as well as the random time  $g = \sup\{t : M_t = \overline{M}_t\}$ , are shown to play an important role in the characterization of honest times (i.e. ends of predictable sets) and in the theory of progressive enlargements of filtrations (see also [13]). One key formula in [16] is the multiplicative decomposition of the Azéma's supermartingale associated with  $g$ :

$$\mathbb{P}[g > t | \mathcal{F}_t] = \frac{M_t}{\overline{M}_t}. \quad (1.1)$$

One may wonder if a simple multiplicative decomposition such as (1.1) still holds in more general situations? One such situation occurs in the resolution of the Skorokhod stopping problem by Azéma and Yor ([3], see for example the survey [20] for an overview) where  $M_\infty = \psi(\overline{M}_\infty)$ , with  $\psi$  some measurable function.

The goal of this paper is twofold:

- to show that the multiplicative decomposition formula (1.1) as well Doob's maximal identity naturally extend in the framework of local submartingales of the class  $(\Sigma)$ ;
- to outline the importance of honest times in the theory of stochastic processes: it is a remarkable fact that the knowledge of the Azéma supermartingales (as well as the law) of such random times gives information about the underlying process.

More precisely, this paper is organized as follows:

In Section 2.1, we recall some useful results about honest times as well as an elementary additive characterization of Azéma's supermartingale associated with the end of a predictable set.

In Section 2.2, we consider conveniently stopped processes  $(X_{t \wedge T})$  of the class  $(\Sigma)$ , with possibly  $T = \infty$ , and which satisfy  $X_T = \psi(A_T)$ , for some measurable function  $\psi$ . We show how the results of section 2.1 can be used to obtain a multiplicative decomposition for the Azéma's supermartingale associated with the last zero of  $(X_{t \wedge T})$  and use this decomposition to compute the law of  $A_T$ . As a corollary, we obtain a generalization of the results in [16] for nonnegative local martingales which satisfy  $M_\infty = \psi(\overline{M}_\infty)$ .

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## 2. A GENERALIZATION OF DOOB'S MAXIMAL IDENTITY

**2.1. Useful results about honest times.** Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  be a filtered probability space, satisfying the usual assumptions, and let  $L$  be the end of an  $(\mathcal{F}_t)$  predictable set  $\Gamma$ , i.e:

$$L = \sup\{t : (t, \omega) \in \Gamma\}.$$

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<sup>1</sup>In fact, this identity, which is a simple consequence of the optional stopping theorem, has been well known for some time and appears for example as an exercise in [23] p.73.

A typical example of such a random time  $L$  is

$$L = \sup \{t \leq 1 : B_t = 0\},$$

where as usual  $(B_t)$  denotes the standard Brownian Motion. Ends of predictable sets are the most studied random times after stopping times (see [18] for examples and more references). One process plays an essential role in the study of the random time  $L$ , namely the supermartingale:

$$Z_t^L = \mathbb{P}(L > t | \mathcal{F}_t), \quad (2.1)$$

associated with  $L$  by Azéma in [1], and chosen to be càdlàg. For simplicity, we make the following assumptions throughout this paper, which we call the (CA) conditions:

- (1) all  $(\mathcal{F}_t)$ -martingales are continuous (e.g: the Brownian filtration).
- (2) the random time  $L$  avoids every  $(\mathcal{F}_t)$ -stopping time  $T$ , i.e.  
 $\mathbb{P}[L = T] = 0$ .

*Remark 2.1.* Under the (CA) conditions, the optional and the predictable sigma fields are equal and the supermartingale  $(Z_t^L)$  is continuous.

Now consider the Doob-Meyer decomposition of  $Z^L$ :

$$Z_t^L = 1 + \mu_t^L - A_t^L \quad (2.2)$$

The process  $(A_t^L)$ , which we shall sometimes note  $(A_t)$  in the sequel, is the dual predictable projection of the increasing process  $\mathbf{1}_{\{L \leq t\}}$ , and

$$\mu_t^L = \mathbb{E}(A_\infty^L | \mathcal{F}_t) - 1$$

**Proposition 2.2** (Azéma [1]). *Let  $L$  be the end of some predictable set; then*

$$L = \sup \{t : Z_t = 1\},$$

*and the measure  $dA_t$  is carried by the set  $\{t : Z_t = 1\}$ . In particular,  $A$  does not increase after  $L$ , i.e.  $A_L = A_\infty$ . Moreover,  $A_\infty$  is distributed with the standard exponential law.*

It follows from this proposition that

$$X_t \equiv 1 - Z_t^L$$

is of the class  $(\Sigma D)$ , with:  $X_0 = 0$  and  $\lim_{t \rightarrow \infty} X_t = 1$ . In fact, the converse of this result is also true. It is stated in [15] (where it was used to derive path decomposition results for diffusion processes) as a consequence of a very useful lemma (although not so well known) which first appeared in the papers of Azéma, Meyer and Yor [2] (section 7) and Azéma and Yor [4] (Proposition 2.2), in a more abstract framework. It will again play an important role in our discussions in this paper. We state a variant of this lemma in which the local martingale  $(N_t)$  in Definition 1.1 does not necessarily start from 0 but from an arbitrary positive number  $x$ :  $N_0 = x \geq 0$ . The proof by Azéma and Yor [4] (Proposition 2.2) still holds and we simply reproduce it.

**Lemma 2.3.** *Let  $(X_t)$  be a submartingale of the class  $(\Sigma D)$  with  $X_0 = x \geq 0$  (i.e.  $N_0 = x$  in Definition 1.1) and let*

$$L = \sup \{t : X_t = 0\},$$

with the convention that  $\sup\{\emptyset\} = 0$ . Assume further that:

$$\mathbb{P}(X_\infty = 0) = 0.$$

Then:

$$X_t = \mathbb{E}(X_\infty \mathbf{1}_{\{L \leq t\}} | \mathcal{F}_t). \quad (2.3)$$

*Proof.* Since  $(X_t)$  is continuous, the set  $\{t : X_t = 0\}$  is a predictable closed set. Let us remark that:

$$X_\infty \mathbf{1}_{\{L \leq t\}} = X_{d_t},$$

where

$$d_t \equiv \inf \{s > t : X_s = 0\}.$$

Hence, we have:

$$\begin{aligned} \mathbb{E}(X_\infty \mathbf{1}_{\{L \leq t\}} | \mathcal{F}_t) &= \mathbb{E}(X_{d_t} | \mathcal{F}_t) \\ &= \mathbb{E}(N_{d_t} | \mathcal{F}_t) + \mathbb{E}(A_{d_t} | \mathcal{F}_t). \end{aligned}$$

Now, from the optional stopping theorem, we have:

$$\mathbb{E}(N_{d_t} | \mathcal{F}_t) = N_t.$$

Moreover, as  $(dA_t)$  is carried by the set  $\{t : X_t = 0\}$ , we have:

$$A_{d_t} = A_t.$$

We can thus conclude that:

$$\mathbb{E}(X_\infty \mathbf{1}_{\{L \leq t\}} | \mathcal{F}_t) = N_t + A_t = X_t,$$

and this completes the proof.  $\square$

*Remark 2.4.* In their recent papers, D. Madan, B. Roynette and M. Yor ([11, 12]) have obtained some nice and striking results relating the price of some options to the Azéma supermartingale associated with some last passage times. The very first step of this work can be viewed through Lemma 2.3. Indeed, consider  $(M_t)$  a continuous and nonnegative local martingale starting from  $z$  and such that  $\lim_{t \rightarrow \infty} M_t = 0$ . In the pricing of put options, quantities such as  $\mathbb{E}[(K - M_T)^+]$  or  $\mathbb{E}[(K - M_T)^+ | \mathcal{F}_t]$  for some  $t \leq T$  often arise.

For any  $K > 0$ , consider the local submartingale  $X_t = (K - M_t)^+$ . It is easy to see that  $X$  satisfies the assumptions of Lemma 2.3 with  $x = (K - z)^+$  and  $X_\infty = K$ . Here,  $L \equiv g_K = \sup\{t : M_t = K\}$ . An application of Lemma 2.3 yields (these expressions first appeared in [11, 12]):

$$(K - M_t)^+ = K \mathbb{P}[g_K \leq t | \mathcal{F}_t].$$

Consequently, for  $t \leq T$ , we have:

$$\mathbb{E}[(K - M_T)^+ | \mathcal{F}_t] = K \mathbb{P}[g_K \leq T | \mathcal{F}_t],$$

and

$$\mathbb{E}[(K - M_T)^+] = K \mathbb{P}[g_K \leq T].$$

For other proofs and more results with financial models in view, the interested reader should refer to [11, 12].

Moreover, at  $t = 0$ , the above representation gives:

$$\mathbb{P}[g_K = 0] = \frac{(K - z)^+}{K}.$$

But  $\{g_K = 0\} = \{\overline{M}_\infty < K\}$  and hence

$$\mathbb{P}[\overline{M}_\infty < K] = \frac{(K - z)^+}{K},$$

which is Doob's maximal identity. More generally, if  $(X_t)$  satisfies the assumptions of Lemma 2.3 with  $X_\infty = K$  where  $K > 0$  is a constant, then  $\mathbb{P}[\overline{X}_\infty > 0] = \frac{x}{K}$ .

*Remark 2.5.* In a forthcoming paper [19], some other applications of honest times in financial modelling are developed.

*Remark 2.6.* More generally, as already mentioned, Lemma 2.3 leads to a very natural characterization of Azéma's supermartingales: roughly speaking, any submartingale  $X$  of the class  $(\Sigma)$  that converges to a nonzero constant  $K$ , that we can without loss of generality take to be one, is equal to  $1 - \mathbb{P}[L > t | \mathcal{F}_t]$ , where  $L$  is the last time  $X$  hits zero. This characterization corresponds to Theorem 2.3 in [15] and is systematically used there to express some well known submartingales of the class  $(\Sigma)$  in terms of some associated last passage times.

**2.2. Azéma's supermartingale associated with the last zero.** In [17], we computed the law of  $A_\infty$  when  $X$  is of the class  $(\Sigma D)$ ; in particular, we were interested in stopping times of the form

$$T \equiv \inf \{t : \varphi(A_t) X_t \geq 1\}, \quad (2.4)$$

for certain nonnegative Borel functions. In particular, we observed in [17] that  $(\varphi(A_t) X_t)$  is of the class  $(\Sigma)$ , and may be represented as:

$$\varphi(A_t) X_t = \int_0^t \varphi(A_u) dN_u + \Phi(A_t), \quad (2.5)$$

where  $\Phi(x) = \int_0^x dz \varphi(z)$  (since  $X$  is continuous, this last formula can also be easily deduced from standard balayage arguments as exposed in [23], chapter VI). We also showed that if  $\varphi$  is a nonnegative locally bounded Borel function, such that  $\int_0^\infty dz \varphi(z) = \infty$ , then the stopping time  $T$  is almost surely finite:

$$\int_0^\infty dz \varphi(z) = \infty \Rightarrow T < \infty. \quad (2.6)$$

Here, we shall develop further the study of  $X$  on  $[0, T]$ , using Lemma 2.3 and Proposition 2.2. More precisely, we shall look for the Azéma's supermartingale associated with the random time

$$g_T \equiv \sup \{t < T : X_t = 0\}.$$

**Proposition 2.7.** *Let  $\varphi$  be a nonnegative locally bounded Borel function, such that  $\varphi(x) > 0$ ,  $\forall x > 0$  and such that  $\int_0^\infty dz \varphi(z) = \infty$  and let  $X$  be a local submartingale of the class  $(\Sigma)$ , such that  $A_\infty = \infty$ . Let  $T$  be defined as in (2.4) and let*

$$g_T \equiv \sup \{t < T : X_t = 0\}.$$

Then,

$$\mathbb{P}(g_T \leq t | \mathcal{F}_t) = \varphi(A_{t \wedge T}) X_{t \wedge T} = \int_0^{t \wedge T} \varphi(A_u) dN_u + \Phi(A_{t \wedge T}), \quad (2.7)$$

and  $g_T$  avoids all  $(\mathcal{F}_t)$  stopping times. Consequently,

$$\mathbb{P}(A_T > x) = \exp\left(-\int_0^x dz \varphi(z)\right).$$

*Proof.* From (2.6),  $T < \infty$ , and consequently,  $\varphi(A_T) X_T = 1$ . Now, since  $(\varphi(A_{t \wedge T}) X_{t \wedge T})$  is of the class  $(\Sigma)$ , we can apply Lemma 2.3 (with  $X_\infty = 1$ ) to obtain the identity (2.7). From (2.5), we also have:  $\mathbb{P}(g_T \leq t | \mathcal{F}_t) = \int_0^{t \wedge T} \varphi(A_u) dN_u + \Phi(A_{t \wedge T})$ , and hence the dual predictable projection of  $(\mathbf{1}_{g_T \leq t})$  is  $(\Phi(A_{t \wedge T}))$ . Now, since all  $(\mathcal{F}_t)$  martingales are continuous and  $(\Phi(A_{t \wedge T}))$  is continuous, all  $(\mathcal{F}_t)$  stopping times are predictable, and  $g_T$  avoids all  $(\mathcal{F}_t)$  stopping times. Indeed, for any  $(\mathcal{F}_t)$  stopping time  $R$ ,

$$\mathbf{E}[\mathbf{1}_{\{g_T = R\}}] = \mathbb{E}[\Delta(\Phi(A_{R \wedge T}))] = 0.$$

Thus we get  $P(g_T = R) = 0$ .

The fact that  $\mathbb{P}(A_T > x) = \exp(-\int_0^x dz \varphi(z))$  is a consequence of the fact that  $\Phi(A_T)$  is distributed as a random variable with the standard exponential law (see Proposition 2.2).  $\square$

*Remark 2.8.* Applying Corollary 4.2 in [16], one obtains:  $\mathbb{P}(g_T > t | \mathcal{F}_t) = \frac{Y_t}{\bar{Y}_t}$ , with  $(Y_t)$  some continuous and nonnegative local martingale starting from 1 and converging to 0 at infinity. Moreover,  $Y_t$  and  $\bar{Y}_t$  can be expressed in terms of  $N_t$  and  $A_t$ :

$$\begin{aligned} Y_t &= \exp\left(\int_0^t \frac{dM_s}{Z_s} - \frac{1}{2} \int_0^t \frac{d\langle M \rangle_s}{Z_s^2}\right) \\ \bar{Y}_t &= \exp(\Phi(A_{t \wedge T})); \end{aligned}$$

with  $M_t = 1 + \int_0^{t \wedge T} \varphi(A_u) dN_u$  and  $Z_t = \varphi(A_{t \wedge T}) X_{t \wedge T}$ . The multiplicative expression (2.7) is thus much simpler.

It may happen that  $T = \infty$ , or in other words,  $X_\infty = \varphi(A_\infty)$ , for some nonnegative Borel functions, in which case we have the following proposition:

**Proposition 2.9.** *Let  $\varphi$  be a nonnegative Borel function such that  $1/\varphi$  is locally bounded, and let  $X$  be a submartingale of the class  $(\Sigma D)$ , such that  $X_\infty = \varphi(A_\infty)$ . Define:*

$$g \equiv \sup\{t : X_t = 0\}.$$

If  $\varphi(A_\infty) > 0$ , a.s., then:

$$\mathbb{P}(g \leq t | \mathcal{F}_t) = \frac{X_t}{\varphi(A_t)} = \int_0^t \frac{dN_u}{\varphi(A_u)} + \int_0^{A_t} \frac{dz}{\varphi(z)}. \quad (2.8)$$

Consequently,

$$\mathbb{P}(A_\infty > x) = \exp\left(-\int_0^x \frac{dz}{\varphi(z)}\right).$$

*Proof.* From Lemma 2.3, we have:

$$X_t = \mathbb{E}(X_\infty \mathbf{1}_{\{g \leq t\}} | \mathcal{F}_t) = \mathbb{E}(\varphi(A_\infty) \mathbf{1}_{\{g \leq t\}} | \mathcal{F}_t).$$

But from Proposition 2.2, we also have:

$$\varphi(A_\infty) \mathbf{1}_{\{g \leq t\}} = \varphi(A_t) \mathbf{1}_{\{g \leq t\}},$$

and consequently,

$$X_t = \varphi(A_t) \mathbb{P}(g \leq t | \mathcal{F}_t),$$

and (2.8) follows easily from (2.5). Eventually, the law of  $A_\infty$  follows from the fact that the dual predictable projection of  $\mathbf{1}_{\{g \leq t\}}$ , which is  $\int_0^{A_t} \frac{dz}{\varphi(z)}$ , taken at  $t = \infty$ , follows the standard exponential law.  $\square$

*Remark 2.10.* We note that taking  $\varphi \equiv K > 0$  a constant, we recover the results discussed in the remarks following Lemma 2.3. We moreover see that  $A_\infty$  follows an exponential law with parameter  $1/K$ .

Now, we give some applications of Propositions 2.7 and 2.9 to the case when  $M_\infty = \psi(\overline{M}_\infty)$ , with  $\psi$  a nonnegative Borel function. The next Theorem generalizes Lemma 2.1 and Proposition 2.2 in [16]. Without loss of generality, we can assume that  $M_0 = 1$ . With the notation

$$\overline{M}_t \equiv \sup_{u \leq t} M_u,$$

we have:

**Theorem 2.11.** *Let  $(M_t)$  be a continuous nonnegative local martingale starting from 1, and such that  $M_\infty = \psi(\overline{M}_\infty)$ , with  $\psi$  a nonnegative Borel function such that  $\psi(x) < x$ ,  $\forall x \geq 1$ . Define*

$$g \equiv \sup \{t : M_t = \overline{M}_t\}.$$

Then,

$$\begin{aligned} \mathbb{P}(g \leq t | \mathcal{F}_t) &= \frac{\overline{M}_t - M_t}{\overline{M}_t - \psi(\overline{M}_t)} \\ &= - \int_0^t \frac{dM_u}{\overline{M}_u - \psi(\overline{M}_u)} + \int_1^{\overline{M}_t} \frac{dz}{z - \psi(z)}. \end{aligned}$$

Hence, the dual predictable projection of the raw increasing process  $(\mathbf{1}_{\{g \leq t\}})$ , is  $(\int_0^{\overline{M}_t} \frac{dz}{z - \psi(z)})$ . Consequently, we have:

$$\mathbb{P}(\overline{M}_\infty > x) = \exp\left(- \int_1^x \frac{dz}{z - \psi(z)}\right), \quad \forall x \geq 1. \quad (2.9)$$

In particular, when  $\psi \equiv 0$ , i.e. when  $M_\infty = 0$ , we recover the following results which appear in [16]:

$$\begin{aligned} \mathbb{P}(g > t | \mathcal{F}_t) &= \frac{M_t}{\overline{M}_t}, \\ \mathbb{P}(\overline{M}_\infty > x) &= \frac{1}{x}, \quad \forall x \geq 1. \end{aligned}$$

*Proof.* Let us define

$$X_t \equiv 1 - \frac{M_t}{\overline{M}_t}.$$

An application of Itô's formula, combined with the fact that  $d\overline{M}_t$  is carried by the set  $\{t : M_t = \overline{M}_t\}$ , yields:

$$X_t = - \int_0^t \frac{dM_u}{\overline{M}_u} + \log(\overline{M}_t),$$

and since  $\{t : X_t = 0\} = \{t : M_t = \overline{M}_t\}$ , we easily deduce that  $X$  (which is bounded by 1) is of the class  $(\Sigma D)$ . With the notations of Proposition 2.8, we have  $A_t = \log(\overline{M}_t)$  and  $\varphi(x) \equiv 1 - \frac{\psi(\exp(x))}{\exp(x)}$ . Since  $\psi(x) < x$ , we have  $\varphi(A_\infty) > 0$ , and the results of the corollary follow from an application of Proposition 2.9 and some elementary calculations.  $\square$

*Remark 2.12.* The situation described in the previous Theorem often occurs in the Azéma-Yor solution to Skorokhod's embedding problem which relies upon the construction of a Brownian martingale  $M_t = B_{t \wedge T}$  such that  $M_T = \psi(\overline{M}_T)$  (see [3]).

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