Large deviations and interacting random walks and random surfaces

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The first topic addressed in this course is the so called polymer measure in three dimension, sometimes called Edwards' model. The rigorous construction of this goes back to Westwater who in two celebrated papers in the early eighties proved that a suitably regularized version converges if the regularization is taken away. It perhaps needs some justification that such an "old fashioned" topic is taken up again. The Westwater model did in fact not gain much attention in the probability community, despite the fact that it is certainly more interesting than the two dimensional one (constructed originally by Varadhan) on which many papers have been written. The reason probably is that Westwater's approach is so complicated (at least for people not familiar with methods from quantum field theory), that essentially nobody seems to have taken the pains to study his papers. Until very recently, there was barely any paper in the probability community discussing properties of the three dimensional polymer measure. Not really much later than the Westwater approach, there has been an alternative one, first in the context of quantum field theory, and then also for the polymer problem, with two contributions by Brydges, Fröhlich and Sokal, and the latter one by Bovier, Felder and Fröhlich. Their approach is *much* simpler, but also had considerable shortcomings. The main one was that it was impossible to speak of *the* polymer measure, because the proof gave only boundednes properties of finite dimensional distributions, from which the existence of convergent subsequences could be derived. It is clear that having only such boundednes or tightness properties, one can never prove for instance, that the limit coincide with the process constructed by Westwater. In a paper of mine in 1992, most of these shortcomings had been removed, and really the convergence had been proved, besides of other things. Unfortunately, the technical details then still got somewhat involved, although the basic strategy is quite transparent. The topic was vigorously taken up by X.Y. Zhou who then wrote a number of papers (mostly with Albeverio, and one with me) on the topic, extending the approach for instance to arbitrary coupling constants, identifying the measure with the one constructed by Westwater, and proving limit theorems for self-repellent random walks converging to the measure. It was an extremely sad event when Zhou died suddenly in 1996, shortly before a planned visit to Zürich, where we wanted to work on the topic.

The second Chapter will deal with self-attracting random walks. The most natural example would be to just change the sign of the coupling constant in the standard (weakly) self-avoiding case, but it is easy to see that this is not an interesting object as the attraction would be far too strong. So one is lead to models with weaker interaction, namely where the coupling constant decays proportional in time. Somewhat surprisingly, this model has a collapse transition in two and more dimensions, changing from a collapsed state for a large coupling constant, to a diffusive one for weak couplings. The diffusive phase had been discussed by Brydges and Slade. Of particular interest is the two dimensional case, which is closely related to the polymer measure (in two dimension). A part of the second Chapter will be devoted to problems around the Wiener sausage. A self-attracting path measure (starting for instance with the Brownian motion) is obtained by transforming the measure by favoring paths with small Wiener sausage. It turns out that the path measure obtained in this way, leads to a kind of droplet construction, where the droplet describes in which region of the space the paths have to concentrate under the new measure. This droplet is somewhat trivial, being just a ball, a fact which is related to the standard isoperimetric problem. That there is such a droplet in a weak sense, at least after some compactification procedure, is an immediate consequence of the classical work of Donsker and Varadhan on the Wiener sausage. The real issue is however if there is a corresponding confinement in L_∞. Recently, after previous work in the two-dimensional case by Sznitman and by myself, Th. Povel had been able to prove that the droplet concentrates in L_∞ near the optimal droplet in all dimensions.

The behavior of this model also depends crucially on the coupling constant chosen. It turns out that a model with decaying coupling constant is just diffusive if the decay is too fast. There is a critical case where the "droplet picture" starts to dissolve, which is quite interesting, and has recently been investigated by M. van den Berg, F. den Hollander and me. This will be presented in the last Section of the Chapter.

The third Chapter discusses a number of topics which are all connected with localization-delocalization phenomena, which are induced by an interaction of a random walk or a random surface with a wall. In particular, the case of a random interface interacting with a wall is quite a fascinating topic, with many open problems. The Chapter covers a model of a so called heteropolymer (in one dimension) with a localization-delocalization phase transition, and furthermore wetting transitions in dimension one (where it is very easy) and dimension three (where it is absent).

Some comments about the degree in which technical details will be presented during this course. Some of the proofs presented here would be technically very lengthy if given in all the details. For instance, a full and complete proof of the construction of the three dimensional polymer measure would still require considerable time, but in fact, some of the calculations and estimates are quite repetitive (with small variations), and it would only be tiring if all of them would be presented. As a rule, I am trying to present for most of the results some of the very core arguments in details.

1 Polymer measures in three dimensions

1.1 Introduction

An outstanding open problem in probability theory is the determination of the mean end to end distance of a standard self-avoiding random walk on the d-dimensional lattice $\mathbb{Z}^d$ for $d = 2, 3$ (and 4).

Given $n \in \mathbb{N}$, let Ω_n be the set of paths ω of length n:

$$\Omega_n \stackrel{def}{=} \{\omega = (\omega_0, \omega_1, \dots, \omega_n) : \omega_i \in \mathbb{Z}^d, |\omega_i - \omega_{i-1}| = 1 \,\text{for}\, 1 \leq i \leq n\},$$

and the set of self-avoiding paths

$$\Omega_n^{SA} \stackrel{def}{=} \{\omega \in \Omega_n : \omega_i \neq \omega_j \text{ for } i \neq j\}$$

.The problem is to give precise information about the asymptotic behavior of $|\Omega_n^{SA}|$, the number of self-avoiding paths, and about the mean length of self-avoiding paths:

$$\langle \|\omega_n\| \rangle_{SA} \stackrel{def}{=} \sum_{\omega \in \Omega_n^{SA}} \|\omega_n\| / |\Omega_n^{SA}|$$

where $\| \quad \|$ is the Euclidean length. From arguments in theoretical physics (conformal field theory, expansion techniques) it is believed that $\langle \|\omega_n\| \rangle_{SA}$ scale, with $n^{3/4}$ for $d = 2$, and with n^ν for some ν slightly less than 3/5 (≈ 0.582) for $d = 3$. Also, the scaling limits, i.e. the asymptotic distribution of $\omega_n / \langle \|\omega_n\| \rangle_{SA}$ should be non Gaussian (see [?]). Starting with dimension 4, the scaling limits are becoming Gaussian, with a slight correction to ordinary central limit scaling for $d = 4$, where $\langle \|\omega_n\| \rangle_{SA}$ is thought to be of order $\sqrt{n}\sqrt[8]{\log n}$. The case of $d \geq 5$ is completely settled: Starting with work by Brydges and Spencer [?] who introduced the lace expansion, and culminating with Hara and Slade [?]. An excellent monograph on these and related topics is [?]. There is no (published) proof for $d = 4$ which is not (directly) tractable by lace expansions (see [?], [?] for partial results).

I will not give any discussion of these techniques here. One of the results I present is a very weakly interactive case for $d = 3$, where the interaction is so weak that one has ordinary scaling, but where nevertheless the scaling limit of $(\omega_{[nt]} / \langle \|\omega_n\| \rangle)_{0 \leq t \leq 1}$, which is shown to exist, is not Gaussian, but instead the so called Edwards' model first constructed rigorously by Westwater [?].

We now introduce the so called weakly self-avoiding random walks. Here, all paths in Ω_n obtain positive weight, but the ones with many intersections get "punished". This is achieved by choosing a parameter $\lambda \in (0,1)$. Then every path $\omega \in \Omega_n$ gets its relative weight decreased by a factor $(1-\lambda)$ for every self intersection, i.e. we define the probability measure on Ω_n by

$$\hat{P}_{n,\lambda}(\omega) \stackrel{def}{=} \prod_{0\le i<j\le n} (1-\lambda 1_{\omega_i=\omega_j}) \ / \ Z_{n,\lambda},$$

where $Z_{n,\lambda} = \sum_{\omega\in\Omega_n} \prod_{0\le i<j\le n} (1-\lambda 1_{\omega_i=\omega_j})$.

(Remark: Through these notes, we will always use $\hat{P}$ to denote measures on path spaces obtained from "simple" random walk measures by introducing interactions, self-repelling in this chapter, and self-attracting in the next.) We rewrite the above measure by setting (with a slight abuse of notation)

$$\hat{P}_{n,\beta}(\omega) = \exp\left[-\frac{\beta}{2}\sum_{i,j=1}^{n} 1_{\omega_i=\omega_j}\right] \Big/ \ Z_{n,\beta}, \tag{1.1}$$

where $\beta = -\log(1-\lambda) \in (0,\infty)$, and $Z_{n,\beta}$ being the appropriate norming. Remark that the diagonal part in the summation is cancelling. We can also rewrite the interaction:

$$\sum_{i,j=1}^{n} 1_{\omega_i=\omega_j} = \sum_{x\in\mathbb{Z}^d} \ell_n(x,\omega)^2,$$

where $\ell_n(x,\omega)$ is the discrete local time

$$\ell_n(x,\omega) \stackrel{def}{=} \sum_{j=0}^{n} 1_{\omega_j=x}.$$

This is the so called Domb-Joyce model. It is supposed to have essentially the same properties as the strictly self-avoiding walk, and it is essentially as challenging, except that for instance the diffusive behavior for $d \ge 5$ is less involved if β is small enough. (See the recent paper by van der Hofstad, den Hollander and Slade [?].)

The above expression for the Domb-Joyce model naturally leads to the question if similar models exist starting with the Brownian motion instead of the random walk and how the relations between this and the discrete models are.

We start with the Wiener measure P_T on $C_0^d(T)$, the set of continuous paths $\omega : [0,T] \to \mathbb{R}$, starting at 0, and we want to define the polymer measure formally by

$$\hat{P}_{T,\beta}(d\omega) = \frac{1}{Z_{T,\beta}} \exp(-\frac{\beta}{2} \int_0^T dt \int_0^T ds\ \delta(\omega_t - \omega_s)) P_T(d\omega), \tag{1.2}$$

where δ is the Dirac function. There is evidently some trouble defining this, as the formal expression

$$\int_0^T ds \int_0^T dt\ \delta(\omega_t - w_s) = \int dx\ \ell_T(x,\omega)^2,$$

where

$$\ell_T(x,\omega) = \int_0^T \delta(\omega_s - x) ds,$$

i.e. the L_2-norm of the local time, only makes sense for $d = 1$. The trouble is revealed also by formally calculating the expectation under Wiener measure

$$E_T \int_0^T ds \int_0^T dt\ \delta(\omega_t - \omega_s) = 2 \int\int_{0 \le s \le t \le T} dsdt\ p_{t-s}(0), \tag{1.3}$$

where $p_u(x)$ is the transition density of Brownian motion, i.e.

$$p_u(x) = (2\pi u)^{-d/2} e^{-\frac{\|x\|^2}{2u}}.$$

However, the right hand side of (**??**) is evidently divergent for $d > 1$. There are a number of ways in which one can try to remedy the situation. The first idea, but not the easiest one, is to step back to the Domb-Joyce model and to try to make some limiting procedures with the lattice spacing going to 0, and an appropriate dependence of β on n. This is possible, but is somewhat delicate, and has only recently been done in a completely satisfactory way [**?**] for $d = 3$. I will discuss that below. Another approach is to replace δ by a smoothed version, e.g. $p_\varepsilon, \varepsilon > 0$, and then let $\varepsilon \to 0$. For $d = 2$ this was the way in which Varadhan proved the existence of the polymer measure (**??**). The most convenient way however is to use some gap regularization. Observe that the right hand side of (**??**) is divergent only because of the integration near the diagonal. If we leave a gap between s, t, e.g. integrating only over $s + \varepsilon \leq t,\ \varepsilon > 0$, then this stays finite. It is in fact known that

$$J^{\varepsilon}_{0,T}(\omega) = \int_0^{T-\varepsilon} ds \int_{s+\varepsilon}^{T} dt\ \delta(\omega_t - \omega_s)$$

is well defined, P_T - a.s. As this is still only a formal expression, some comments are in order. We can define, for every $a > 0$,

$$J^{\varepsilon,a}_{0,T}(\omega) = \int_0^{T-\varepsilon} ds \int_{s+\varepsilon}^{T} dt\ p_a(\omega_t - \omega_s),$$

and then (with fixed $\varepsilon > 0$) let $a \to 0$. This limit is what we denote by $J^{\varepsilon}_{0,T}$. The limit has nice properties, e.g. it is a.s. continuous in ε, T. We will not go into a discussion of these properties, but simply refer to the relevant literature, e.g. [**?**]. We then define our regularized Edwards' model by

$$\hat{P}^{\varepsilon}_{T,\beta}(d\omega) = \exp(-\beta J^{\varepsilon}_{0,T}(\omega)) P_T(d\omega)/Z_{T,\beta,\varepsilon}. \tag{1.4}$$

Theorem 1.1 *For $T, \beta > 0$, $d = 2, 3$, the limit*

$$\hat{P}_{T,\beta} = \lim_{\varepsilon \to 0} \hat{P}^{\varepsilon}_{T,\beta}$$

exists as a weak limit of probability measures on $C^d_o(T)$.

We will focus on the case $d = 3$ which is considerably more delicate than the case $d = 2$. The theorem is essentially due to Westwater [**?**]. The only difference is that he took a slightly different gap regularization. The procedure Westwater follows is however extremely difficult. We give some comments on it below. We explain here an approach which is much easier and is based on so called skeleton inequalities. This method has been introduced by Brydges, Fröhlich and Sokal [**?**] in Euclidean φ^4_d quantum field theory and had then been adapted to the polymer problem in [**?**]. From a probabilistic point of view, there had however been a number of shortcomings. The most serious one was that no convergence could be proved, but only boundednes properties. There was then no possibility for an identification of the process for instance with the one constructed by Westwater. The above authors also had used a lattice regularization, and it was not even clear that there are limits which are rotationally symmetric. For this reason, the approach was thought to be simple but that it would give only weak results, especially in quantum field theory. However, at least for the polymer case, some of the shortcomings can be remedied, and a modification proving convergence has been developed in [?]. It is however not yet clear if similar modifications can be made for φ_d^4 models.

In [?], the polymer measure was constructed only for a small coupling parameter β. This restriction has later been removed in [?].

Here are some comments on the approach by Westwater [?]. He uses no regularization of the δ-function but a slightly different gap regularization. Take $T = 1$ (for notational simplicity). Then

$$X_0(\omega) = \int_0^{1/2} ds \int_{1/2}^1 dt \; \delta(\omega_t - \omega_s)$$

is P_1-a.s. well defined (in the sense described above). On the next level, one defines

$$X_{1,1}(\omega) = \int_0^{1/4} ds \int_{1/4}^{1/2} dt \; \delta(\omega_t - \omega_s), \quad X_{1,2}(\omega) = \int_{1/2}^{3/4} ds \int_{3/4}^{1} dt \; \delta(\omega_s - \omega_t),$$

and then of course

$$X_{n,i}(\omega) = \int_{(i-1)2^{-n}}^{(2i-1)2^{-n}} ds \int_{(2i-1)2^{-n}}^{i2^{-n}} dt \; \delta(\omega_s - \omega_t),$$

$n \geq 1, 1 \leq i \leq 2^n$. These variables have a number of simple properties. For fixed n, the 2^n variables $X_{n,i}$ are evidently independent. Furthermore, the law of $X_{n,i}$ by simple Brownian rescaling is the same as that of $X_{n-1,i}/\sqrt{2},\ n \geq 2$ for $d = 3$. The *main* difficulty is that for different n, the $X_{n,i}$ are not independent. Westwater proves that there is a near independence between $X_{n,\cdot}$ and $X_{m,\cdot}$ if $|m - n|$ is large. In other words, there is near independence between short and long range self intersections. Westwater then proves, using this property, that $\lim_{N\to\infty} \hat{P}_{1,\beta}^{N,WW}$ exists where

$$\hat{P}_{1,\beta}^{N,WW}(d\omega) = \exp(-\beta \sum_{n=0}^{N} \sum_{j=1}^{2^n} X_{n,j}(\omega)) P_1(d\omega)/Z_\beta^N.$$

The main disadvantage of the Westwater approach is that it is extremely complicated which is mainly due to the fact that it makes bad use of the fact that $X_{n,i} \geq 0$. A further enormous complication arises because the $X_{n,i}$ do not have exponential moments. It has recently been proved by Albeverio and Zhou [?] that the Westwater process coincides with the one of Theorem **??**. This might look obvious, but in fact, the removal of the gap is a subtle process, and it will become apparent that it is not true that any gap regularized version is close to the polymer measure regardless how the gap looks like.

One of the motives to investigate the continuous polymer measures had certainly been the hope that they shed some light on the discrete model. The relation is however quite delicate. To see what the appropriate scaling should be, we will perform some formal calculations. We consider the polymer measure on a time slot $[0, T]$ with a coupling parameter which may depend on T. Formally

$$d\hat{P}_{T,\beta_T} = \exp(-\beta_T \int_0^T ds \int_s^T dt \; \delta(\omega_t - \omega_s)) P_T(d\omega)/Z.$$

Performing Brownian scaling

$$\tilde{\omega}_t = \omega_{tT}/\sqrt{T}, \; t \leq 1,$$

and using

$$\int_0^T ds \int_s^T dt \; \delta(\omega_t - \omega_s) = T^{2-\frac{d}{2}} \int_0^1 ds \int_s^1 dt \; \delta(\tilde{\omega}_s - \tilde{\omega}_t)$$

we see that for $\beta_T = \beta T^{\frac{d}{2}-2}$, the distribution of the rescaled path under $\hat{P}_{T,\beta_T}$ is just $\hat{P}_{1,\beta}$. (This is of course not a rigorous proof, but the statement is correct). Anyway, this suggests that starting with a standard random walk $(\omega_0 = 0, \omega_1, \ldots, \omega_T)$ on $\mathbb{Z}$, and the weakly self-avoiding walk

$$\tilde{P}_{T,\beta}(\omega) = \exp(-\beta \sum_{0 \leq i < j \leq T} 1_{\omega_i = \omega_j})/Z_{T,\beta}.$$

one has

Theorem 1.2 *Assume $d \leq 3$ and $\beta > 0$. Then*

$$\lim_{T \to \infty} \tilde{P}_{T,\beta T^{2-d/2}} Y_T^{-1} = \hat{P}_{1,\beta}$$

where $Y_T : \Omega_T \to C_o^d(T)$ is defined by $Y_T(\omega)(i/T) = \omega_i/\sqrt{T}$, and linearly interpolated between.

The above Theorem is easy for $d = 1$, has been proved by Stoll [?] for $d = 2$, and in [?] for $d = 3$. It is of course far from the "real" question, namely what happens with $\tilde{P}_{T,\beta}$ for fixed β as $T \to \infty$. On the other hand, even in the above "very weakly" self-avoiding case, the limiting measure is singular with respect to any Wiener measure, as has been proved by Westwater. Remark that in the two dimensional case, the T-dependence of β_T is $\beta_T = \beta/T$. This will be important in Chapter II.

There are considerable technical difficulties to prove Theorem **??** for $d = 3$. The main problem is to show that the short range intersections, where the random walk does not quite look like a Brownian motion, do not disturb the limiting picture. We will not give a proof here. It is essentially a modification of the arguments in the proof of Theorem **??** but requires some additional nontrivial arguments.

It is to be expected that the limiting behavior of the weakly self-avoiding model (i.e. $\tilde{P}_{T,\beta}$ for fixed $\beta, T \to \infty$) is by some rescaling related to the $\beta \to \infty$) behavior of the polymer measure $\hat{P}_{1,\beta}$. This is even for the $d = 1$ case far from trivial, and has only recently been solved (by van der Hofstad and den Hollander).

We give an outline of the rest of this chapter. We entirely focus on $d = 3$ which is the most delicate case. In Section **??** we discuss the boundednes properties of the so called two point function. This follows closely the approach in [?] and [?], but there are some differences. First, we avoid using Laplace transforms in time. Proving things in Laplace transformed versions is technically simpler but then one has the trouble to invert the result. This inversion is not done in the above mentioned papers of Brydges, Fröhlich, Sokal and Bovier, Felder and Fröhlich. We also derive relatively sharp pointwise estimates (in contrast to L_p-estimates).

I will present some details of the proof of Theorem **??**, but not all. First, I take some fairly evident properties of intersection local times for granted. These are modifications of classical results proved by Rosen. For details I refer to the Appendix of [?]. The basic inequalities are explained in details but the calculations are somewhat repetitive and I will not give all of them..

The boundednes properties immediately imply the tightness of the measures, as $\varepsilon \to 0$. With the inequalities derived in **??**, it is however not possible to prove convergence. In Section **??**, we derive some alternative inequalities which are more delicate to handle, but with which it is possible to prove convergence.

The approach in [?] and [?] had originally been purely perturbative, but by an observation of [?], this can be extended to arbitrary $\beta > 0$.

1.2 The skeleton inequalities and boundednes properties

Let $0 \leq t < T < \infty, \varepsilon > 0$, and set

$$J^{\varepsilon}_{t,T}(\omega) = \int_t^{T-\varepsilon} ds_1 \int_{s_1+\varepsilon}^{T} ds_2 \; \delta(\omega_{s_1} - \omega_{s_2})$$

which can be defined as the a.s. limit

$$\lim_{a\downarrow 0} \int_t^{T-\varepsilon} ds_1 \int_{s_1+\varepsilon}^{T} ds_2 \; p_a(\omega_{s_1} - \omega_{s_2}).$$

The existence of this limit can e.g. be proved by Fourier techniques (see [?]). We will below perform some formal manipulations with δ-functions, which all can easily been justified (for a fixed ε-gap) by replacing δ by p_a and letting $a \to 0$. All the serious trouble is coming when discussing the $\varepsilon \to 0$ limit, and we will focus on that.

We consider the so called two point functions $\overline{g}^{\varepsilon}_{T,\beta}(x)$ defined to be the density of the measure

$$E_T(\exp(-\beta J^{\varepsilon}_{0,T}); \; \omega_T \in dx)$$

on $\mathbb{R}^{\not\Vdash}$. It is convenient to write this formally as

$$\overline{g}^{\varepsilon}_{T,\beta}(x) = E_T(\exp(-\beta J^{\varepsilon}_{0,T})\delta(\omega_T - x)).$$

We write $\overline{g}$ because these quantities have to be slightly modified later on, and we will switch then to g.

Evidently, we have for $0 \leq t \leq T$

$$p_t * \overline{g}^{\varepsilon}_{T-t,\beta} = E\exp(-\beta J^{\varepsilon}_{t,T})\delta(\omega_T - x).$$

Setting $t = 0$ gives $\overline{g}_T$ and $t = T$ gives p_T. Invoking the fundamental theorem of calculus, we therefore arrive at

$$\begin{aligned} p_T(x) - \overline{g}^{\varepsilon}_{T,\beta}(x) &= \int_0^T dt \frac{d}{dt}(p_t * \overline{g}^{\varepsilon}_{T-t,\beta})(x) \\ &= \int_0^T dt \; E(-\beta \frac{d}{dt} J^{\varepsilon}_{t,T} \exp(-\beta J^{\varepsilon}_{t,T})\delta(\omega_T - x)). \end{aligned}$$

Now, $\frac{d}{dt} J^{\varepsilon}_{t,T}(\omega) = -\int_{t+\varepsilon}^{T} ds\ \delta(\omega_s - \omega_t)$, if $t \le T - \varepsilon$, and 0 otherwise, and we therefore get

$$p_t(x) - \overline{g}^{\varepsilon}_{T,\beta}(x) = \beta \int_0^{T-\varepsilon} dt \int_{t+\varepsilon}^{T} ds\, E(\delta(\omega_s - \omega_t) \exp(-\beta J^{\varepsilon}_{t,T}) \delta(\omega_t - x)). \tag{1.5}$$

The manipulation may look somewhat cavalier, but they are harmless. We will derive some concrete inequalities involving $\overline{g}$. This can be done by replacing δ by p_a in all the manipulations, and letting $a \to 0$ in the end. We will however stick to the δ notation which is evidently more convenient. We will often drop ε, β in the notations but they should be remembered to be present. On the right hand side of (**??**), we can split the interaction on $[t, T]$ into the self-interactions on $[t, s]$ and $[s, T]$ and the interactions between these intervals:

$$J^{\varepsilon}_{t,T} = J^{\varepsilon}_{t,s} + J^{\varepsilon}_{s,T} + J^{\varepsilon}_{t,s;s,T}, \tag{1.6}$$

where

$$J^{\varepsilon}_{t,s;s,T} = \iint\limits_{\substack{t \le s_1 \le s \le s_2 \le T \\ s_2 - s_1 \ge \varepsilon}} ds_1 ds_2\ \delta(\omega_{s_1} - \omega_{s_2}). \tag{1.7}$$

In, there is no interaction inside the interval $[0, t]$, and also none between this interval and the next. However, there is an interaction left between $[t, s]$ and $[s, T]$, which is given by the third summand on the right hand side of (**??**). Without this interaction, the right hand side of (**??**) would just be

$$\int_0^{T-\varepsilon} dt \int_{t+\varepsilon}^{T} ds \int dy\ p_t(y) \overline{g}_{s-t}(0) \overline{g}_{T-s}(x - y).$$

(We have dropped β, ε in $\overline{g}$ for notational convenience.) The trouble is evidently coming from the presence of the interaction. This is now handled by some simple inequalities which use the fact that the interaction term (**??**) is nonnegative. We therefore get the two inequalities

$$e^{-\beta J^{\varepsilon}_{t,s;s,T}} \ge 1 - \beta J^{\varepsilon}_{t,s;s,T}, \tag{1.8}$$

and

$$e^{-\beta J^{\varepsilon}_{t,s;s,T}} \leq 1 - \beta J^{\varepsilon}_{t,s;s,T} + \frac{\beta^2}{2}(J^{\varepsilon}_{t,s;s,T})^2. \tag{1.9}$$

Implementing the second summand in the right hand side of (**??**) gives a contribution

$$\begin{aligned} &-\beta^2 \int_{A(\varepsilon)} ds_1 ds_2 ds_3 ds_4 \int dy \int dz\, p_{s_1}(y) \\ &\times E(e^{-\beta J^{\varepsilon}_{s_1,s_4}}\ \delta(\omega_{s_2} - (z-y))\delta(\omega_{s_{s_3}})\delta(\omega_{s_4} - (z-y)))\overline{g}_{T-s_4}(x-z), \end{aligned} \tag{1.10}$$

where $A(\varepsilon) = \{(s_1, s_2, s_3, s_4) :\ \ 0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq T, s_3 - s_1 \geq \varepsilon, s_4 - s_2 \geq \varepsilon\}$.

It should now be observed that $J^{\varepsilon}_{s_1,s_4}$ contains all the interaction on the interval $[s_1, s_4]$. It looks that we have gained noting as things are becoming more and more complicated. However, dropping the remaining interaction between the intervals $[s_1, s_2]$, $[s_2, s_3]$, $[s_3, s_4]$ gives an estimate in the right direction if we use this together with (**??**), simply because

$$J^{\varepsilon}_{s_1,s_4} \geq J^{\varepsilon}_{s_1,s_2} + J^{\varepsilon}_{s_2,s_3} + J^{\varepsilon}_{s_3,s_4}.$$

As the remaining propagator will be crucial, we give it a new name:

$$\overline{G}^{\varepsilon}_T(x) = \iint\limits_{\substack{0\leq s_1\leq s_2\leq T\\ s_2\geq\varepsilon, T-s_1\geq\varepsilon}} ds_1 ds_2\ \overline{g}_{s_1}(x)\overline{g}_{s_2-s_1}(x)\overline{g}_{s_3-s_2}(x). \tag{1.11}$$

We will also need the corresponding propagator where the $\overline{g}$ are replaced by the free propagator p, but where the restriction on the integration are kept. This is denoted by

$$P^{\varepsilon}_T(x) = \iint\limits_{\substack{0\leq s_1\leq s_2\leq T\\ s_2\geq\varepsilon, T-s_1\geq\varepsilon}} p_{s_1}(x)p_{s_2-s_1}(x)p_{T-s_2}(x).$$

In order to recast the first inequality of (**??**), we still have to look at the contribution of 1. Implementing this part into (**??**) just means that we forget about the interaction between $[t, s]$ and $[s, t]$. We therefore get our first basic inequality:

$$p_T(x) - g_T(x) \geq \beta \iint\limits_{0\leq s_1\leq s_2\leq T} ds_1 ds_2\, \overline{g}_{s_2-s_1}(0)(p_{s_1} * \overline{g}_{T-s_2})(x) \qquad (1.12)$$
$$- \beta^2 \iint\limits_{0\leq s_1\leq s_2\leq T} ds_1 ds_2\, (p_{s_1} * \overline{G}_{s_2-s_1} * \overline{g}_{T-s_2})(x).$$

To get an upper bound, we have to expand the interaction between the legs in (**??**) in the same way as before, and we have also to take into account the third summand of the second inequality in (**??**). The reader will convince himself quickly that all the contributions are of the same form, namely

$$\int_{A_3(\varepsilon)} d\underline{s}\ (p_{s_1} * [\overline{g}_{\Delta s_3}((\overline{g}_{\Delta s_1}\overline{g}_{\Delta s_2}) * (\overline{g}_{\Delta s_4}\overline{g}_{\Delta s_5}))] * \overline{g}_{\Delta s_7})(x). \qquad (1.13)$$

where $A_3(\varepsilon)$ is some subset of $\{\underline{s} = (s_1, s_2, \ldots, s_6) : 0 \leq s_1 \leq s_2 \leq \ldots \leq s_6 \leq T\}$ with a number of ε-gap conditions, whose exact form will be no longer of any importance, and $\Delta s_i = s_{i+1} - s_i$ ($s_7 = T$). This contribution pops up from multiplying out the square of the third summand in (**??**), dropping afterwards all the remaining interactions between the time slots there, and by expanding the interaction between the time slots in (**??**) once. The inequalities evidently all go in the correct direction to yield

$$p_T(x) - g_T(x) \leq \beta \iint\limits_{0\leq s_1\leq s_2\leq T} ds_1 ds_2\, \overline{g}_{s_2-s_1}(0)(p_{s_1} * \overline{g}_{T-s_2})(x)$$
$$- \beta^2 \iint\limits_{0\leq s_1\leq s_2\leq T} ds_1 ds_2\, (p_{s_1} * \overline{G}_{s_2-s_1} * \overline{g}_{T-s_2})(x) \qquad (1.14)$$
$$+ 3\beta^3 \int\limits_{A_3(\varepsilon)} d\underline{s}\ (p_{s_1} * [\overline{g}_{\Delta s_3}((\overline{g}_{\Delta s_1}\overline{g}_{\Delta s_2}) * (\overline{g}_{\Delta s_4}\overline{g}_{\Delta s_5}))] * \overline{g}_{\Delta s_7})(x).$$

It is worthwhile to pause and consider if anything has been achieved with the inequalities (**??**) and (**??**). A moment's reflection reveals that this is not the case. For instance, the first diagram contains an integration over a "loop" $\overline{g}_s(0)$ over a time $\geq \varepsilon$. If for the moment, we let drop the interaction completely, we have $\int_\varepsilon p_s(0)ds$ which is divergent as $\varepsilon \to 0$. One might think that the interaction could help and $\int\limits_{\varepsilon} \overline{g}_s(0)ds$ would be convergent, but this is not the case. For similar reasons, the second summand is divergent for $\varepsilon \to 0$ (but actually only marginally). We are more fortunate with the third summand on the right hand side of (**??**). If we drop all the interactions we arrive at

$$\int\limits_{A_3(\varepsilon)} ds\, (p_{s_1} * [p_{\Delta s_3}((p_{\Delta s_1} p_{\Delta s_2}) * (p_{\Delta s_4} p_{\Delta s_5}))] * p_{\Delta s_6})(x),$$

and if we drop also all the gap conditions (this gives an estimate from above), we arrive at

$$\int\limits_{0 \le s_1 \le s_2 \le s_3 \le s_4 \le s_5 \le s_6 \le T} ds\{p_{s_1} * [p_{\Delta s_3}((p_{\Delta s_1} p_{\Delta s_2}) * (p_{\Delta s_4} p_{\Delta s_5}))] * p_{\Delta s_6}\}(x),$$

and it is elementary to check that this is convergent! (For $d = 3$.) Therefore, there might be some hope that the third summand on the r.h.s. of (**??**) might be o.k. The way to get also the first two summands right is to modify the definition of $\overline{g}$ slightly by introducing so called counterterms which are cancelling the divergencies. We will then prove pointwise boundednes and decay properties by an appropriate recursion Ansatz, assuming β is small enough.

As these counterterms are supposed to cancel the loop and “three leg” divergency, we define them by the corresponding objects for the free propagator:

$$\kappa_1(a) = \int_{\varepsilon}^{1} ds\, p_s(0) \tag{1.15}$$

and

$$\begin{aligned} \kappa_2(\varepsilon) &= \int_0^1 ds\ \|P_s^{\varepsilon}\|_1 \\ &= \iiint\limits_{\substack{0 \le s_1 \le s_2 \\ s_2 \ge \varepsilon, s_3 - s_1 \ge \varepsilon}} ds_1 ds_2 ds_3 \int dx\ p_{s_1}(x) p_{s_2 - s_1}(x) p_{s_3 - s_1}(x) \end{aligned}$$

$\kappa_1(\varepsilon)$ is of course just $2(2\pi)^{-3/2}(\frac{1}{\sqrt{\varepsilon}} - 1)$. κ_2 is slightly more complicated to evaluate. Remark first that

$$\int dx\, p_{t_1}(x) p_{t_2}(x) p_{t_3}(x) = (2\pi)^{-3}[t_1 t_2 + t_1 t_3 + t_2 t_3]^{-1},$$

and therefore

$$\kappa_2(\varepsilon) = (2\pi)^{-3} \int_0^1 dt_1 \int_0^1 dt_3 \int_{(\varepsilon - t_1)\vee(\varepsilon-t_3)\vee 0}^1 dt_3 [(t_1+t_3)t_2 + t_1 t_3]^{-3/2} \tag{1.16}$$

$$= (2\pi)^{-3} \int_0^1 dt_1 \int_0^1 dt_3 (t_1+t_3)^{-1} \frac{2}{\sqrt{t_1 t_3 + (\varepsilon - t_1)\vee(\varepsilon - t_3)\vee 0}}$$

$$= (2\pi)^{-3} \int_0^1 du \frac{1}{u} \int_0^u \frac{2dv}{\sqrt{v(u-v) + (\varepsilon - v)\vee(\varepsilon - v + u)\vee 0}}$$

$$= (2\pi)^{-3} \int_\varepsilon^1 du \frac{1}{u} \int_0^u \frac{2dv}{\sqrt{v(u-v)}} + 0(1) = (2\pi)^{-2} |\log \varepsilon| + 0(1).$$

We therefore see that this is just barely divergent. The divergence of $\kappa_2(\varepsilon)$ is actually making all the trouble for $d = 3$. It should be remarked that $\kappa_1(a)$ is essentially just $EJ^\varepsilon_{0,1}$ and $\kappa_2(\varepsilon)$ the variance. If the variance stays bounded as $\varepsilon \to 0$, one can apply what in quantum field theory is called vacuum renormalization, i.e. one just has to replace J by $J - EJ$ getting something which is convergent. This is the approach of Varadhan for $d = 2$ [?].The renormalized interaction is now just

$$R^{\varepsilon,\beta}_{t,T} = \beta J^\varepsilon_{t,T} - \beta(T-t)\kappa_1(\varepsilon) + \beta^2 (T-t)\kappa_2(\varepsilon)$$

and we put

$$g^\varepsilon_{T,\beta}(x) = E_T(\exp(-R^{\varepsilon,\beta}_{0,T})\delta(\omega_T - x)).$$

We again apply

$$p_T(x) - g^\varepsilon_{T,\beta}(x) = \int_0^T dt \frac{d}{dt} E(\exp(-R^{\varepsilon,\beta}_{t,T})\delta(\omega_T - x)).$$

There are only small changes to our inequalities (**??**) and (**??**). The presence of the counterterms just gives the additional summand

$$(-\beta\kappa_1(\varepsilon) + \beta^2\kappa_2(\varepsilon)) \int_0^T dt\ p_t * g_{T-t}(x) \tag{1.17}$$

to both sides.

The main pointwise estimate for small $\beta > 0$ is

Proposition 1.3 *There exists $C_1 > 0$ and $\beta_0 > 0$, such that for all $\beta \in [0, \beta_0], T \leq 1, x \in \mathbb{R}^3, \varepsilon > 0$ one has*

$$|p_T(x) - g^{\varepsilon}_{T,\beta}(x)| \leq C\beta\sqrt{T}p_{2T}(x).$$

The proof of this estimate is by a recursion argument using our basic inequalities (**??**) and (**??**) with the appropriate corrections coming from the counterterms, i.e. (**??**). We set

$$K_0(\varepsilon, \beta) = \sup_{0<T\leq 1} \sup_{x\in\mathbb{R}^3} \frac{|g^{\varepsilon}_{T,\beta}(x) - p_T(x)|}{\sqrt{T}p_{2T}(x)}.$$

Remark first, that $K_0(\varepsilon, \beta)$ is finite for fixed $\varepsilon > 0, \beta \geq 0$. In fact, for $t \leq \varepsilon$, the interaction is 0, so

$$g^{\varepsilon}_{T,\beta}(x) = p_T(x)\exp(\beta T\kappa_1(\varepsilon) - \beta^2 T\kappa_2(\varepsilon)),$$

so the sup over $0 < T \leq \varepsilon$ is certainly finite, as $p_T(x)$ decays faster at $|x| \sim \infty$ than $p_{2T}(x)$. For the same reason, the supremum is also finite on $\varepsilon \leq T \leq 1$. $K_0(\varepsilon, \beta)$ looks the right quantity for Proposition **??**, but for technical reasons, we have to slightly change it, and we set

$$K(\varepsilon, \beta) = K_0(\varepsilon, \beta) \vee |\int_0^1 (p_s(0) - g^{\varepsilon}_{s,\beta}(0))ds|.$$

The reason is that in order to estimate K_0, one has to use estimates on $\int_0 (p_s - g_s)ds$. Evidently, this quantity itself cannot be controlled by K_0. This is a slightly awkward point, and for that reason we have to work with K instead of K_0.

The main work for proving Proposition **??** is then contained in

Proposition 1.4 *There exists a polynomial $\phi(x)$ with nonnegative coefficients, such that for all $\varepsilon > 0, \beta \in [0, 1]$, one has*

$$K(\varepsilon, \beta) \leq \beta\ \phi(K(\varepsilon, \beta)).$$

The proof of this is a bit lengthy and tedious but essentially rather straightforward. We give details of some parts of the estimates, namely the ones involving the divergent "three leg" diagram. In the next section where we prove convergence, we then focus on the other divergent part. Before we begin with that, we show how Proposition **??** implies Proposition **??**. We still need a further result

Lemma 1.5 *For any fixed $\beta > 0$, the function $(0,1) \ni \varepsilon \to K(\varepsilon,\beta)$ is continuous.*

This follows by rather a straightforward adoption of well known techniques concerning intersection local times (see [?], and the Appendix of [?]). We will not give a proof here. The Proposition **??** and Lemma **??** imply Proposition **??** in the following way:

Let

$$\varrho(\beta) \stackrel{\text{def}}{=} \inf\{x \geq 0 : x = \beta\phi(x)\}.$$

If β is small enough then we have $\varrho(\beta) \leq c\beta$. We have $K(\varepsilon,\beta) = 0$ for $\varepsilon = 1$, and as $K(\varepsilon,\beta)$ is continuous in $\varepsilon > 0$, it can never cross $\varrho(\beta)$. We therefore get the estimate $K(\varepsilon,\beta) \leq c\beta$ for all $\varepsilon > 0$ if β is small enough. This proves Proposition **??**.

To come now to the proof of Proposition **??**, we get, using our inequalities (**??**) and (**??**) with the correction (**??**) from the counterterms:

$$|p_T(x) - g_T(x)| \leq \beta \left| \int\limits_{0\leq s_1\leq s_2\leq T} ds_1 ds_2 \; g_{\Delta s_1}(0) p_{s_1} * g_{\Delta s_2}(x) - \kappa_1(\varepsilon) \int_0^T ds \; p_s * g_{T-s}(x) \right| \tag{1.18}$$

$$+ \beta^2 \left| \int\limits_{0\leq s_1\leq s_2\leq T} ds_1 ds_2 \, p_{s_1} * G_{\Delta s_1} * g_{\Delta s_2}(x) - \kappa_2(\varepsilon) \int_0^T ds \; p_s * g_{T-s}(x) \right|$$

$$+ 3\beta^3 \int d\underline{s} \; (p_{s_1} * [g_{\Delta s_3}((g_{\Delta s_1} g_{\Delta s_2}) * (g_{\Delta s_4} g_{\Delta s_5}))] * g_{\Delta s_7})(x).$$

Here $G_t^\varepsilon(x)$ is formed in the same way as $\overline{G}_T^\varepsilon(x)$, but just without the bar, meaning that the appropriate counterterms are included. In the last summand, the integration is over time slots for the vertices of the diagram which sum to T. We can drop the various ε-gap restrictions in that contribution, getting an upper bound. In contrast, the other two summands retain the gap restrictions.

We will not present all the estimates in details, but will show how to perform them for the second summand.

Lemma 1.6

$$\left|\int\limits_{0\le s_1\le s_2\le T} ds_1 ds_2\ p_{s_1} * G_{\Delta s_1} * g_{\Delta s_2}(x) - \kappa_2(\varepsilon)\int_0^T ds\ p_s * g_{T-s}(x)\right| \le \phi(K) p_{2T}(x) T^{2/3}.$$

Notations We use $\phi(x)$ as a generic polynomial with positive coefficients, not necessarily the same at different occurrences. K is always $K(\varepsilon, \beta)$. We also use C as a generic positive constant, also not necessarily the same at different occurrences, which does not depend on ε, β.

Remark 1.7 *The estimate is better than necessary for our estimate. I had not been able to get anything better than $\sqrt{T}$ in the estimate involving the first summand on the right hand side of (**??**). Whether or not this is optimal, I do not know.*

In order to prove the Lemma, we split things into three parts:

$$\left|\int\limits_{0\le s_1\le s_2\le T} ds_1 ds_2\ p_{s_1} * G_{\Delta s_1} * g_{\Delta s_2}(x) - \kappa_2(\varepsilon)\int_0^T ds\ p_s * g_{T-s}(x)\right| \tag{1.19}$$

$$\le \left|\int\limits_{0\le s_1\le s_2\le T} ds_1 ds_2\ (p_{s_1} * (G_{\Delta s_1} - P_{\Delta s_1}) * g_{\Delta s_2}(x))\right|$$

$$+ \left|\int\limits_{0\le s_1\le s_2\le T} ds_1 ds_2\ (p_{s_1} * P_{\Delta s_1} * g_{\Delta s_2}(x) - p_{s_2} * g_{\Delta s_2}(x) \times \|P_{\Delta s_1}\|_1)\right|$$

$$+ \left|\int\limits_{0\le s_1\le s_2\le T} ds_1 ds_2 p_{s_2} * g_{T-s_2}(x) \times \|P_{\Delta s_1}\|_1 - \int_0^1 ds\, \|P_{\Delta s_1}\|_1 \int_0^T ds\ p_s * g_{T-s}(x)\right|$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

(Remember that G_s and P_s have the gap conditions and particularly are nonzero only if their time length is $\ge \varepsilon$). The third summand I_3 is very easy, we begin with that. The only difference between the two contributions inside is that in the first, the integration over Δs_1 is restricted to $\varepsilon \leq \Delta s_1 \leq s_2$. Therefore

$$\begin{aligned}
I_3 &= \int_0^T ds\,(p_s * g_{T-s})(x) \int_s^1 \|P_u\|_1 du \\
&\leq C \int_0^T ds(p_s * g_{T-s})(x)|\log(s)| \\
&\leq Cp_T(x) \int_0^T |\log(s)|ds + C \int_0^T ds\,(p_s * |g_{T-s} - p_{T-s}|)(x)|\log s| \\
&\leq CT^{3/4} p_T(x) + CK \int_0^T ds(p_s * p_{2T-2s})\sqrt{T-s}|\log s| \\
&\leq Cp_{2T}(x)[T^{3/4} + \int_0^T \sqrt{T-s}|\log s|] \\
&\leq CT^{3/4} p_{2T}(x).
\end{aligned}$$

Next, we estimate I_1, which is more complicated. We first split

$$g_{t_1}(x)g_{t_2}(x)g_{t_3}(x) - p_{t_1}(x)p_{t_2}(x)p_{t_3}(x)$$

as a sum of expressions of the form $h_{t_1}(x)h_{t_2}(x)h_{t_3}(x)$, where h_t is either $p_t(x)$ or $g_t(x) - p_t(x)$, but where at least one of the h's is the latter. For definiteness, let us look at

$$(g_{t_1}(x) - p_{t_1}(x))p_{t_2}(x)p_{t_3}(x)$$

which we estimate in absolute value by

$$\begin{aligned}
&\sqrt{t_1} K p_{2t_1}(x)p_{t_2}(x)p_{t_3}(x) \\
&= \sqrt{2t_1} K [2t_1t_2 + 2t_1t_3 + t_2t_3]^{-3/2} p_\sigma(x),
\end{aligned}$$

where

$$\sigma = \frac{2t_1t_2t_3}{2t_1t_2 + 2t_1t_3 + t_2t_3}. \tag{1.20}$$

We also replace $g_{\Delta s_2}$ in I_1 by $p_{\Delta s_2}$ and the difference, where we again estimate the latter by $\sqrt{\Delta s_2} K p_{2\Delta s_2}(x)$. Evidently, the more of the $\sqrt{\Delta s}$ terms we have, the better, so we look what happens if we just have one. Such a contribution to an upper bound of I_1 is

$$CK \int_{\substack{0\le s_1\le s_2\le s_3\le s_4\le T\\ s_3-s_1\ge\varepsilon, s_4-s_2\ge\varepsilon}} ds_1ds_2ds_3ds_4 \frac{\sqrt{\Delta s_1}}{(2\Delta s_1\Delta s_2+2\Delta s_1\Delta s_3+\Delta s_2\Delta s_3)^{3/2}} \times (p_{s_1+\Delta s_4} * p_\sigma)(x) \tag{1.21}$$

where $\Delta s_i := s_{i+1}-s_i$ and σ is from (**??**), t_i replaced by Δs_i. Of course, there are also summand with K^2, K^3, K^4 in the estimates (and correspondingly more Δs_i), but these can be estimated similarly.

Let us look at what happens with the expression (**??**). Keeping $\Delta s_1, \Delta s_2, \Delta s_3$ fixed and integrating over s_1 gives just a factor $t \overset{\text{def}}{=} T-(\Delta s_1+\Delta s_2+\Delta s_3) = (s_1+\Delta s_4)$. Furthermore, a simple estimate for σ is $\sigma \le \sum_i \Delta s_i = T-t$, which yields

$$tp_t(x) \le t^{-1/2}Tp_{T-\sigma}(x).$$

Therefore,

$$\begin{aligned}(\mathbf{??}) &\le CKT^{3/2}p_T(x)\int_{t_i\ge 0,\sum t_i<1} \frac{\sqrt{t_1}dt_1dt_2dt_3}{\sqrt{1-\sum t_i}(t_1t_2+t_1t_3+t_2t_3)^{3/2}}\\ &\le CKT^{3/2}p_T(x).\end{aligned}$$

The other expressions get similar estimates, but we cannot always have $p_T(x)$. However, $p_{2T}(x)$ is o.k., too, of course. Also, there are expressions with K^2, K^3, K^4. Altogether, we get

$$I_1 \le \phi(K)T^{3/2}p_{2T}(x),$$

which is much better than required.

There remains to estimate I_2 (which is the only place where we have to increase T to $2T$ in the estimate). Even if we drop all ε restrictions, $P_t^\varepsilon(x)$ is evidently finite for $x \ne 0$, and has L_1-norm $\|P_t^0\|_1 = c/t$. Despite the fact that this is divergent for $t \to 0$, it is nevertheless true that for $t \sim 0$, is $P_t^\varepsilon(x)$ is essentially concentrated close to 0, and therefore there is not much difference between $p_s * P_t^\varepsilon * g_u$ and $(p_s * g_u) \times \|P_t^\varepsilon\|_1$.

If we set

$$\tau(s) = [(\Delta s_1\Delta s_2+\Delta s_1\Delta s_3+\Delta s_2\Delta s_3]^{-3/2},$$

$$\sigma(s) = \Delta s_1 \Delta s_2 \Delta s_3 \tau^{2/3},$$

where as usual $\Delta s_i = s_{i+1} - s_i$, then

$$p_{\Delta s_1}(x) p_{\Delta s_2}(x) p_{\Delta s_3}(x) = (2\pi)^{-3} \tau(s) p_{\sigma(s)}(x).$$

Therefore

$$I_2 \leq C \int ds\, \tau(s)(|p_{s_1+\sigma(s)} - p_{s_4}| * g_{T-s_4})(x),$$

where the integration is over $0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq T$, and where we have dropped the ε-gap restrictions. We split the above integral in the part, where $s_1 + \sigma \leq s_4/2$ and the complement of this. On $\{s_1 + \sigma \leq s_4/2\}$ there is actually no cancellation needed and we estimate $|p_{s_1+\sigma} - p_{s_4}| \leq p_{s_1+\sigma} + p_{s_4}$, and of course, we also estimate

$$g_{T-s_4} \leq p_{T-s_4} + K\sqrt{T-s_4}\, p_{2T-2s_4} \leq \phi(K) p_{2T-2s_4}.$$

Therefore,

$$\begin{aligned}
&\int_{\{s_1+\sigma<s_4/2\}} ds\, \tau(s)(|p_{s_1+\sigma(s)} - p_{s_4}| * g_{T-s_4})(x) \\
&\leq \phi(K) \int_{\{s_1<s_4/2\}} ds\, \tau(s)(p_{2T-s_4} + p_{2T-2s_4+s_1+\sigma})(x) \\
&\leq \phi(K) \int_{\{s_1<s_4/2\}} ds\, \tau(s) p_{2T}(x) + \phi(K) \int_{\{s_1<s_4/2, s_4>T/2\}} ds\, \tau(s) p_{2T-2s_4+s_1+\sigma}(x).
\end{aligned}$$

The integration of τ over s_2, s_3 for given s_1, s_4 is just $\|q_{s_4-s_1}\|_1 = c|s_4 - s_1|$, and so the first summand is $\phi(K) T\, p_{2T}(x)$. As for the second, it is

$$\begin{aligned}
&\leq \phi(K) \int_{\{s_1<s_4/2, s_4>T/2\}} \tau(s)(T - s_4 + s_1)^{-3/2} e^{-|x|^2/2T} \\
&\leq \phi(K) e^{-|x|^2/2T} \int_0^{3T/4} \frac{du}{\sqrt{u}(T-u)} \leq \phi(K) T p_{2T}(x).
\end{aligned}$$

Altogether we get

$$\int_{\{s_1+\sigma<s_4/2\}} ds\,\tau(s)(|p_{s_1+\sigma(s)} - p_{s_4}| * g_{T-s_4})(x) \le \phi(K)Tp_{2T}(x).$$

It remains to estimate the integral over $\{s_1 + \sigma(s) \ge s_4/2\}$. In that case it would be disastrous to take the absolute value inside. Instead we use the elementary estimate

$$|p_u(x) - p_v(x)| \le c|u - v|v^{-1}p_{2v}(x),$$

when $v/2 \le u \le v$. Using this, we get

$$\begin{aligned}
&\int_{\{s_1+\sigma\ge s_4/2\}} ds\,\tau(s)(|p_{s_1+\sigma(s)} - p_{s_4}| * g_{T-s_4})(x) \\
&\le \int ds\ \tau(s)(s_4 - s_1 - \sigma)\frac{1}{s_4}p_{s_4} * g_{T-s_4}(x) \\
&\le \phi(K)p_{2T}(x) \int ds\,\tau(s)(s_4 - s_1)\frac{1}{s_4} = \phi(K)Tp_{2T}(x),
\end{aligned}$$

the integrals of course all restricted to $0 \le s_1 \le s_2 \le s_3 \le s_4 \le T$.

Altogether, we have $I_2 \le \phi(K)Tp_{2T}(x)$. We have therefore completely proved Lemma **??**.

The estimates of the other pieces in (**??**) are running along similar lines. The first summand is actually $\le \beta\phi(K)\sqrt{T}p_{2T}(x)$. In the course to prove this one needs K at one place and not just K_0. The fourth summand is estimated by observing that the diagram with lines given by the free propagator is convergent and can be estimated by $CT^{3/2}p_{2T}(x)$, and replacing successively the $g's$ by the $p's$ one catches only additional factors $\phi(K)$. Altogether, one therefore estimates the last summand by $\beta^3\phi(k)T^{3/2}p_{2T}(x)$. Summing these estimates, one obtains

$$K_0(\varepsilon, \beta) \le \beta\phi(K).$$

I will not give the details of these estimates which are a bit tedious but essentially straightforward (see [**?**]). It then still remains to estimate

$$|\int_0^1 (p_T(0) - g_T(0))dT| \le \beta\phi(K). \tag{1.22}$$

As there is a slight slip in the argument in [**?**], we give the proof here. We use the same expansion which underlies the estimate (**??**). As the second and third summand on the right hand side of (**??**) are at most $\beta^2\phi(K)T^{3/4}p_{2T}(x)$, and $T^{3/4}p_{2T}(0)$ is integrable at 0, it suffices to prove

$$\left|\int_0^1 dT \int\limits_{0\le s_1\le s_2\le T} p_{s_1} * g_{\Delta s_2}(0) g_{\Delta s_1}(0) - \int_0^1 dT \int_0^T ds\ p_s * g_{\Delta s}(0) \int_\varepsilon^T p_s(0)ds\right| \le \phi(K).$$

The left hand side of this expression is

$$\left|\int\limits_{\substack{u+v+t\le 1\\ v\ge\varepsilon}} dudvdt(p_u * g_t)(0)g_v(0) - \int\limits_{u+t\le 1} dudvdt(p_u * g_t)(0)\int_\varepsilon^1 dvp_v(0)\right|$$

$$\le \left|\int\limits_{u+t\le 1-\varepsilon} dudt(p_u * g_t)(0) \int\limits_{\varepsilon}^{1-u-t} dv(p_v - g_V)(0)\right|$$

$$+ \left|\int\limits_{u+t\le 1} dudt(p_u * g_v)(0) \int\limits_{1-u-t}^{1} p_v(0)dv\right|$$

$$\le K \int\limits_{u+t\le 1} dudt(p_u * g_t)(0) + \int\limits_{u+t\le 1} dudt(p_u * g_t)(0) \int\limits_{1-u-t}^{1} dv|p_v(0) - g_v(0)|$$

$$+ C \int\limits_{u+t\le 1} dudt(p_u * g_t)(0)(1-u-t)^{-1/2}.$$

We now estimate g_t by $p_t + \phi(K)p_{2t} \le \phi(K)p_{2t}$ and get that the above expression is

$$\le \phi(K)\left\{\int\limits_{u+t\le 1} dudt(u+t)^{-3/2}(1-u-t)^{-1/2}\right\} = \phi(K).$$

(**??**) is therefore proved which implies now Proposition **??**. We therefore have Proposition **??**.

We now show that the relevant information contained in Proposition **??** can be boosted by a rescaling argument due to Albeverio and Zhou, [**?**] to cover any $\beta > 0$.

Proposition 1.8 *Let $\beta \geq 0, T \leq 1$. Then there exist constants $c_1(\beta), \ldots, c_4(\beta) > 0$ with*

a) $g^{\varepsilon}_{T,\beta}(x) \leq c_1(\beta) p_{c_2(\beta)T}(x), \quad \forall \varepsilon > 0.$

b) $c_3(\beta) \leq \|g^{\varepsilon}_{T,\beta}\|_1 \leq c_4(\beta)$.

The basis is the following simple rescaling property

Lemma 1.9 *For all $T, \beta, \varepsilon > 0$*

$$g^{\varepsilon}_{T/2,\beta}(x) = g^{2\varepsilon}_{T,\beta/\sqrt{2}}(\sqrt{2}x)\exp(a(\varepsilon,\beta,T)),$$

where $\sup\limits_{\varepsilon>0, T\leq 1} |a(\varepsilon,\beta,T)| < \infty$ *for all $\beta > 0$.*

Proof

$$g^{\varepsilon}_{T/2,\beta}(x) = E\left(\exp\left[-\beta J^{\varepsilon}_{0,T/2}(\omega) + \frac{\beta T}{2}\kappa_1(\varepsilon) - \frac{\beta^2 T}{2}\kappa_2(\varepsilon)\right]\delta(\omega_{T/2} - x)\right)$$

$$= E\left(\exp\left[-\frac{\beta}{\sqrt{2}}J^{2\varepsilon}_{0,T}(\omega) + \frac{\beta T}{\sqrt{2}}\frac{\kappa_1(\varepsilon)}{\sqrt{2}} - \left(\frac{\beta}{\sqrt{2}}\right)^2 T\kappa_2(\varepsilon)\right]\delta(\omega_T - \sqrt{2}x)\right),$$

$$\frac{1}{\sqrt{2}}\kappa_1(\varepsilon) = \frac{1}{\sqrt{2}}\frac{2}{(2\pi)^{3/2}}\left(\frac{1}{\sqrt{\varepsilon}} - 1\right) = \kappa_1(2\varepsilon) + \left(1 - \frac{1}{\sqrt{2}}\right)\frac{2}{(2\pi)^{3/2}},$$

$$\kappa_2(\varepsilon) = \kappa_2(2\varepsilon) + 0(1).$$

Implementing this, we get the conclusion. ■

Proof *of Proposition* **??***a)* From Proposition **??** we already know that there exists $\beta_0 > 0$ such that the statement is true for $\beta \leq \beta_0$. We now prove that if the statement is correct for $\beta \leq \hat{\beta}$, then it is correct for $\beta \leq \sqrt{2}\hat{\beta}$.

Let $\beta \leq \sqrt{2}\hat{\beta}, T \leq 1, \varepsilon > 0$. Then

$$\begin{aligned} g^{\varepsilon}_{T,\beta}(x) &= E\exp[-R^{\varepsilon,\beta}_{0,T}]\delta(\omega_T - x) \\ &= E\exp[-R^{\varepsilon,\beta}_{0,T/2} - R^{\varepsilon,\beta}_{T/2,T}]\delta(\omega_T - x) \\ &= (g^{\varepsilon}_{T/2,\beta} * g^{\varepsilon}_{T/2,\beta})(x) \\ &\leq e^{2a(\varepsilon,\beta,T)}(g^{2\varepsilon}_{T,\beta/\sqrt{2}}(\sqrt{2}\cdot) * g^{2\varepsilon}_{T,\beta/\sqrt{2}}(\sqrt{2}\cdot))(x) \\ &\leq 2^{3/2}e^{a(\varepsilon,\beta,T)}c_1 p_{2c_2}(\sqrt{2}x), \end{aligned}$$

where $c_i = c_i(\beta/\sqrt{2})$. As $p_a(\sqrt{2}x) \le c p_a(x)$, this proves the claim. ■

Proof *of Proposition* **??***b)* The upper bound follows from part a), so it remains to prove the lower bound. First remark that from Lemma **??** we get

$$\|g^{\varepsilon}_{T/2,\beta}\|_1 \ge c(\beta)\|g^{2\varepsilon}_{T,\beta/\sqrt{2}}\|_1. \tag{1.23}$$

We again use "induction" on β. Assume that the lower bound in Proposition **??**b) is correct for $\beta \le \hat{\beta}$, and assume $\beta \le \sqrt{2}\hat{\beta}$.

Let $P_{(2)}$ (with corresponding expectation $E_{(2)}$) be the product measure of two independent Brownian motions of length $T/2$. If ω_1, ω_2 are two paths, we write

$$J^\varepsilon(\omega_1,\omega_2) = \int_0^{T/2} ds \int_0^{T/2} dt 1_{s+t\ge\varepsilon}\delta(\omega_{1,s} - \omega_{2,t}).$$

Then

$$\|g^{\varepsilon}_{T,\beta}\|_1 = E^{(2)} \exp\left[-R^{\varepsilon,\beta}_{0,T/2}(\omega_1) - R^{\varepsilon,\beta}_{0,T/2}(\omega_2) - \beta J^\varepsilon(\omega_1,\omega_2)\right].$$

Let $\hat{P}^{\varepsilon,\beta}$ be the polymer measure (with gap ε) on paths of length $T/2$, and $\hat{P}^{\varepsilon,\beta}_{(2)}$ be the corresponding product measure. Then

$$\begin{aligned}\|g^{\varepsilon}_{T,\beta}\|_1 &= \|g^{\varepsilon}_{T/2,\beta}\|_1^2 \hat{E}^{\varepsilon,\beta}_{(2)} \exp(-\beta J^\varepsilon)\\ &\ge c(\beta)^2 \|g^{2\varepsilon}_{T,\beta/\sqrt{2}}\|_1^2 \exp(-\beta \hat{E}^{\varepsilon,\beta}_{(2)} J^\varepsilon),\end{aligned}$$

by (**??**). By the induction assumption, we have $\|g^{2\varepsilon}_{T,\beta/\sqrt{2}}\|_1 \ge c_1(\beta)$. In order to prove the result, we only have to estimate

$$\hat{E}^{\varepsilon,\beta}_{(2)} J^\varepsilon = \int_0^{T/2} ds \int_0^{T/2} dt 1_{s+t\ge\varepsilon} E_{(2)} \left\{ e^{-R^{\varepsilon,\beta}_{0,T/2}(\omega_1)} e^{-R^{\varepsilon,\beta}_{0,T/2}(\omega_2)} \delta(\omega_{1,s} - \omega_{2,t}) \right\} \Big/ \|g^{\varepsilon,\beta}_{T/2}\|_1^2$$

from above, and we again estimate $\|g^{\varepsilon,\beta}_{T/2}\|_1^2$ from below with (**??**) and the induction assumption.

$$\begin{aligned}&E_{(2)} \left\{ \exp(-R^{\varepsilon,\beta}_{0,T/2}(\omega_1) - R^{\varepsilon,\beta}_{0,T/2}(\omega_2))\delta(\omega_{1,s} - \omega_{2,t}) \right\}\\ &\qquad \le \int dx g^{\varepsilon,\beta}_s(x) g^{\varepsilon,\beta}_t(x) \|g^{\varepsilon,\beta}_{T/2-s}\|_1 \|g^{\varepsilon,\beta}_{T/2-t}\|_1\\ &\qquad \le c(\beta) \int dx p_{c_1(\beta)s}(x) p_{c_1(\beta)t}(x)\\ &\qquad \le c(\beta) p_{c_1(t)(s+t)}(0) = c_2(\beta)(s+t)^{-3/2}.\end{aligned}$$

Integrating over s,t gives the desired claim. ■

We can already derive an important conclusion

Proposition 1.10 *For all $\beta > 0$ the family of measures*

$$\{\hat{P}^{\varepsilon}_{1,\beta}\}_{\varepsilon>0}$$

is tight.

Proof The counterterms play of course no rôle for the measures. So

$$\hat{P}^{\varepsilon}_{1,\beta}(d\omega) = \exp(-R^{\varepsilon,\beta}_{0,1}(\omega))P_1(d\omega)/\|g^{\varepsilon,\beta}_1\|_1.$$

Therefore for $\leq t < t+s \leq 1$ by Proposition **??**

$$\begin{aligned}\int |\omega_t - \omega_{t+s}|^4 \hat{P}^{\varepsilon}_{1,\beta}(d\omega) &\leq C(\beta) \int |\omega_t - \omega_{t+s}|^4 \exp(-R^{\beta,\varepsilon}_{0,1}(\omega))P_1(d\omega) \\ &\leq C(\beta) \int |\omega_t - \omega_{t+s}|^4 \exp(-R^{\beta,\varepsilon}_{0,t} - R^{\beta,\varepsilon}_{t,t+s} - R^{\beta,\varepsilon}_{t+1,1})P_1(d\omega) \\ &= C(\beta) \int g_t(x) g_s(y-x) g_{1-t-s}(z-y)|x-y|^4 \\ &\leq C(\beta)|t-s|^2.\end{aligned}$$

This tightness follows now by standard criteria. ■

1.3 The convergence of $P^{\varepsilon}_{T,\beta}, \varepsilon \to 0$

It is somehow evident that the inequalities presented in Section **??** are not able to prove convergence. The reason simply is that the difference of the upper and lower bounds deviate by the contribution

$$\int d\underline{s}\ (p_{s_1} * [g_{\Delta s_3}((g_{\Delta s_1} g_{\Delta s_2}) * (g_{\Delta s_4} g_{\Delta s_5}))] * g_{\Delta s_7})(x),$$

which does not go to 0 as $\varepsilon \to 0$, but only stays finite. We would be much better of, if one of the integrations involved would be only over an interval which becomes small with ε.

The idea to achieve something of this type is to differentiate with respect to the gap width $\varepsilon > 0$. To do this, the gap regularization is evidently much better suited than e.g. a lattice regularization. As mentioned in the introduction to this Chapter, it can also be proved that the lattice regularization measure converges to the same limit. However, there are considerable additional difficulties popping up and we will not go into that. It should also be remarked that the inequalities we will get by differentiating with respect to $\varepsilon > 0$ are somewhat more delicate to handle for reasons which will become clear. We will heavily rely on the boundednes (and tightness) properties already obtained.

Let $\psi : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be bounded and smooth, and for $0 \le s < t \le 1$ define $\Psi_{s,t} : \Omega \to \mathbb{R}$ by

$$\Psi_{s,t}(\omega) = \exp(\int_s^t \psi(u, \omega_u) du).$$

The functions $\Psi = \Psi_{0,1} : \Omega \to \mathbb{R}$ will be convenient for us. They form a convergence determining class, i.e. if we prove that

$$\lim_{\varepsilon \to 0} \int \Psi d\hat{P}^{\varepsilon}_{T,\beta} \tag{1.24}$$

exists (for suitable T, β), then we have proved convergence of the measures, given of course the tightness which is already proved. We fix $T = 1$. Given the validity of the estimates in Proposition **??**, we prove the convergence of the expression (**??**): Let

$$\varrho_\psi(\varepsilon) = \int \Psi \exp(-R^{\beta,\varepsilon}_{0,1}) dP_1.$$

Proposition 1.11 *For any bounded function ψ and all $\beta > 0$ there exists an integrable function $i : (0,\infty) \to (0,\infty)$ such that for any $\varepsilon_2 > \varepsilon_1$*

$$\varrho_\psi(\varepsilon_2) - \varrho_\psi(\varepsilon_1) \ge - \int_{\varepsilon_1}^{\varepsilon_2} i(\varepsilon) d\varepsilon.$$

The bound together with the bounds in Proposition **??** immediately prove Theorem **??** (for $d = 3$). Indeed as the $\varrho_\psi(\varepsilon)$ stay bounded by Proposition **??**, Proposition **??** implies that $\lim_{\varepsilon \to 0} \varrho_\psi(\varepsilon)$ exists. This together with the tightness proved in Proposition **??** proves the convergence of the measures.

We fix now ψ bounded and smooth (with bounded derivatives of all desired order, say) and we write just $\varrho(\varepsilon)$. We give some detailed explanations how the bound Proposition **??** is obtained. First, we simply write

$$\varrho(\varepsilon_2) - \varrho(\varepsilon_1) = \int_{\varepsilon_1}^{\varepsilon_2} \frac{d\varrho}{d\varepsilon} d\varepsilon.$$

We actually do not want to prove that $\frac{d\varrho}{d\varepsilon}$ exists. This can be circumvented in the same way as in Section **??**: We replace all δ function by $p'_a s$, derive the necessary inequalities and finally let go $a \to 0$ in the end. This is evidently somewhat cumbersome to write down, so we pretend that we can work directly with the δ function. Differentiating gives

$$\frac{d}{d\varepsilon}\varrho(\varepsilon) = \int_0^{1-a} ds E(e^{-R_{0,1}^{\beta,\varepsilon}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) + (\beta\kappa_1'(\varepsilon) - \beta^2\kappa_2'(\varepsilon))\varrho(\varepsilon) \quad (1.25)$$

The crucial point is now as follows. $R_{0,1}^{\beta,\varepsilon}$ of course still contains all the interactions, and we somehow want to expand that out like in the previous Section. Especially, we want to expand out the interactions between the interval $[s, s+\varepsilon]$ and its complement. This will lead to contributions which cancel the nonintegrability of the counterterms. The delicacy is coming from the fact that we are *not* allowed to expand the interaction of the time before s and after $s+\varepsilon$ out in any way. Although these contributions are finite, they would, if expanded not completely exactly, lead to a destruction of all the cancellations. We therefore have to control these cancellations in the presence of the interactions of the time before s and after $s+\varepsilon$. Let

$$R_{0,1}^{\beta,\varepsilon} = \tilde{R}_{0,1}^{s,\beta,\varepsilon} + \beta J_{0,s;s+\varepsilon}^{\varepsilon} + \beta J_{s,s+\varepsilon;s+1,1}^{\varepsilon}, \quad (1.26)$$

where

$$\tilde{R}_{0,1}^{s,\beta,\varepsilon} = R_{0,s}^{\beta,\varepsilon} + R_{s+\varepsilon,1}^{\beta,\varepsilon} + \beta J_{0,s;s+\varepsilon,1} - \beta\varepsilon\kappa_1(\varepsilon) + \beta\varepsilon\kappa_2(\varepsilon). \quad (1.27)$$

The presence of the $J_{0,s;s+\varepsilon,1}$-summand in (**??**) is making a lot of trouble. Of course, we will like to argue that dropping the two last summands on the r.h.s. of (**??**) is cancelling with $\kappa_1'(\varepsilon)$, and expanding once is cancelling with $\kappa_2'(\varepsilon)$. However, this will be a cancellation of divergent terms (as $\varepsilon \to 0$) and as in Section **??**, some surgery will be needed to operate the divergency out. The crucial point is that we do not want to expand out any interaction unless it is an interaction connecting an ε-piece to something else. In this way we get estimates which after the cancellation of the divergencies become controllable for $\varepsilon \to 0$.

We give details here for the first part where the contribution coming from $\tilde{R}$ cancels with κ_1'. We then give a sketch how the rest is done.

We use i as a generic function $(0,\infty) \to (0,\infty)$ which is integrable near 0, not necessarily the same at different occurrences.

Proposition 1.12

$$\int_0^{1-\varepsilon} dsE(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) + \kappa_1'(\varepsilon)\varrho(\varepsilon) \geq -i(\varepsilon).$$

(We usually drop $\beta, \varepsilon' s$ at places where they obviously have to be, e.g. in $\tilde{R}$.)

Proof *of Proposition* **??** We set

$$Y_s = Y_s^\varepsilon = \int_{\substack{u \leq s \leq v \leq 1-\varepsilon \\ r-u \leq \varepsilon}} \delta(\omega_u - \omega_v) du dv,$$

and $\theta_\varepsilon(\omega)(t) = \omega(t-\varepsilon)$. Then

$$E(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) = E(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi_{0,s}\Psi_{s+\varepsilon,1}(1+0(\varepsilon))).$$

The $1+0(\varepsilon)$ is just the $\Psi_{s,s+\varepsilon}$. There is evidently no interaction inside $[s, s+\varepsilon]$ because of the gap, and in $\tilde{R}$ we have left out the interaction of the "loop" with the rest. We take separately the expectation over $\delta(\omega_s - \omega_{s+\varepsilon})$ which is just $p(\varepsilon) = -\kappa_1'(\varepsilon)$, but then the interactions do no longer quite fit, because we no longer have any gap condition between the path before and after s, after having cut the loop out. To restore this, we have to correct by Y_s: Using (**??**) we get

$$\begin{aligned} &E(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) \\ &= p_\varepsilon(0)E(e^{-R_{0,1-\varepsilon}-\beta Y_s}\Psi e^{\beta\varepsilon\kappa_1(\varepsilon)}e^{-\beta^2\varepsilon\kappa_2(\varepsilon)}(1+0(\varepsilon)) \\ &= p_\varepsilon(0)E(e^{-R_{0,1-\varepsilon}-\beta Y_s}\Psi(1+\beta\varepsilon\kappa_1(\varepsilon))(1+0(\varepsilon|\log\varepsilon|)). \end{aligned}$$

It is evident from the considerations in Section **??** that $E\exp(-R^s_{0,1-\varepsilon}-\beta Y^\varepsilon_s)$ stays bounded (as $\varepsilon\to 0$), and so we can neglect the $0(\varepsilon|\log\varepsilon|)$ contribution as $p_\varepsilon(0)\varepsilon|\log\varepsilon|$ is integrable at 0. However $p_\varepsilon(0)\varepsilon\kappa_1(\varepsilon)$ is not integrable, a fact of which we will be pleased as it will cancel the contribution coming from Y_s. As $Y_s\geq 0$, we get

$$e^{-\beta Y_s}\geq 1-\beta Y_s,$$

and therefore

$$\int_0^{1-\varepsilon} ds\ E(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})\Psi)$$

$$\geq p_\varepsilon(0)(1+\beta\varepsilon\kappa_1((\varepsilon))\int_0^{1-\varepsilon} ds\left\{E(e^{-R_{0,1-\varepsilon}}\Psi)-\beta E(Y_s e^{-R_{0,1-\varepsilon}}\Psi)\right\}-i(\varepsilon).$$

It is plausible that

$$E(e^{-R_{0,1-\varepsilon}}\Psi)=E(e^{-R_{0,1}}\Psi)+0(\varepsilon)=\varrho(\varepsilon)+0(\varepsilon),$$

but it is not quite obvious how to prove this. We would need something like a bound for

$$\frac{d}{dv}E(e^{-R_{0,v}}\Psi).$$

That is close to what we have done in Section **??**, but there the integration over v was important. (That we differentiate here with respect to the upper boundary in contrast to the lower is of course of no relevance.) One can however squeeze out essentially of what has been done in Section **??**, a slightly worse bound which is good enough for our purpose:

Lemma 1.13 *There exists $\delta>0$ such that*

$$|E(e^{-R_{0,1}}\Psi)-E(e^{-R_{0,1-\varepsilon}}\Psi)|\leq c\varepsilon^{1/2+\delta}.$$

We will not give a proof here as it is essentially a repetition of some of the steps of Section **??** (see p. 96 of [?]). With the help of Lemma **??**, the proof of Proposition **??** is now easily finished: We have

$$
\begin{aligned}
E(Y_s e^{-R_{0,1-\varepsilon}}\Psi) &= \int\limits_{\substack{u\le s\le v\\ v-u\le\varepsilon}} dudv\; E(\delta(\omega_u-\omega_v)e^{R_{0,1-\varepsilon}}\Psi)\\
&\le \int\limits_{\substack{u\le s\le v\\ v-u\le\varepsilon}} p_{v-u}(0)E(e^{-R_{0,1-\varepsilon-(v-u)}}\Psi)+0(\varepsilon^{3/2})
\end{aligned}
$$

where we have just dropped the interactions between the interval $[u,v]$ and the rest, and the "less than ε" interaction after readjusting time. This increases the expression. The readjustment of Ψ gives only an $0(\varepsilon^{3/2})$ contribution which we can incorporate into $i(\varepsilon)$. By Lemma **??**, we can replace $R_{0,1-\varepsilon-(v-u)}$ by $R_{0,1}$, making an error which again can be incorporated into $i(\varepsilon)$. Therefore, we get

$$
\begin{aligned}
&\int_0^{1-\varepsilon} ds\; E(e^{-\tilde{R}^s_{0,1}}\delta(\omega_s-\omega_{s-\varepsilon})\Psi)\\
&\ge p_\varepsilon(0)(1+\beta\varepsilon\kappa_1(\varepsilon))\varrho(\varepsilon)\left(1-\beta\int_0^{1-\varepsilon} ds \int\limits_{\substack{u\le s\le v\\ v-u\le\varepsilon}} p_{v-u}(0)dudv\right)-i(\varepsilon)\\
&= p_\varepsilon(0)\varrho(\varepsilon)-i(\varepsilon),
\end{aligned}
$$

as $\varepsilon\kappa_1(\varepsilon)=\int_0^{1-\varepsilon} ds \int\limits_{\substack{u\le s\le v\\ v-u\le\varepsilon}} p_{v-u}(0)dudv+0(\varepsilon)$. As $p_\varepsilon(0)=-\kappa_1'(\varepsilon)$, this proves Proposition **??**. ■

¿From Proposition **??**, we get

$$
\begin{aligned}
\frac{d}{d\varepsilon}\rho(\varepsilon) \ge -\beta^2\int_0^{1-\varepsilon} ds\, E\left(\mathrm{e}^{-\tilde{R}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})\left[J_{0,s;s,s+\varepsilon}+J_{s,s+\varepsilon;s+\varepsilon,1}\right]\Psi\right)\\
\times(1+O(1/\sqrt{\varepsilon}))-\lambda^2\frac{d}{d\varepsilon}\kappa_2(\varepsilon)-i(\varepsilon).
\end{aligned}
\tag{1.28}
$$

In order to finish the proof of Proposition **??**, it therefore only remains to show that there is some cancellation between the first and the second summand on the right hand side of the above inequality, which leads to something integrable in ε. $\frac{d}{d\varepsilon}\kappa_2(\varepsilon)$ is of order $1/\varepsilon$, so it is clear that not much cancellation is needed. This helps very much, and allows for application of relatively crude estimates. On the other hand, it is also clear that the cancellation is here somewhat more subtle than the one in Proposition **??**, because the three leg diagram is more delicate to handle than the loop one.

I will not give the details here of the estimates, as it is a bit repetitive of what had been done in Proposition **??** (and in the last Section). Here a short outline: One of the problems is of course that $J_{0,s;s,s+\varepsilon}$ and $J_{s,s+\varepsilon;s+\varepsilon,1}$ contain interactions which go outside the interval $[s, s+\varepsilon]$, so they come into conflict with $\tilde{R}^s_{0,1}$. As remarked at the beginning of this section, it is not possible to cancel or expand the interaction inside $\tilde{R}^s_{0,1}$ which ties the part before and that after s. However, it turns out that we can essentially neglect the interactions inside $\tilde{R}^s_{0,1}$ which come into conflict with the above J-terms. What helps here a lot is the fact that the divergency is only logarithmic, and one can work with somewhat crude estimates. What one does is to choose some parameter $0 < \gamma < 1$, whose value is not very important, and cut out from $\tilde{R}^s_{0,1}$ all the interactions with the intervals $[s-\varepsilon^\gamma, s]$ and $[s+\varepsilon, s+\varepsilon+\varepsilon^\gamma]$. However, we retain (this is crucial) the interaction between $[0, s-\varepsilon^\gamma]$ and $[s+\varepsilon+\varepsilon^\gamma, 1]$. This surgery cost an error which can be incorporated into $i(\varepsilon)$. This is essentially an argument like the one involving the Lemma **??** above. Likewise, we drop inside $J_{0,s;s,s+\varepsilon}$ the interaction between $[0, s-\varepsilon^\gamma]$ and $[s, s+\varepsilon]$, and similarly for $J_{s,s+\varepsilon;s+\varepsilon,1}$. In this way, we keep the interactions inside (**??**) separated, and we can now operate the divergency out, cancelling with the derivative of κ_2. There arise now the same problems we had encountered in the last section, namely, that in contrast with the loop diagram, one has to "glue" the two loose ends together, after taking out the three-leg diagram, but this can essentially be handled in the same way as we did it in details in the last section. One then still has to restore the interaction with the now "void" interval $[s-\varepsilon^\gamma, s+\varepsilon+\varepsilon^\gamma]$, and one has to show that this gives again an error which can be incorporated into $i(\varepsilon)$.

The whole procedure is a bit messy and needs some care, but it should be fairly evident that with the tricks already developed, this can be done, and (**??**) can be proved in this way, leading then to Proposition **??**. For further details, see the [**?**].

2 Self-attracting random walks, large deviations, and collapse transitions

We discus in this chapter a number of problems on self-attracting path interactions which are all closely related to large deviation theory. A simple case of an attraction would be to just change sign in the (weakly) self repellent case of chapter I. For technical reasons, it is convenient to work with continuous time but discrete state space Markov processes. Therefore, we consider the standard symmetric random walk on $\mathbb{Z}^d$ starting in 0 having holding times with expectation $1/d$. The path measure on the space $D_\infty = D([0,\infty), \mathbb{Z}^d)$ of right continuous piecewise constant paths is denoted by P. As usual, we write $X_t(\omega) = \omega_t$, $\omega \in D_\infty$ for the evaluation map. We then would try to transform the path measure in the same way as the weakly repellent case, just having the opposite sign of the coupling constant:

$$\hat{P}_T(d\omega) = \exp\left[\beta \int_0^T ds \int_0^T dt 1_{\omega_t=\omega_s}\right] P_T(d\omega) \Big/ Z_{T,\beta}, \quad \beta > 0,$$

However, it is easy to see that this is not an interesting object, as the self-attraction is far too strong. In fact, a path staying just all the time at 0 gets a weight $\exp\left[\beta T^2\right]$, whereas the entropic cost for doing that is only of order $\exp\left[-cT\right]$. It is therefore evident that as $T \to \infty$ the path measures just concentrates with probability going to 1 on the path identical to 0. A more interesting object is obtained when having the interaction only of order $1/T$. Therefore, we define for $\beta > 0$:

$$\hat{P}_{T,\beta}(d\omega) = \exp\left[\frac{\beta}{T} \int_0^T ds \int_0^T dt\, 1_{\omega_t=\omega_s}\right] P_T(d\omega) \Big/ Z_{T,\beta}. \tag{2.1}$$

This path measure has been investigated in two papers [?] and [?]. In the first one, it was shown that the for $d \geq 2$, the measure behaves diffusively if β is small enough, and in the second, it was shown that for $d = 1$, and in all dimensions if β is large enough, the path measures is localized in the sense that the end points ω_T have fluctuations of order one, but which stay non trivial in the $T \to \infty$ limit. Therefore, for $d \geq 2$, there is what is called a collapse transition if β grows from small values to large ones. We will give the argument for the diffusive behavior in section 2, and discuss the localized phase in section 3. For lack of space and time, I cannot present that here in all the details, but I will give a technically simplified version of the key step of the argument in [?].

There are other models which have a similar behavior. One case is Brownian motion transformed by the Wiener sausage in such a way that large volumes of the sausage are suppressed. For a random walk the role of volume of the sausage is played by the number $N_T(\omega)$ of sites visited up to time T, and for these, this would correspond in transforming the path measure P in the following way:

$$d\hat{P}_{T,\beta}(\omega) = \exp\left[-\beta N_T(\omega)\right] dP(\omega)/\, Z_{T,\beta},$$

where

$$Z_{T,\beta} = E\left(\exp\left[-\beta N_T(\omega)\right]\right).$$

It had been proved in [?] (and in [?] for the Wiener sausage) that for $d = 2$ the path measure is concentrated on paths which stay inside a ball of radius of order $T^{1/4}$. This is closely related to the classical analysis of Donsker and Varadhan of the leading order asymptotic behavior of $Z_{T,\beta}$. Sznitmans results and techniques have been extended very recently to arbitrary dimension by Povel [?]. We will give a discussion of these results in section ??. Sznitman's approach uses the enlargement of obstacles techniques (see [?]). The approach in [?] is more combinatorial by "path counting", and is rather involved. The problem amounts to a droplet construction, where the macroscopic droplet is trivial, namely just a ball. It is remarkable that one can prove that in all dimensions the microscopic droplet approaches the macroscopic in $L_\infty-$norm, whereas the corresponding analytic variational problem which can be reduced to the ordinary isoperimetric problem which characterizes the droplet, is stable only in L_1 (for $d \geq 3$). We will not be able to present the full details here, but we will give a discussion of this aspect.

This model has no collapse transition: For all $\beta > 0$, the path measures lives on a droplet of scale $T^{1/(d+2)}$. However, an interesting and somewhat unexpected features shows up if we make the self-attracting weaker by replacing β by a coupling constant which go to 0 as $T \to \infty$. Fix $\alpha > 0$ and define

$$Z_{T,\beta} = E_T\left(\exp\left[-\frac{\beta}{T^\alpha} N_T(\omega)\right]\right).$$

One way to estimate this is just Jensen's inequality, which gives the trivial estimate

$$Z_{T,\beta} \geq \exp\left[-\frac{\beta}{T^\alpha} E_T\left(N_T(\omega)\right)\right].$$

It is well know that for $d \geq 3$, asymptotically $E_T\left(N_T(\omega)\right) \sim \kappa T$, where κ is the escape probability for a discrete time random walk from a single point. For $\alpha = 0$, this estimate is very bad as it is known from the classical work of Donsker and Varadhan [?] that $Z_T \approx \exp\left[-cT^{d/(d+2)}\right]$. It turns out that the Jensen estimate is essentially sharp as soon as $\alpha > 2/d$. As α crosses $2/d$, there is a collapse transition: For $\alpha < 2/d$, the path measure is localized on scale $T^{\frac{1+\alpha}{2+d}}$, and for $\alpha > 2/d$, it is just diffusive. Remark that the critical case $\alpha = 2/d$ ($d \geq 3$), would correspond to the path measure living on a subdiffusive scale $T^{1/d}$. This critical case has recently been investigated in [?] and there are some quite interesting features. For instance, it turns out that there is a collapse transition from small to large β. I will discuss this critical case in section **??**.

There are several motivations for the investigation of these problems. In the physical literature, the main interest in collapse transitions are for models which have a mixed attractive and repulsive interaction. Mathematically, absolutely nothing is known, not even about the diffusive behavior in high dimensions. For the physical background, see [?], [?]. As an example, consider the interacting random walk (in discrete time, say), defined by

$$\hat{P}_{n,\beta,\gamma}(\omega) = \exp\left[-\beta \sum_{1\leq i<j\leq n} 1_{\omega_i=\omega_j} + \gamma \sum_{1\leq i<j\leq n} 1_{|\omega_i-\omega_j|=1}\right],$$

$\beta, \gamma > 0$. One would expect that if $\gamma \ll \beta$, and at least in high dimension, the repulsion dominates the attraction, and the model would just be diffusive. There is however no proof of this, and it appears that the lace expansion with which the diffusive behavior for $\gamma = 0$ has been proved is completely powerless as soon as there is a positive γ. In the physical literature, there is a collapse transition predicted if one changes the parameter γ.

Some of the investigations above had been motivated by the problem to determine the effective mass of the so-called polaron, which is a (one-dimensional) model with a Kac-type potential and a continuous symmetry (actually a shift degeneracy). This is still (mathematically) an open problem. For a heuristic derivation, see [?].

Convention: During this chapter, we use $c, c_1, c_2, ...$ as generic constants not necessarily the same at different occurences. They may depend on the dimesion, and on a fixed coupling constant, but on nothing else, except when indicated clearly.

2.1 Introduction, and a simple "maximum entropy principle"

To start with, we consider the following trivial problem. Let $X_1, X_2, \ldots$ be a sequence of independent coin tossings: $P(X_i = 0) = P(X_i = 1) = \frac{1}{2}$. If $\alpha > \frac{1}{2}$, then by the Bernoulli law of large numbers

$$P(S_n/n \geq \alpha) \to 0,$$

as $n \to \infty$, where $S_n = \sum_{i=1}^{n} X_i$. Question: what is the limiting distribution of X_1, if we condition on the event $\{S_n/n \geq \alpha\}$? The answer is evident:

$$\lim_{n\to\infty} P(X_1 = 1 \,| S_n/n \geq \alpha) = \alpha.$$

Similarly, the conditioned distribution of $x_1, \ldots, x_{k(n)}$ converges to coin tossing if $k(n) = \sigma(1)$.

We consider a slightly more general problem. We assume that the X_i are i.i.d. random variables, taking values in some Polish space S equipped with its Borel field $\mathcal{S}$. P is the product measure of the law μ_o of the X_i on $\Omega = (S, \mathcal{S})^{\mathbb{N}}$, with the X_i being the projections $\Omega \to S$. Let

$$L_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$$

be its empirical distribution. Let further $F : \mathcal{M}_1^+(S) \to [-\infty, \infty)$ be an upper semicontinuous function. ($\mathcal{M}_1^+(S)$ is the set of probability measure on $(S, \mathcal{S})$). We will assume that F is bounded above, but it may take the value $-\infty$. We consider the transformed measure on Ω

$$d\hat{P}_n = \frac{1}{Z_n} \exp\left[nF(L_n)\right] dP.$$

$$Z_n = E\left(\exp[nF(L_n)]\right)$$

By Sanov's Theorem, and the upper semicontinuity of F, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log Z_n \leq b_F \stackrel{\text{def}}{=} \sup_{\mu} \left[F(\mu) - \int \log\left(\frac{d\mu}{d\mu_o}\right) d\mu \right], \tag{2.2}$$

and if F_{lc} is the lower continuous modification of F, then

$$\liminf_{n\to\infty} \frac{1}{n} \log Z_n \geq b_{F_{\text{lc}}}. \tag{2.3}$$

$\int \log\left(\frac{d\mu}{d\mu_o}\right) d\mu$ has to be understood to be ∞ if μ is not absolutely continuous w.r.t. μ_o, or if the logarithm is not in $L_1(\mu)$. For these standard results, see [**?**].

Proposition 2.1 *Assume $b_F = b_{F_{\text{lc}}}$ Then the sequence $\{\hat{P}_n\}$ is tight in the weak topology on the set of probability measures on Ω. Any limiting probability measure Q has a representation $Q = \int \mu^{\mathbb{N}} \Gamma(d\mu)$, where Γ is a probability measure on $\mathcal{M}_1^+(S)$ which is concentrated on*

$$K_F = \left\{ \mu : F(\mu) - \int \log\left(\frac{d\mu}{d\mu_o}\right) d\mu = b_F \right\}$$

Remark 2.2 *This is a very weak formulation of a so called propagation of chaos result. For much stronger variants (under more restrictive conditions on F), see [**?**]*

Proof The proof is a very easy application of the Sanov Theorem. From the well known fact that the rate function $I(\mu) = \int \log\left(\frac{d\mu}{d\mu_o}\right) d\mu$ has compact level sets, it follows that K_F is a compact subset of $\mathcal{M}_1^+(S)$. Moreover, if $U_\varepsilon(K_F)$ is an open neighborhood of K_F, it follows from (**??**), (**??**) and the assumption $b_F = b_{F_{\text{lc}}}$ that

$$\lim_{n\to\infty} \hat{P}_n(L_n \in U_\varepsilon(K_F)) = 1.$$

In fact, $\hat{P}_n(L_n \notin U_\varepsilon(K_F)) = E\left(\exp\left[nG(L_n)\right]\right) / E\left(\exp\left[nF(L_n)\right]\right)$, where we set $G \stackrel{\text{def}}{=} F$ on $(U_\varepsilon(K_F))^c$ and $-\infty$ otherwise. Then the denominator behaves in leading order as $\exp[nb_F]$, whereas the numerator can be estimated from above in leading order by $\exp\left[n\sup_{\mu\notin U_\varepsilon(K_F)}\left[F(\mu)-\int\log\left(d\mu/d\mu_o\right)d\mu\right]\right] \ll \exp[nb_F]$.

Therefore, the sequence $\left(\hat{P}_n L_n^{-1}\right)_{n\geq 1}$ of probability measures on $\mathcal{M}_1^+(S)$ is tight and any limit measure is supported by K_F. Now, we decompose

$$\hat{P}_n(\cdot)=\int_{\mathcal{M}_1^+(S)}\hat{P}_n(\cdot|L_n)\,d\hat{P}_n L_n^{-1}.$$

Evidently, we have $\hat{P}_n(\cdot|L_n)=P_n(\cdot|L_n)$, which is just drawing without replacement. It is well know that for large n, drawing without replacement is asymptotically the same as drawing with replacement, if we consider only $o(n)$ drawings (which is much more than we need for weak topology considerations). Therefore, in the weak topology (and also in some stronger ones), $P_n(\cdot|L_n)$ is close to $L_n^{\mathbb{N}}$. From this, we easily see that the sequence $\{\hat{P}_n\}_{n\geq 1}$ is tight (as a sequence of probability measures on Ω), and every limit point is of the required form. ■

The above Proposition evidently applies to the coin tossing example at the beginning. The empirical distribution there is just the relative number of 1's in the sequence, and we take $F=0$ if this is $\geq\alpha$, and $-\infty$ otherwise. It should however be remarked that already quite simple modification of this trivial example can become quite delicate, as is revealed by the following example (see [**?**], where it is mentioned as a difficult example not solvable by the methods developed there):

Exercise 2.3 *Start with the coin tossing sequence of length* n *as above, and define*

$$T_n=\sum_{i=1}^{n-1}1_{\{X_i=1,X_{i+1}=1\}}.$$

Then determine

$$\lim_{n\to\infty}P(X_1=1|T_n/n\geq\alpha)$$

for $\alpha>1/4$.

The exercise falls into a category of problems running under the heading "equivalence of ensembles", in that case between some sort of microcanonical and grand canonical ones. There is still no really general and satisfactory theory for that (see for instance [**?**]).

Remark 2.4 *If K_F contains just one point, say μ, then the Proposition states that $\hat{P}_n$ converges to $\mu^{\mathbb{N}}$. If K_F contains more than one point, then one usually has to derive finer asymptotics in order to evaluate the limit law of $\hat{P}_n$. The situation we encounter in some of the following sections is more delicate than the one in Proposition* **??***, mainly because there K_F contains more than one point (and is not even compact).*

Let us now start with discussing the self-attracting random walk. P is the law of the standard symmetric random walk (in continuous time), starting at 0, with holding times of expectation $1/d$, and we define the transformed path measure $\hat{P}_{T,\beta}$ by (**??**). It is formally convenient to have $\hat{P}_{T,\beta}$ defined as a measure on paths of infinite length, i.e. on $D_\infty = D\left([0,\infty),\mathbb{Z}^d\right)$. Of course, after time T it is just an ordinary random walk. Remark that the Hamiltonian $\frac{1}{T}\int_0^T ds \int_0^T dt \, 1_{\omega_t=\omega_s}$ can be written as $T\,\|l_T\|_2^2$ where l_T is the normalized local time:

$$l_T(x) \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T 1_{\{X_s=x\}} ds,$$

and $\|l_T\|_2^2 = \sum_x l_T(x)^2$. Clearly, l_T is a random probability measure on $\mathbb{Z}^d$. It satisfies a weak LDP (see e.g. [**?**]):

Proposition 2.5 *a) For any weakly open subset U of $\mathcal{M}_1^+(\mathbb{Z}^d)$ one has:*

$$\liminf_{T\to\infty} \frac{1}{T}\log P(l_T \in U) \geq \inf_{\mu\in U} I(\mu).$$

b) For any weakly compact subset K of $\mathcal{M}_1^+(\mathbb{Z}^d)$ one has:

$$\limsup_{T\to\infty} \frac{1}{T}\log P(l_T \in K) \leq \inf_{\mu\in K} I(\mu)$$

where $I(\mu) = \frac{1}{2}\sum_{\langle x,y\rangle}(\sqrt{\mu(x)} - \sqrt{\mu(y)})^2$, where summation is over (unordered) nearest neighbor pairs x,y.

¿From this Proposition, we easily get:

Proposition 2.6

$$\lim_{T\to\infty}\frac{1}{T}\log Z_{T,\beta} = \lim_{T\to\infty}\frac{1}{T}\log E\exp\left[\beta T\left\|l_T\right\|_2^2\right] \tag{2.4}$$
$$= b(\beta) \overset{\text{def}}{=} \sup_\mu\left[\beta\sum_x \mu(x)^2 - I(\mu)\right].$$

Proof This is essentially Varadhan's Lemma but there is a slight problem. If $F:\mathcal{M}_1^+(\mathbb{Z}^d)\to\mathbb{R}$ is continuous, and has the property that $\{\mu : F(\mu)\geq c\}$ is compact for all c, then by Varadhan's Lemma

$$\lim_{T\to\infty}\frac{1}{T}\log E\exp\left[TF(l_T)\right] = \sup_\mu\left(F(\mu)-I(\mu)\right).$$

In our case, we take $F(\mu)=\sum_x \mu(x)^2$, but this evidently does not satisfy the above compactness property. There is however a very simple trick. Consider the periodized situation, where we replace $\mathbb{Z}^d$ by a finite discrete torus $T_R = \{0,\ldots,R-1\}^d$, and correspondingly a symmetric random walk with periodic boundary conditions on this torus. We can just map the old random walk by "winding it up" in an evident way. Then we have

$$\left\|l_T\right\|_2^2 \leq \left\|l_T^R\right\|_2^2,$$

where $l_T^R(x)$ is the local time for the wound up random walk on the torus. Now, for the random walk on the torus, we evidently have a full LDP as $\mathcal{M}_1^+(T_R)$ itself is compact, with a rate function $I^R(\mu)=\frac{1}{2}\sum_{\langle x,y\rangle}(\sqrt{\mu(x)}-\sqrt{\mu(y)})^2$, the only difference being the summation is now over nearest neighbors on the torus. Therefore

$$\limsup_{T\to\infty}\frac{1}{T}\log Z_{T,\beta} \leq \limsup_{T\to\infty}\frac{1}{T}\log E\exp\left[\beta T\left\|l_T^R\right\|_2^2\right]$$
$$= b^R(\beta) \overset{\text{def}}{=} \sup_\mu\left[\beta\sum_x \mu(x)^2 - I^R(\mu)\right],$$

and it is easy to see that $\lim_{R\to\infty} b^R(\beta) = b(\beta)$. Therefore, we get

$$\limsup_{T\to\infty} \frac{1}{T} \log Z_{T,\beta} \leq b(\beta).$$

The lower bound is no problem and follows from the weak LDP (and the continuity of the functional). ■

It should be remarked that the above monotonicity argument is rather special. We will encounter in section 3 a situation where such a procedure cannot immediately be applied, and where things become then more delicate. Having the above large deviation property, a natural first question is to ask whether or not there are minimizers of the variational problem. This is directly connected with the question if $b(\beta) > 0$.

Proposition 2.7 *a) If $d = 1$, then $b(\beta) > 0$ for all $\beta > 0$.*
b) If $d \geq 2$ then there exists $\beta_{cr}(d) > 0$ such that $b(\beta) > 0$ for $\beta > \beta_{cr}(d)$ and $b(\beta) = 0$ for $\beta < \beta_{cr}(d)$.

Proof Evidently, $b(\beta)$ is increasing in β, and furthermore, $b(\beta) > 0$ if β is large enough. This simply follows from the fact that $I(\delta_0)$ is finite. Therefore, it remains to prove that for $d = 1$, we have $b(\beta) > 0$ for all β, and that for $d \geq 2$, there exists a constant $c > 0$ such that

$$\sum_x \mu(x)^2 \leq cI(\mu). \tag{2.5}$$

We start with the one dimensional case. We define a sequence of measures which become flat and flatter:

$$\mu_n(x) = \frac{\max(1 - |x|/n, 0)^2}{\xi_n},$$

where ξ_n is the appropriate norming. Evidently, $\xi_n \sim cn$. Therefore, $\sum_x \mu_n(x)^2 \sim c/n$, and $I(\mu_n) \sim cn^{-2}$. Therefore, $\beta \sum_x \mu_n(x)^2 > I(\mu_n)$ for any $\beta > 0$ if n is large enough. This proves a).

The inequality (**??**) simply follows from the (discrete version of the) standard Sobolev inequality

$$\|g\|_4^4 \leq C \|g\|_2^2 \|\nabla g\|_2^2 ,$$

applied to $\mu(x) = g^2(x)$. Here $\bigtriangledown$ denotes the discrete gradient. ■

It turns out that if $b(\beta) > 0$, then there are solutions of the variational problem. Let

$$K_\beta \stackrel{\text{def}}{=} \left\{ \mu \in \mathcal{M}_1^+(\mathbb{Z}^d) : \beta \sum_x \mu(x)^2 - I(\mu) = b(\beta) \right\}.$$

One of the basic difficulties we will encounter is that K_β is shift invariant: Any shift of an element of K_β is again in K_β. We summarize the basic facts about this set.

Proposition 2.8 *Assume $b(\beta) > 0$. Then*
a) $K_\beta \neq \emptyset$.
b) Any $\mu \in K_\beta$ satisfies $\mu(y) > 0$ for all $y \in \mathbb{Z}^d$.
c) There exist $c, \gamma > 0$ such that for any $\mu \in K_\beta$ there exists $x_\mu \in \mathbb{Z}^d$ with $\mu(y - x_\mu) \leq c \exp[-\gamma |y|]$ for all y.

The proof is not difficult, but a bit lengthy. I will not give it here (see [?]).

A natural question is if there is uniqueness modulo shifts as soon as $b(\beta) > 0$. Unfortunately, I don't know the answer, not even for $d = 1$. Corresponding uniqueness questions for variational problems in the continuous setting on $\mathbb{R}^d$ have a very long history with many results. However, the knowledge about similar questions on $\mathbb{Z}^d$ is essentially zero. One of the difficulties in the discrete situation is that standard symmetrization techniques do not work. The discrete problems seem to be inherently more delicate than the continuous ones. Take for instance the variational problem in the one-dimensional case, but in the continuous situation. This just is the problem to maximize

$$\beta \int g(x)^4 dx - \frac{1}{2} \int g'(x)^2 dx,$$

subject to the condition $\int g(x)^2 dx = 1$. It is easy to see that modulo shifts, there is just one solution of the Euler equation

$$4\beta g(x)^3 + \Delta g(x) = \lambda g(x)$$

which decays to 0 at infinity and satisfies $\int g(x)^2 dx = 1$ (just $\sqrt{\beta/2}/\cosh(\beta x)$ and its shifts). On the other hand, the discrete problem, namely to maximize

$$\beta \sum g(x)^4 - \frac{1}{2} \sum (g(x+1) - g(x))^2$$

subject to $\sum g(x)^2 = 1$ (we have replace $\mu(x)$ by $g(x)^2$) has countably many such solutions. I have no formal proof of this, but playing on the computer one "sees" them. (Computer simulations however indicate that among these solution there are just two candidates as maximizers. Both are symmetric, although we don't have a proof that the maximizers have to be symmetric.)

It is not difficult to see that one has uniqueness if β is large enough. This is just coming from the fact that for "$\beta = \infty$", the solutions are unique, namely just the δ_x. By a perturbation argument around $\beta = \infty$ one can prove that uniqueness persists for large β :

Proposition 2.9 *If $\beta \geq 2d$ then*

$$K_\beta = \left\{ \theta_x \mu : x \in \mathbb{Z}^d \right\}$$

for some single $\mu \in \mathcal{M}_1^+(\mathbb{Z}^d)$, where $\theta_x : \mathcal{M}_1^+(\mathbb{Z}^d) \to \mathcal{M}_1^+(\mathbb{Z}^d)$ is the usual shift $\theta_x \mu(y) = \mu(y - x)$.

This is Proposition 1.19 of [**?**]. I will not repeat the proof here. There is actually nothing special about $2d$. The proof allows for slightly lower values of β.

In view of the Proposition **??**, it is natural to conjecture that for $\beta > \beta_{cr}(d)$, one has that $\hat{P}_{T,\beta}$ behaves for large T such that l_T is close to some element in K_β. This is in fact true (see Proposition **??** below). What makes things delicate is that K_β contains infinitely many elements. It will turn out that there are infinitely many elements of K_β which will get positive limiting weight under $\hat{P}_T l_T^{-1}$. In the case where one has uniqueness modulo shifts actually all elements of K_β get positive weight. However, for $\mu \in K_\beta$ which lie for out, these weight will be small, uniformly in T. A preformulation of the main result on this collapsed phase is the following

Theorem 2.10 *Assume* $b(\beta) > 0$. *Then*
a) $\left(\hat{P}_{T,\beta}\, l_T^{-1}\right)_{T>0}$ *is tight in* $\mathcal{M}_1^+(\mathcal{M}_1^+(\mathbb{Z}^d))$.
b) There exists $c(\beta) > 0$ *such that*

$$\sup_T \int \exp\left[c(\beta)\,\|\omega_T\|\right] d\hat{P}_{T,\beta} < \infty.$$

c) If there is uniqueness modulo shift, then $\lim_{T\to\infty} \hat{P}_{T,\beta}\, \omega_T^{-1}$ *exists.*

It is natural to conjecture that for $\beta < \beta_{cr}$, $\hat{P}_{T,\beta}$ just behaves diffusively, but there is no full proof of that. What Brydges and Slade in [**?**] proved is that for $d \geq 2$ there exists $\beta_o(d) \leq \beta_{cr}(d)$ such that for $\beta < \beta_o(d)$ there is diffusive behavior (with some complications for $d = 2$). We will threat this in the next section. In section **??**, we will then come to the large β i.e. collapsed case.

The two sections afterwards are devoted to related problems for the Wiener sausage.

2.2 The diffusive phase for self-attracting random walks

I am presenting part of the arguments in [**?**] for the existence of a diffusive phase for dimensions $d \geq 2$. The two dimensional case is the most interesting one, and it is related to the topics discussed in chapter 1. I give a detailed discussion of the case $d \geq 3$. For abbreviation, we set

$$\gamma_T = \frac{1}{T}\int_0^T dt \int_0^T ds 1_{\omega_s = \omega_t} = T\,\|l_T\|_2^2\,.$$

Let $p_s(y)$ be the transition probabilities for our random walk. It is well known that

$$p_s(y) \leq c_1 \min\left[1, |s|^{-d/2}\right] \exp\left[-c_2 |y|/s\right] \tag{2.6}$$

and

$$G(y) \stackrel{\text{def}}{=} \int_0^\infty p_s(y) ds \leq c \min\left(|y|^{-d+2}, 1\right) \tag{2.7}$$

for $d \geq 3$.

Lemma 2.11 *Assume* $d \geq 3$.
a) There exists $\beta_o(d) > 0$ *such that*

$$\sup_{T>0} E\left(\exp\left[\beta\gamma_T\right]\right) < \infty$$

for $\beta < \beta_o$.
b) $E\left(\gamma_T - E\gamma_T\right)^2 = o(1)$ *as* $T \to \infty$.

Proof a) By Jensen's inequality, we have

$$\exp\left[\beta\gamma_T\right] \leq \sum_y l_T(y) \exp\left[\beta T l_T(y)\right] = \frac{1}{T} \sum_y \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \left(\int_0^T ds 1_{\omega_s = y}\right)^{n+1}$$
$$= \frac{1}{T} \sum_y \sum_{n=0}^{\infty} \beta^n (n+1) \int_{0 \leq s_1 < \ldots < s_{n+1} \leq T} 1_{\left\{\omega_{s_1} = y, \ldots, \omega_{s_{n+1}} = y\right\}} ds_1 \ldots ds_{n+1}.$$

Therefore,

$$E\left(\exp\left[\beta\gamma_T\right]\right) \leq \frac{1}{T} \sum_y \sum_{n=0}^{\infty} \beta^n (n+1) \int_{0 \leq s_1 < \ldots < s_{n+1} \leq T} p_{s_1}(y) p_{s_2 - s_1}(0) \ldots p_{s_{n+1} - s_n}(0)$$
$$= \frac{1}{T} \sum_{n=0}^{\infty} \beta^n (n+1) \int_{0 \leq s_1 < \ldots < s_{n+1} \leq T} p_{s_2 - s_1}(0) \ldots p_{s_{n+1} - s_n}(0) \leq \sum_{n=0}^{\infty} \beta^n (n+1) G(0)^n,$$

which is finite if $\beta < G(0)^{-1}$. This proves a)

In order to prove b), remark first that

$$E\gamma_T = \frac{2}{T} \sum_y \int_0^T ds \int_s^T dt \, p_s(y) p_{t-s}(0)$$
$$= \frac{2}{T} \int_0^T ds \int_s^T dt \, p_{t-s}(0) = 2G(0) + o(1).$$

Therefore, we have to prove

$$E\left(\gamma_T^2\right) = 4G(0)^2 + o(1).$$

$$\gamma_T^2 = \frac{4}{T^2} \int_{0 \leq s_1 < t_1 \leq T} ds_1 \, dt_1 \int_{0 \leq s_2 < t_2 \leq T} ds_2 \, dt_2 \, 1_{\omega_{s_1} = \omega_{t_1}} 1_{\omega_{s_2} = \omega_{t_2}}.$$

When calculating the expectation, we have to distinguish between the cases where the two intervals $[s_1, t_1]$ and $[s_2, t_2]$, are disjoint, one contains the other, and when they nontrivially overlap, respectively. The first one is the main contribution:

$$\frac{8}{T^2}\int_{0\le s_1<t_1\le s_2<t_2\le T} ds_1\,dt_1\,ds_2\,dt_2\,p_{t_1-s_2}(0)p_{t_2-s_2}(0) = 4G(0)^2 + o(1).$$

It is readily checked that the other contributions are negligible:

$$R_1(T) \stackrel{\text{def}}{=} \frac{1}{T^2}\int_{0\le s_1\le s_2<t_2\le t_1\le T} ds_1\,dt_1\,ds_2\,dt_2 E\left(1_{\omega_{s_1}=\omega_{t_1}}1_{\omega_{s_2}=\omega_{t_2}}\right) = o(1)$$

$$R_2(T) \stackrel{\text{def}}{=} \frac{1}{T^2}\int_{0\le s_1\le s_2<t_1\le t_2\le T} ds_1\,dt_1\,ds_2\,dt_2 E\left(1_{\omega_{s_1}=\omega_{t_1}}1_{\omega_{s_2}=\omega_{t_2}}\right) = o(1).$$

We check this for the last case.

$$R_2(T) = \frac{1}{T^2}\sum_y \int_{0\le s_1\le s_2<t_1\le t_2\le T} ds_1\,dt_1\,ds_2\,dt_2\,p_{s_2-s_1}(y)p_{t_1-s_2}(y)p_{t_2-t_1}(y).$$

For $d \ge 4$, we can estimate the r.h.s. by $(1/T)\sum_y G(y)^3 = O(1/T)$, but for $d = 3$, this is divergent, and one has to argue slightly more carefully. Using (**??**), one gets

$$\begin{aligned} R_2(T) &\le \frac{c}{T}\sum_y \left(\int_0^T p_s(y)ds\right)^3 \le \frac{c_1}{T}\sum_y \frac{1}{|y|^3}\exp\left[-c_2\frac{|y|}{T}\right] \\ &= O\left(\frac{\log T}{T}\right). \end{aligned}$$

This proves the Lemma. ■

With this Lemma, one now easily gets the following result:

Theorem 2.12 *Assume $d \ge 3$ and $\beta < \beta_o(d)$. Then, using Brownian scaling,*

$$\rho_T(\omega)(t) \stackrel{\text{def}}{=} \omega(tT)/\sqrt{T},\ \omega \in D\left([0,\infty), \mathbb{R}^d\right),$$

one has

$$\lim_{T\to\infty} \hat{P}_{T,\beta}\rho_T^{-1} = P_\infty,$$

weakly, where P_∞ is the standard Wiener measure.

Proof This is immediate from the estimates in the Lemma: Let $\Phi : D\left([0,\infty),\mathbb{R}^d\right) \to \mathbb{R}$ be continuous and bounded. Then

$$\lim_{T\to\infty}\int \Phi\, d\left(\hat{P}_{T,\beta}\rho_T^{-1}\right) = \lim_{T\to\infty}\int \exp\left[\beta\gamma_T\right]\Phi\, d\left(P\rho_T^{-1}\right)\Big/\int \exp\left[\beta\gamma_T\right] d\left(P\rho_T^{-1}\right)$$

$$= \lim_{T\to\infty}\int \exp\left[\beta(\gamma_T - E(\gamma_T))\right]\Phi d\left(P\rho_T^{-1}\right)\Big/\int \exp\left[\beta(\gamma_T - E(\gamma_T))\right] d\left(P\rho_T^{-1}\right)$$

$$= \lim_{T\to\infty}\int \Phi d\left(P\rho_T^{-1}\right) = \int \Phi dP_\infty.$$

The third equality is coming from the fact that $\gamma_T - E_T(\gamma_T)$ converges to 0 in probability, according the Lemma b), and the exponential moment estimate from Lemma a). ■

Remark 2.13 *It is also not difficult to prove that one has convergence of all moments of finite dimensional distributions (see [**?**]).*

The two dimensional case is more delicate and the limiting measure is more interesting. We step back to the discussion of chapter 1. There we had argued the for $d = 2, 3$, the (formal) rescaling property of the polymer measure $d\hat{P}^{\text{Polym}}_{T,\beta} = \exp\left[-\beta J_{0,T}\right] dP^{\text{Wiener}}/Z$, which is

$$\hat{P}^{\text{Polym}}_{T,\beta T^{-(4-d)/2}}\rho_T^{-1} = \hat{P}^{\text{Polym}}_{1,\beta},$$

indicates that the self-repellent random walk with coupling parameter $\beta/T^{-(4-d)/2}$ should converge, after a Brownian scaling, toward the polymer measure with coupling parameter β. This is the content of Theorem **??** for $d = 3$ and has been proved for $d = 2$ by Stoll [**?**]. The renormalization needed to define the two dimensional polymer measure is just the subtraction of the logarithmically divergent loop diagram, i.e. just by subtracting the expectation. This is an old result of Varadhan [**?**] who proved (with a different regularization) that

$$Y_T = \lim_{\varepsilon\to 0}\left(J^\varepsilon_{0,T} - EJ^\varepsilon_{0,T}\right)$$

exists in L_2, and $E\exp\left[-\beta Y_T\right] < \infty$ for all $\beta \geq 0$. The polymer measure for $d = 2$ is therefore just

$$d\hat{P}^{\text{Polym}}_{T,\beta} = \exp\left[-\beta Y_T\right] dP^{\text{Wiener}}/Z.$$

Somewhat surprisingly, Y_T has also a positive exponential moment, as has been proved by LeGall:

Proposition 2.14 *([?]) There exists $\beta_o(2) > 0$ such that*

$$E\left(\exp\left[\beta Y_1\right]\right) < \infty$$

for $\beta < \beta_o(2)$.

Therefore, the polymer measure exists (for $d = 2$ not for $d = 3$) also with the "wrong sign" if β is small. This makes it plausible that Stoll's result stays correct also in the attractive case. This is in fact true and is the content of the following result by Brydges and Slade:

Theorem 2.15 *Assume $d = 2$. Then there exists $\beta_o(2) > 0$ such that for $0 \leq \beta < \beta_o(2)$*

a) $\sup_{T>0} E^{\mathrm{RW}} \exp\left[\beta\left(\gamma_T - E\gamma_T\right)\right] < \infty$

b) $\lim_{T\to\infty} \hat{P}_{T,\beta}\rho_T^{-1} = \hat{P}_{T,-\beta}^{\mathrm{Polym}}$

I will not give the details. There are a number of interesting observations:

- The renormalization is necessary. Evidently $E^{\mathrm{RW}} \exp\left[\beta\gamma_T\right] \to \infty$ for $\beta > 0$, simply because $\gamma_T \to \infty$ in probability.
- In contrast to the situation for $d \geq 3$, the limit in the two dimensional case depends on β.

2.3 The collapsed phase for self-attracting random walks

In this section, we will discuss the Theorem **??**. We will give a partly more precise version, identifying the limit, in the case where one has uniqueness modulo shifts. For the rest of this section, we assume

Condition 2.16 $K_\beta = \left\{\theta_x \mu_o : x \in \mathbb{Z}^d\right\}$ *for some μ_o.*

This especially applies to $\beta \geq 2d$, but it may well always be true. To cover the case where it fails is only technically a bit more cumbersome but requires no new ideas. It must however be remarked that in case of nonuniquenes, we don't have a proof that the measures converge, but the tightness properties is true as well. The proof given here incorporates some technical simplifications compared with the one given in [**?**] which make it more transparent (I hope).

Theorem 2.17 *Assume $b(\beta) > 0$ and Condition **??**. Then*
a) $\lim_{T\to\infty} \hat{P}_T l_T^{-1} = \sum_{x\in\mathbb{Z}^d} \sqrt{\mu_o(x)}\delta_{\theta_x\mu_o} \Big/ \sum_{x\in\mathbb{Z}^d} \sqrt{\mu_o(x)}$
b) $\lim_{T\to\infty} \hat{P}_T X_T^{-1} = \left(\sqrt{\mu_o} * \sqrt{\mu_o}\right) \big/ \sum_x \left(\sqrt{\mu_o} * \sqrt{\mu_o}\right)(x)$

The proof splits into three parts, which will be presented in the subsections. I will give a complete proof except for the analytic properties of the variational problem.

A crucial first step is to prove that under $\hat{P}_T$, the local times l_T concentrates with high probability to a neighborhood of K_β. We had coined this the "tube property", because K in the case of uniqueness and $d = 1$ is a sort of an infinite line, so a neighborhood looks like a tube.

The weak topology is metricized by the total variation norm $\|\cdot\|_{\mathrm{TV}}$. If A is a subset of $\mathcal{M}_1^+(\mathbb{Z}^d)$, then we write $U_\varepsilon(A)$ for the ε-neighborhood in total variation of A. So the statement is

Proposition 2.18 *For any $\varepsilon > 0$*

$$\lim_{T\to\infty} \hat{P}_{T,\beta}(l_T \notin U_\varepsilon(K_\beta)) = 0.$$

Even after having proved this "tube property" the path measure could still float around very freely. The next and crucial step is the proof that this does not happen.

Proposition 2.19 *For any $\eta > 0$ there exists $S(\eta) \in \mathbb{N}$ such that for all $\varepsilon > 0$*

$$\limsup_{T\to\infty} \hat{P}_{T,\beta} \left(l_T \notin U_\varepsilon \left(\{\theta_x\mu_o : |x| \leq S(\eta)\}\right)\right) \leq \eta.$$

¿From this tightness property, the convergence easily follows as will explained in subsection **??**.

2.3.1 The tube property: Proof of Proposition ??

It is quite evident that Proposition **??** should be true, but there is a very annoying problem to prove it. First observe that

$$\hat{P}(l_T \notin U) = \frac{E\left(1_{l_T \notin U} \exp\left[\beta T \|l_T\|_2^2\right]\right)}{E\left(\exp\left[\beta T \|l_T\|_2^2\right]\right)}.$$

For the estimation of the numerator, we define $F : \mathcal{M}_1^+(\mathbb{Z}^d) \to [-\infty, \infty)$ by $F(\mu) = \|\mu\|_2^2$ if $\mu \notin U$ and $F(\mu) = -\infty$ otherwise. As F is upper semicontinuous, we would expect to get

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T} \log E\left(1_{l_T \notin U} \exp\left[\beta T \|l_T\|_2^2\right]\right) &= \limsup_{T\to\infty} \frac{1}{T} \log E\left(\exp\left[\beta T F(l_T)\right]\right) \\ &\leq \sup\left\{\beta F(\mu) - I(\mu)\right\} \\ &= \sup_{\mu \notin U}\left\{\beta \|\mu\|_2^2 - I(\mu)\right\}\end{aligned}$$

The right hand side of this is evidently strictly smaller than $b(\beta)$, so this would prove the claim. The above inequality is however not quite clear because we only have a weak LDP at our disposal. We can try to remedy the situation by using a compactification argument, i.e. wind the random walk on the torus in the same way as we did in the proof of Proposition **??**. The problem is that in our situation, the monotonicity argument does not work out such nicely. We would like to argue as follows: Fix some (large) $R \in \mathbb{N}$ and consider again the wound up random walk on the torus T_R. Denote the corresponding set of solutions of the variational problem by $K^R = \left\{\mu \in \mathcal{M}_1^+(T_R) : \beta \|\mu\|_2^2 - I^R(\mu) = b^R(\beta)\right\}$. For a given neighborhood U_ε of K we would like to find $\varepsilon' > 0$ such that for any large enough R

$$1_{l_T \notin U_\varepsilon(K)} \exp\left[\beta \|l_T\|_2^2\right] \leq 1_{l_T^R \notin U_{\varepsilon'}(K^R)} \exp\left[\beta \left\|l_T^R\right\|_2^2\right], \tag{2.8}$$

in which case we would get the desired inequality, by estimating

$$\lim_{T\to\infty} \frac{1}{T} \log E\left(1_{l_T^R \notin U_{\varepsilon'}(K^R)} \exp\left[\beta \left\|l_T^R\right\|_2^2\right]\right) \leq \sup_{\mu \notin U_{\varepsilon'}(K^R)} \left(\beta \|\mu\|_2^2 - I^R(\mu)\right)$$

which is easily seen to be $< b(\beta)$ if R is large enough. However, (**??**) is not quite correct, as evidently there are probability measures which on the full space are far away from K, but which become close when wound up. On the other hand, it should be clear that such measures must be somewhat weird, and one should be able to control the event where (**??**) fails separately. This is indeed the case.

The proof is based on a reflection trick. Let $i \in \mathbb{N}$ and consider the space of paths D_T of paths (right continuous pure jump) of length T. We define a reflection $\varphi_i(\omega)$ of paths $\omega \in D_T$ at the hyperplane $H_i \stackrel{\mathrm{def}}{=} \left\{(i, i_2, \ldots, i_d) : (i_2, \ldots, i_d) \in \mathbb{Z}^{d-1}\right\}$ simply by switching any excursion which moves strictly to the right of the hyperplane to the left. Remark that we start left of the hyperplane as we assume $i \geq 0$. Therefore, after the switching, the whole path is at the left of the hyperplane, or on it. It is easy to estimate the density of $P_T \varphi_i^{-1}$ with respect to P_T. Let $n_{T,i}(\omega)$ be the number of times, the path visits the plane, coming from outside it. Then

Lemma 2.20 $dP_T \varphi_i^{-1} / dP_T \leq 2^{n_{T,i}}$.

This is fairly evident: The switching costs at most a factor 2 "in entropy" for every visit of the plane.

One important and easy property we are using is that "finite size" approximations of the variational problem are approximating the infinite one very well. For a proof of the following Lemma we also refer the reader to [**?**].

Lemma 2.21 *a)* $\lim_{R \to \infty} b^R(\beta) = b(\beta)$.
b) K^R *is close to* K *in the following sense: For any* $\varepsilon > 0$ *there exists* $R_o(\varepsilon)$ *such that for any* $R \geq R_o$ *one has*
b1) For any $\mu \in K$*, the wound up measure* μ^R *measure on the torus* T_R *is within* $\varepsilon-$*distance of some* $\nu \in K^R$.
b2) For any $\nu \in K^R$ *one can cut the torus open in such a way (i.e. identify it with the subset* $\{1, \ldots, R\}^d \subset \mathbb{Z}^d$ *such that if* ν *is extended by* 0 *to the whole of* $\mathbb{Z}^d$*, it is within distance less than* ε *to* K.

The Lemma states that analytically, the tube property holds, and we have to prove the probabilistic counterpart. We first state an immediate consequence of the above Lemma **??** and the LDP on the finite torus.

Lemma 2.22 *Given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all R large enough*

$$\limsup_{T\to\infty} \frac{1}{T} \log \hat{P}_T \left(l_T^R \notin U_\varepsilon(K^R) \right) \leq -\delta(\varepsilon).$$

Remark that at this stage, no uniformity in R of the estimates for $\hat{P}_T \left(l_T^R \notin U_\varepsilon(K^R) \right)$ is claimed. To prove such an uniformity is essentially the task we have in order to finish the proof of Proposition **??**.

The idea is as follows: Take $R \gg 1$. Assume we are having a path such that $l_T(\omega)$ is not close to K. We however know from Lemma **??** that l_T^R lies with large $\hat{P}_T$-probability close to K^R. By Lemma **??**, for large enough R, K^R looks much like the translates on the torus of our basic $\mu_o \in K$ (somewhat chopped to fit it onto the torus). Therefore our path, except with very small $\hat{P}_T-$probability, has to distribute its l_T-mass on several essentially disjoint translates of μ_o. The problem is of course that this may happen on an increasing number, with growing T, which looks at first glance difficult to control. Nevertheless, between these chunks of translates of μ_o on which l_T is sitting, there must be vast regions essentially not visited. We select a hyperplane H_i which is not often visited. Then the reflected path has essentially the same probability as the old one (not quite, of course, but this is measured by Lemma **??**). As we have enough "empty" space, we can choose the hyperplane in such a way that after the reflection $l_T(\varphi_i(\omega))$ is *not* close to K^R. Therefore, such a behavior of ω is excluded by Lemma **??**.

As there are several things which have to tally, we will spell out the details.

If $\delta > 0$ is small enough, we have by our condition **??**

$$U_\delta(K) = \bigcup_{l\in\mathbb{Z}^d} U_\delta(\theta_l \mu_o),$$

and similarly on the torus (if R is large enough). Therefore, if $\varepsilon, \delta > 0$ are small enough we have

$$\begin{aligned} &\hat{P}_T \left(l_T^R \in U_\varepsilon(K^R), l_T \notin U_\delta(K) \right) \qquad (2.9) \\ &= \hat{P}_T \left(\bigcup_{k\in T_R} \left\{ l_T^R \in U_\varepsilon(\theta_k \mu_o^R) \right\}, l_T \notin \bigcup_\ell U_\delta(\theta_\ell \mu_o) \right) \\ &\leq \sum_{k\in T_R} \hat{P}_T \left(l_T^R \in U_\varepsilon(\theta_k \mu_o^R), l_T \notin \bigcup_\ell U_\delta(\theta_\ell \mu_o) \right). \end{aligned}$$

Let also $\eta > 0$. From Proposition **??** and Lemma **??** it is geometrically evident that if $\varepsilon \leq \varepsilon_o(\delta, \eta)$ (small enough) and $R \geq R_o(\varepsilon, \delta, \eta)$ (large enough) then for any $\mu \notin \bigcup_\ell U_\delta(\theta_\ell \mu_o)$ with $\mu^R \in U_\varepsilon(\theta_k \mu_o^R)$ there exists a hyperplane H_i with $0 \leq i \leq R$ such that $(\hat{\mu}_i)^R \notin U_{\delta/2}(K^R)$, $\left| \|\hat{\mu}_i\|_2^2 - \|\mu\|_2^2 \right| \leq \eta$ and $\mu(H_i) \leq \varepsilon$, where $\hat{\mu}_i$ is the measure where the mass right of the hyperplane is reflected to the left. Therefore, we have

$$\left\{ l_T^R \in U_\varepsilon(\theta_k \mu_o^R), l_T \notin \bigcup_\ell U_\delta(\theta_\ell \mu_o) \right\} \tag{2.10}$$
$$\subset \bigcup_{0 \leq i \leq R} \left\{ (l_T \circ \varphi_i)^R \notin U_{\delta/2}(K^R), \left| \|l_T \circ \varphi_i\|_2^2 - \|l_T\|_2^2 \right| \leq \eta, l_T(H_i) \leq \varepsilon \right\},$$

implying

$$\hat{P}_T \left(l_T^R \in U_\varepsilon(K^R), l_T \notin \bigcup_\ell U_\delta(\theta_\ell \mu_o) \right) \tag{2.11}$$
$$\leq R^d \hat{P}_T \left(\bigcup_{0 \leq i \leq R} \left\{ (l_T \circ \varphi_i)^R \notin U_{\delta/2}(K^R), \left| \|l_T \circ \varphi_i\|_2^2 - \|l_T\|_2^2 \right| \leq \eta, l_T(H_i) \leq \varepsilon \right\} \right)$$
$$\leq R^{d+1} \max_{0 \leq i \leq R} \hat{P}_T \left((l_T \circ \varphi_i)^R \notin U_{\delta/2}(K^R), \left| \|l_T \circ \varphi_i\|_2^2 - \|l_T\|_2^2 \right| \leq \eta, l_T(H_i) \leq \varepsilon \right).$$

We would like to replace the condition $l_T(H_i) \leq \varepsilon$ by a condition on $n_{T,i}$. This can be done by still adjusting the ε. By Lemma **??** below it follows that for any $\hat{\varepsilon} > 0$ one has for $\varepsilon > 0$ small enough (depending on $\hat{\varepsilon}$)

$$P\left(l_T(H_i) \leq \varepsilon, n_{T,i} > \hat{\varepsilon} T\right) \leq \exp\left[-(\beta + 2)T\right], \tag{2.12}$$

and therefore

$$\hat{P}_T\left(l_T(H_i) \leq \varepsilon, n_{T,i} > \hat{\varepsilon} T\right) \leq \exp\left[-T\right].$$

For fixed $\hat{\varepsilon}$, and δ, η, we can choose $\varepsilon_o(\hat{\varepsilon}, \delta, \eta)$ such that for $\varepsilon < \varepsilon_o$ the above inequality is true. We can therefore replace the condition $l_T(H_i) \leq \varepsilon$

in (**??**) by $n_{T,i} \le \hat{\varepsilon} T$, making a negligible error. Next, we estimate

$$\begin{aligned}
&\hat{P}_T\left((l_T \circ \varphi_i)^R \notin U_{\delta/2}(K^R), \left|\|l_T \circ \varphi_i\|_2^2 - \|l_T\|_2^2\right| \le \eta, n_{T,i} \le \hat{\varepsilon} T\right) \\
&\le \frac{1}{Z_T} E\left(\mathrm{e}^{\beta T \|l_T\|_2^2}; (l_T \circ \varphi_i)^R \notin U_{\delta/2}(K^R), \left|\|l_T \circ \varphi_i\|_2^2 - \|l_T\|_2^2\right| \le \eta, n_{T,i} \le \hat{\varepsilon} T\right) \\
&\le \frac{\mathrm{e}^{\beta\eta T}}{Z_T} E\left(\mathrm{e}^{\beta T \|l_T \circ \varphi_i\|_2^2}; (l_T \circ \varphi_i)^R \notin U_{\delta/2}(K^R), n_{T,i} \le \hat{\varepsilon} T\right) \\
&\le \frac{\mathrm{e}^{\beta\eta T}}{Z_T} E\left(2^{n_{T,i}} \mathrm{e}^{\beta T \|l_T\|_2^2}; l_T^R \notin U_{\delta/2}(K^R), n_{T,i} \le \hat{\varepsilon} T\right) \\
&\le \mathrm{e}^{\beta\eta T} 2^{\hat{\varepsilon} T} \hat{P}_T\left(l_T^R \notin U_{\delta/2}(K^R)\right).
\end{aligned}$$

Therefore, for given $\beta, \delta > 0$ we choose $\eta, \hat{\varepsilon}$ small enough such that the decay of $\hat{P}_T\left(l_T^R \notin U_{\delta/2}(K^R)\right)$ which is guaranteed by Lemma **??** beats $\mathrm{e}^{\beta\eta T} 2^{\hat{\varepsilon} T}$, and then for $\varepsilon \le \varepsilon_o(\beta, \eta, \delta)$, and then R large enough, one gets the desired estimate for $\hat{P}_T\left(l_T \notin U_\delta(K)\right)$.

Lemma 2.23 *If ζ_i, $i \ge 1$, is a sequence of exponentially distributed random variables, with parameter 1, then for $t \le 1$*

$$P\left(\sum_{i=1}^n \zeta_i \le nt\right) \le \exp[-nh(t)],$$

where

$$\lim_{t \to 0} h(t) = -\infty.$$

Proof This is the standard one dimensional large deviation estimate. The rate function is

$$h(t) = \sup_{\lambda \le 0}\left(\lambda t - \log \int \exp[\lambda x - x] dx\right) = t - 1 + \log \frac{1}{t}.$$

■

2.3.2 Tightness: Proof of Proposition ??

We are proving a superficially weaker result:

Proposition 2.24 *There exists $\varepsilon_o > 0$ such that for all $\varepsilon \leq \varepsilon_o, \eta > 0$ there exist $S(\varepsilon, \eta) \in \mathbb{N}$, $T_o(\varepsilon, \eta) > 0$ with*

$$\hat{P}_T \left(\bigcup_{|x| \geq S(\varepsilon,\eta)} \{l_T \in U_\varepsilon(\mu_x)\} \right) \leq \eta,$$

for $T \geq T_o(\varepsilon, \eta)$, where $\mu_x \stackrel{\text{def}}{=} \theta_x \mu_o$.

Together with the Proposition **??**, this evidently implies the Proposition **??**.

If $r \in \mathbb{N}$, let $C_r \stackrel{\text{def}}{=} \{-r, -r+1, \dots, r\}^d$ and for $x \in \mathbb{Z}^d$, $C_r(x) = C_r + x$. $\partial C_r(x)$ is the inner boundary, i.e.

$$\partial C_r(x) = \{y \in C_r(x) : |y_i - x_i| = r \text{ for some } i\}.$$

We denote by $\tau_r(x)$ the first hitting time of $\partial C_r(x)$ and by $\xi_r(x)$ the time the process spends on $\partial C_r(x)$ after $\tau_r(x)$ before leaving it for the first time. We need some control that the process does not leave $\partial C_r(x)$ too quickly.

Lemma 2.25

$$\lim_{\rho \to 0} \hat{P}_T \left(\xi_r(x) \leq \rho \right) = 0,$$

uniformly in T, r, x.

Proof Define

$$Y_t(\omega) = \begin{cases} \omega_t & \text{for } t \leq \tau \\ \omega_{t+\xi} & \text{for } t > \tau \end{cases}.$$

Then $\{Y_t\}_{t>0}$ and ξ_r are independent under P. Let $l'_T(x) = (1/T) \int_0^T 1_{\{Y_s = x\}}$. Then

$$\left| \|l_T\|_2^2 - \|l'_T\|_2^2 \right| \leq c \frac{\xi}{T}.$$

Therefore,

$$\begin{aligned}E\left(\exp\left[\beta T\left\|l_T\right\|_2^2\right];\xi\le\rho\right)&\le c\beta E\left(\exp\left[\beta T\left\|l'_T\right\|_2^2\right];\xi\le\rho\right)\\&\le c\beta\rho E\left(\exp\left[\beta T\left\|l'_T\right\|_2^2\right];\xi\le 1\right)\\&\le c_1\beta\rho E\left(\exp\left[\beta T\left\|l_T\right\|_2^2\right]\right).\end{aligned}$$

This proves the claim. ■

We need a further technical Lemma:

Lemma 2.26 *Given* $\eta>0$*, there exists* $r_o(\eta)$ *such that*

$$\sup_{T\ge 1}\sum_{r=r_o(\eta)}^{\infty}\hat{P}_T\left(\tau_r(0)\le\sqrt{r}\right)\le\eta.$$

Proof We introduce for $t<T$:

$$l_{t,T}(y)\stackrel{\text{def}}{=}\frac{1}{T-t}\int_t^T 1_{X_u=y}\,du.$$

We have the convex combination $l_T=\frac{t}{T}l_t+\frac{T-t}{T}l_{t,T}$, and therefore

$$\left\|l_T\right\|_2^2=\frac{t^2}{T^2}\left\|l_t\right\|_2^2+\frac{(T-t)^2}{T^2}\left\|l_{T-t}\right\|_2^2+2\frac{t(T-t)}{T^2}\left\langle l_t,l_{T-t}\right\rangle. \tag{2.13}$$

Applying this to $t=\sqrt{r}$ we get $T\left\|l_T\right\|_2^2=T\left\|l_{\sqrt{r},T+\sqrt{r}}\right\|_2^2+O(\sqrt{r})$, and therefore

$$\begin{aligned}E\left(\exp[T\beta\left\|l_T\right\|_2^2];\tau_r\le\sqrt{r}\right)&\le \mathrm{e}^{c\sqrt{r}}E\left(\exp[T\beta\left\|l_{\sqrt{r},T+\sqrt{r}}\right\|_2^2]\right)P\left(\tau_r\le\sqrt{r}\right)\\&=\mathrm{e}^{c\sqrt{r}}Z_{T,\beta}P\left(\tau_r\le\sqrt{r}\right).\end{aligned}$$

If the random walk reaches $C_r(0)$ in time $\le\sqrt{r}$, it has to make at least r jumps in this time. Applying now Lemma **??**, the claim easily follows. ■

Given ε, let $\ell(\varepsilon)$ be chosen such that $\mu_o(C^c_{\ell(\varepsilon)})\le\varepsilon$. The main step in the proof of the Proposition **??** is given by the following result:

Lemma 2.27 *If $\varepsilon > 0$ is small enough, there exists $A(\varepsilon) > 0$, such that for x with $|x|, T, u \geq A(\varepsilon)$ and $\rho \in (0, 1]$*

$$\hat{P}_T \left(l_T \in U_\varepsilon(\mu_x), \xi_{\ell(\varepsilon)}(x) > \rho, \tau_{\ell(\varepsilon)}(x) > u \right) \leq \frac{c}{\rho} \exp\left[-c_1 u\right], \tag{2.14}$$

(with c, c_1 which may depend on β but nothing else, as usual).

Proof We abbreviate $\xi_{\ell(\varepsilon)}(x)$ simply as ξ, and $\tau_{\ell(\varepsilon)}(x)$ as τ during this proof. $A(\varepsilon)$ is chosen in any case bigger than $\ell(\varepsilon)$. Then 0 is outside $B_{\ell(\varepsilon)}(x)$. Remark first that for $|x| > \ell(\varepsilon)$ and $l_T \in U_\varepsilon(\mu_x)$, the process can spend outside of $C_{\ell(\varepsilon)}(x)$ only a total time less than a proportion of εT. Therefore, on $\{l_T \in U_\varepsilon(\mu_x)\}$ we have

$$\tau \leq c\varepsilon T.$$

Therefore

$$\begin{aligned} & E\left(\mathrm{e}^{\beta T \|l_T\|_2^2}; l_T \in U_\varepsilon(\mu_x), \xi > \rho, u < \tau\right) \\ & \leq E\left(\mathrm{e}^{\beta T \|l_T\|_2^2}; l_T \in U_\varepsilon(\mu_x), \xi > \rho, u < \tau \leq c\varepsilon T.\right) \\ & \leq \frac{1}{\rho} \int_u^{c\varepsilon T + \rho} dt\, E\left(\mathrm{e}^{\beta T \|l_T\|_2^2}; l_T \in U_\varepsilon(\mu_x), \xi > \rho, t - \rho < \tau \leq t\right) \\ & \leq \frac{1}{\rho} \int_u^{c\varepsilon T + \rho} dt\, E\left(\mathrm{e}^{\beta T \|l_T\|_2^2}; l_T \in U_\varepsilon(\mu_x), X_t \in \partial C_{\ell(\varepsilon)}(x), t - 1 < \tau\right), \end{aligned}$$

where in the last inequality we have used that on the set $\{t - \rho < \tau \leq t, \xi > \rho\}$ we have $X_t \in \partial C_{\ell(\varepsilon)}(x)$.

We use now (**??**). Remark that for $t \leq c\varepsilon T$, the first summand is bounded by a constant if $\varepsilon \leq 1$, which we of course assume. Let's consider the third summand. On $\{t < \tau\}$, the process is outside $C_{\ell(\varepsilon)}(x)$ up to time t. On the other hand $\|l_{T-t} - l_T\|_{\mathrm{TV}} \leq t/T \leq c\varepsilon$ for $t \leq c\varepsilon T + \rho$, T large enough (depending on ε) and therefore $l_{T-t} \in U_{c\varepsilon}(\mu_x)$ if $l_T \in U_\varepsilon(\mu_x)$. Therefore, on $\{l_T \in U_\varepsilon(\mu_x), t < \tau\}$ one has

$$|\langle l_t, l_{T-t}\rangle| \leq |\langle l_t, \mu_x\rangle| + c\varepsilon \leq c_1\varepsilon,$$

as $\mu_x(C_{\ell(\varepsilon)}(x)^c) \leq \varepsilon$. Implementing this, we get

$$E\left(\mathrm{e}^{\beta T\|l_T\|_2^2}; l_T \in U_\varepsilon(\mu_x), \xi > \rho, u < \tau\right) \tag{2.15}$$
$$\leq \frac{1}{\rho}\int_u^{c\varepsilon T+\rho} dt\, \mathrm{e}^{c\varepsilon\beta t} E\left(\mathrm{e}^{\beta\frac{(T-t)^2}{T}\|l_{t,T}\|_2^2}; l_{t,T} \in U_{c_1\varepsilon}(\mu_x), X_t \in \partial C_{\ell(\varepsilon)}(x)\right).$$

We next claim that for $y \in \partial C_{\ell(\varepsilon)}(x)$

$$E_x\left(\mathrm{e}^{\beta T\|l_T\|_2^2}\middle| \mathcal{F}_{t,T}\right) \geq \exp\left[c_2 t + \beta\frac{(T-t)^2}{T}\|l_{t,T}\|_2^2\right] \tag{2.16}$$

on $\{l_{t,T} \in U_{c_2\varepsilon}(\mu_x), X_t = y\}$, where $\mathcal{F}_{t,T}$ is the σ-field generated by X_s, $t \leq s \leq T$. Before proving this, we show that (**??**) and (**??**) imply the Proposition.

$$Z_{T,\beta} = E_x\left(E_x\left(\mathrm{e}^{\beta T\|l_T\|_2^2}\middle| \mathcal{F}_{t,T}\right)\right)$$
$$\geq \sum_{y\in\partial C_{\ell(\varepsilon)}(x)} E\left(E_x\left(\mathrm{e}^{\beta T\|l_T\|_2^2}\middle| \mathcal{F}_{t,T}\right); l_{t,T} \in U_{c_2\varepsilon}(\mu_x), X_t = y\right)$$
$$\geq \sum_{y\in\partial C_{\ell(\varepsilon)}(x)} E\left(\exp\left[c_2 t + \beta\frac{(T-t)^2}{T}\|l_{t,T}\|_2^2\right]; l_{t,T} \in U_{c_2\varepsilon}(\mu_x), X_t = y\right)$$
$$\geq \mathrm{e}^{c_3 t + c\varepsilon t} E\left(\exp\left[\beta\frac{(T-t)^2}{T}\|l_{t,T}\|_2^2\right]; l_{t,T} \in U_{c_2\varepsilon}(\mu_x), X_t \in \partial C_{\ell(\varepsilon)}(x)\right),$$

if ε is small enough (depending only on β). Therefore, by (**??**)

$$E\left(\mathrm{e}^{\beta T\|l_T\|_2^2}; l_T \in U_\varepsilon(\mu_x), \xi_{r_\varepsilon(x)} > \rho, u < \tau\right)$$
$$\leq \frac{Z_{T,\beta}}{\rho}\int_u^{c\varepsilon T+\rho} \mathrm{e}^{-c_3 t}\, dt \leq c_4\frac{Z_{T,\beta}}{\rho}\,\mathrm{e}^{-c_3 u}.$$

This proves the claim.

It remains to prove (**??**). On the prescribed event, the left hand side of (**??**) is

$$\geq \exp\left[\beta\frac{(T-t)^2}{T}\|l_{T-t}\|_2^2\right] E_x\left(\mathrm{e}^{2t\beta\langle l_t,\mu_x\rangle}\middle| X_t = y\right)\mathrm{e}^{-c\varepsilon t}.$$

We make a transformation of the path measure switching to the measure $P_x^{(\mu_x)}$ of a Markov process starting in x having $Q-$matrix $\left(\frac{1}{2}\sqrt{\mu_x(j)/\mu_x(j)}\right)_{i,j\in\mathbb{Z}^d}$. P_x is well know to be absolutely continuous on $(D_t, \mathcal{F}_t)$ with respect to $P_x^{(\mu_x)}$ with a density

$$\frac{dP_x}{dP_x^{(\mu_x)}}(\omega) = \sqrt{\frac{\mu_x(x)}{\mu_x(\omega_t)}}\exp\left[\int_0^t \frac{\frac{1}{2}\Delta\sqrt{\mu_x(\omega_s)}}{\sqrt{\mu_x(\omega_s)}}\,ds\right], \tag{2.17}$$

where Δ is the discrete Laplacian $\Delta f(x) = \sum_{y:|y-x|=1}(f(y)-f(x))$. We write now $\mu_o(x) = g^2(x)$. The Euler equation for g is

$$4\beta g(x)^3 + \Delta g(x) = \lambda g(x). \tag{2.18}$$

Multiplying with $g(x)$ and summing over x gives

$$\lambda = 4\beta\sum_x g\,(y)^4 - 2I(g^2) \geq 2b(\beta) > 0.$$

On the other hand, if we divide **??** by $g(x)$, we get

$$\int_0^t \frac{\frac{1}{2}\Delta\sqrt{\mu_o(\omega_s)}}{\sqrt{\mu_o(\omega_s)}}\,ds + 2\beta\,\langle l_t,\mu_o\rangle = \lambda.$$

The same is of course true if we replace μ_o by μ_x. Implementing this into (**??**), implies

$$E_x\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle}; X_t = y\right) = \mathrm{e}^{\lambda t/2}\sqrt{\frac{\mu_o(0)}{\mu_o(y-x)}}P_x^{(\mu_x)}(X_t = y) \geq \mathrm{e}^{\lambda t/3}.$$

The last inequality is just coming from the fact that under $P_x^{(\mu_x)}$, the process is ergodic with stationary distribution μ_x. As y is at distance $\ell(\varepsilon)$ from x, the estimates holds for t large enough, $t \geq t_o(\varepsilon)$. Therefore, we only have to choose $A(\varepsilon) \geq t_o(\varepsilon)$. This proves (**??**). ■

The Lemmas **??**, **??** and **??** immediately imply Proposition **??**, which together with Proposition **??** implies the Proposition **??**.

In the next section we will need a result which can be proved by a slight extension of the above:

Proposition 2.28

$$\lim_{m\to\infty} \limsup_{T\to\infty} \sup_{0\le t\le T} \hat{P}_T\left(|X_t|\ge m\right) = 0.$$

2.3.3 Proof of Theorem ??

For small enough $\varepsilon > 0$, the $U_e(\mu_x)$ are all disjoint. We also know from Proposition **??** and Proposition **??** that for any $\eta > 0$ there exist $S(\eta)$ and $m(\eta) > 0$ (not depending on ε !) such that for $T \ge T_o(\varepsilon,\eta)$ one has

$$\sup_{t\le T} \hat{P}_T\left(l_T \in \bigcup_{x:|x|\le S(\eta)} U_\varepsilon(\mu_x), |X_t|\le m(\eta)\right) \ge 1-\eta.$$

We want to prove

$$\lim_{T\to\infty} \hat{P}_T\left(l_T\in U_\varepsilon(\mu_x)\right) = \frac{\sqrt{\mu_o(x)}}{\sum_y \sqrt{\mu_o(y)}}, \tag{2.19}$$

for all small enough ε. We will fix some $t = t(\eta)$, which will *not* depend on ε (provided always that ε is small enough). Given Proposition **??**, this proves Theorem **??**, part a). Now, for x with $|x| \le S(\eta) \le m(\eta)$

$$\begin{aligned}\hat{P}_T\left(l_T\in U_\varepsilon(\mu_x)\right) &= \hat{P}_T\left(l_T\in U_\varepsilon(\mu_x), |X_t|\le m(\eta)\right) + O(\eta)\\ &= \sum_{y:|y|\le 2m} E\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle + \frac{(T-t)^2}{T}\|l_{T-t}\|_2^2}; l_T\in U_\varepsilon(\mu_x), X_t = y\right) + O(\eta)\\ &= \sum_{y:|y|\le 2m} E\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle}\,\middle|\, X_t = y\right)\\ &\times E_y\left(\mathrm{e}^{\frac{(T-t)^2}{T}\|l_{T-t}\|_2^2}; l_T\in U_\varepsilon(\mu_x)\right)\left(1+O(\mathrm{e}^{\varepsilon t})\right) + O(\eta).\end{aligned}$$

By the same discussion as we had in the last section, we have for some $t = t(\eta)$, (not depending on ε)

$$\left| E\left(\mathrm{e}^{2t\langle l_t,\mu_x\rangle}\,\middle|\, X_t = y\right) - \varphi(t)\sqrt{\mu_x(0)}\sqrt{\mu_x(y)}\right| \le \eta,$$

uniformly in $|x| \leq S(\eta)$, $|y| \leq 2\eta$. Remark that $\mu_x(y) = \mu_o(y - x)$ and that μ_o is symmetric. If we choose $\varepsilon \leq \varepsilon_o(\eta)$, we therefore get

$$\left| \hat{P}_T\left(l_T \in U_\varepsilon(\mu_x), |X_t| \leq m(\eta)\right) - \frac{\sqrt{\mu_o(x)}}{\sum_z \sqrt{\mu_o(z)}} \hat{P}_T\left(l_T \in U_\varepsilon(K_\beta)\right) \right| \leq c\eta,$$

and therefore, if then $T \geq T_o(\varepsilon, \eta)$, we have for $|x| \leq S(\eta)$:

$$\left| \hat{P}_T\left(l_T \in U_\varepsilon(\mu_x)\right) - \frac{\sqrt{\mu_o(x)}}{\sum_z \sqrt{\mu_o(z)}} \right| \leq c\eta.$$

This implies (**??**), and therefore part a) of the Theorem.

I will not give the details of part b), as it is by some straightforward modification and extension of the above argument. One has only to introduce another splitting at a time point $T - t$, to ,,separated" the endpoint from the main bulk of the empirical distribution.

Some last remark about what happens if the Condition **??** would fail (a case where I don't know if it occurs at all). In that case it would be difficult to establish a limiting result and one would have to go into finer asymptotics in large deviation in order to determine the relative weights on the different fibres. This has not been done for the present problem (see however [**?**] for the case of sums of i.i.d. random vectors). However, one can easily get some information: The proof of the tightness essentially applies with only small modifications, and one gets at least tightness for instance of the distribution of the endpoint (and the relative distribution inside each fibre of the K_β) without any further assumptions besides $b(\beta) > 0$. For the details, I refer to [**?**].

2.4 A droplet construction for the Wiener sausage

A problem which is closely related to the one in the previous section is connected with the classical large deviation result of Donsker and Varadhan for the volume of the Wiener sausage [**?**]. There is a corresponding result for random walks where the volume of the Wiener sausage is replaced by the number of points visited. I will give a sketch of the some of the problems and results in this section without going into much details.

I stick for the moment to the Wiener sausage: So let $\beta_t, t > 0$, be the standard Brownian motion on $\mathbb{R}^d$, starting in 0. The Wiener sausage is defined by

$$W_T^a = \bigcup_{s \le T} B_a(\beta_s),$$

where $a > 0$ and where $B_a(x)$ is the ball with radius a and center x. All results generalize also to the situation where $B_a(x)$ is replace by $x + C$ where C is an arbitrary set of positive capacity. The volume of the Wiener sausage is then just its Lebesgue measure

$$V_T^a = |W_T^a| \, .$$

The classical result of Donsker-Varadhan states

Theorem 2.29 *For any* $\beta > 0$

$$\lim_{T \to \infty} \frac{1}{T^{d/(d+2)}} \log E\left(\exp\left[-\beta V_T^a\right]\right) = \psi(\beta),$$

where

$$\psi(\beta) = (\omega_d \beta)^{2/(d+2)} \, \lambda_d^{d/(d+2)} \left(\frac{2}{d}\right)^{d/(d+2)} \frac{d+2}{2},$$

ω_d *being the volume of a ball of radius one and* λ_d *is the ground state eigenvalue of the* $\frac{1}{2}\Delta$ *in the ball with radius one with Dirichlet boundary conditions.*

Remark in particular that the limit does not depend on a. This will be different in the next section.

In order to understand the result and especially the at first sight somewhat strange power of T appearing in this LDP, one just has to look at a lower bound. One apparently very crude method is to confine the Brownian motion to stay inside a ball $B_{r_T}(0)$ whose radius r_T has to be determined. For such path the volume of the sausage certainly is not larger than the volume of $B_{r_T+a}(0)$ which is $\omega_d(r_T + a)^d$. On the other hand, it is well known that

$$P\left(\beta_s \in B_{r_T}, s \le T\right) \ge c \exp\left[-\lambda_d \frac{T}{{r_T}^2}\right].$$

Therefore, we get for any choice of r_T :

$$E\left(\exp\left[-\beta V_T^a\right]\right) \geq c\exp\left[-\beta\omega_d(r_T+a)^d - \lambda_d\frac{T}{{r_T}^2}\right].$$

Optimizing over r_T one finds that the optimal radius is $r_T \sim \left(\frac{2\lambda_d T}{d\beta\omega_d}\right)^{1/(d+2)} \stackrel{\text{def}}{=} \rho(\beta)T^{d/(d+2)}$, which gives the lower bound in Theorem **??**. The difficult part of the Theorem is of course to prove an upper bound. It might look somewhat surprising that the above crude argument for the lower bound gives the correct asymptotics, at least in leading order. In order to prove an upper bound one roughly would like to argue as follows:

$$\begin{aligned}E\left(\exp\left[-\beta V_T^a\right]\right) &= \sum_A P(V_T^a = A)\exp\left[-\beta|A|\right] \leq \sum_A P(V_T^a \subset A)\exp\left[-\beta\,|A|\right] \\ &\simeq \sum_A \exp\left[-\beta|A| - \lambda(A)T\right],\end{aligned}$$

where $\lambda(A)$ is the Dirichlet eigenvalue in A. Of course, the summation does not quite make sense, but it should naturally be understood to run over unions of blocks of side length εa, ε small. In the case of a random walk on $\mathbb{Z}^d$, the rôle of V_T^a is played by N_T, the number of points visited by the random walk, and there the above summation is completely sound. In both cases the problem is that the sum is over too many sets. The relevant A's are roughly of diameter $T^{1/(d+2)}$ where both $|A|$ and $\lambda(A)T$ are of order $T^{d/(d+2)}$. Therefore, there are $\exp\left[cT^{d/(d+2)}\right]$ A's which are of the relevant size, so it is clear that one needs some coarse-graining in order to reduce the combinatorial complexity of the summation. (It can however be remarked that the Donsker-Varadhan case is just the border line case where this problem becomes serious). It is natural that such a coarse-graining should be possible as the Brownian motion (or the random walk) is smearing out the empirical measure to some extent. This is also one of the basic ideas of the enlargement of obstacles technique by Sznitman (which works also in the quenched situation not discussed here). The easiest way to get the coarse-graining in the above situation is to use (a simple version of) Talagrand's concentration estimates. This works in situations which had not been treated by other techniques. I will give details of that in the next section. Anyway, if one is ready to believe that such a coarse-graining works, one gets

$$E\left(\exp\left[-\beta V_T^a\right]\right) \stackrel{\log}{\sim} \exp\left[-T^{d/(d+2)}\inf_A\left\{\beta|A| + \lambda(A)\right\}\right],$$

where $\lambda(A)$ is the Dirichlet ground state eigenvalue of $\Delta/2$ in A, and where $\overset{\log}{\sim}$ means that the quotient of the logarithms is going to 1. The variational problem above is a well known one in Mathematical Physics from the beginning of the century, which has been solved independently by Faber and Krahn, who proved that the unique minimizers are the balls. This is closely connected with the isoperimetric problem, and can be reduced to it.

A problem in the spirit of the last section is to determine exactly the behavior of the path measure

$$d\hat{P}_T = \frac{\exp\left[-\beta V_T^a\right] dP}{Z_T}$$

for large T. From the Faber-Krahn Theorem it is natural to expect that the paths under this measure are concentrated in balls of radius about $\rho(\beta)T^{d/(d+2)}$. In particular this should mean that the path stays confined within a region of this order.

The delicacy is however coming from the fact that one has to control events which cannot be controlled by leading order asymptotics in large deviations. To see this, consider the event that the Brownian path rushes off through a small tube (of radius 1, say) to a distance which is very large compared with $T^{d/(d+2)}$, to be specific, say to $\sqrt{T}$, and afterwards settles in the optimal ball. This eccentricity gives a contribution of order $\sqrt{T}$ to the volume of the sausage, which may look large, but which is negligible when compared with the volume of the optimal ball, which is of order $T^{1/(d+2)}$. The probability for rushing (in time $\sqrt{T}$, say) through this narrow tube is for the standard Brownian of order $\exp\left[-\sqrt{T}\right]$, which may look small, but which is very large compared with the probability that the path does what we expect of it, namely to stay within the optimal ball, which is $\exp\left[-cT^{d/(d+2)}\right]$. The path could of course do many other things besides just this "tube eccentricity", and at the outset, it is not clear if one really should expect this confinement to happen.

The problem had first been addressed independently for $d = 2$ in two papers, the first one by Sznitman [?] and then in [?] for the random walk case. (The first versions of the papers came out at exactly the same time.) The confinement has now been proved in a recent paper by Povel [?], which is based on the approach by Sznitman. The results for $d \geq 3$ are still not quite as precise as the one for $d = 2$.

Theorem 2.30 *There exists a function $\delta(T) \to 0$, as $T \to \infty$, such that*
a) for $d = 2$

$$\lim_{T\to\infty} \hat{P}_T \left(\exists x \in B_{\rho T^{1/(d+2)}}(0) : B_{\rho(1-\delta(T))T^{1/(d+2)}}(x) \subset V_T \subset B_{\rho(1+\delta(T))T^{1/(d+2)}}(x)\right) = 1$$

b) (Povel [?]) for $d \geq 3$:

$$\lim_{T\to\infty} \hat{P}_T \left(\exists x \in B_{\rho T^{1/(d+2)}}(0) : V_T \subset B_{\rho(1+\delta(T))T^{1/(d+2)}}(x)\right) = 1.$$

Sznitman's result contains also the limiting distribution of the centering of the optimal ball, which is not at 0, but which is distributed (after rescaling space with $\left(\rho T^{1/(d+2)}\right)^{-1}$) to the normalized ground state eigenfunction of the $\Delta/2$ inside the unit ball.

There is no doubt that the full result is true in all dimensions, and could probably been proved by some additional efforts. The information on $\delta(T)$ is still very modest. The only thing which is known is that one can take some decay of the form $T^{-\alpha}$, for *some* $\alpha > 0$. Bounds for α could be given, but they certainly are not optimal, and by present day's methods, it seems to be completely out of reach to get the precise behavior of the boundary, not even for $d = 2$. An interesting aspect however, is the proof of such a droplet construction in sup-distance in any dimensions.

It is fairly clair that a complete expansion of $E\left(\mathrm{e}^{-\beta V_T}\right)$ up to order $(1 + o(1))$ would be very helpful and desirable for the problem, but this seems to be completely out of reach, too. The methods in [?] do not apply, because V_T as a function of the empirical distribution has very bad continuity properties. The best results so far is the one obtained in [?] for the random walk. The result there is proved in all dimensions (for the random walk), provided the variational problem has a stability of the form of Theorem**??** below. The original version of the enlargement of obstacles technique gave only a slightly weaker result but the newer version [?] gives it, too, for the sausage. The statement is that there exist c_1, c_2, ε (depending on d and β) such that

$$\begin{aligned} \exp\left[-\psi(\beta)T^{d/d+2)} - c_1(\beta)T^{(d-1)/(d+2)}\right] &\leq E\left(\exp\left[-\beta N_T\right]\right) \qquad (2.20) \\ &\leq \exp\left[-\psi(\beta)T^{d//d+2)} + c_2(\beta)T^{(d-\varepsilon)/(d+2)}\right]. \end{aligned}$$

ε can be estimated but presently, there is no hope getting the correct ε. There is a non rigorous calculation in the physics literature [?], claiming that the correct correction is of the form of the lower bound:

$$E\left(\exp\left[-\beta N_T\right]\right) = \exp\left[-\psi(\beta)T^{d/d+2)} + c_1(\beta)T^{(d-1)/(d+2)} + o(T^{(d-1)/(d+2)})\right],$$

but this is based on some Gaussian Ansatz for the field of local times, and I do not know how reliable this prediction is. If correct, this would mean that the correction to the volume order large deviations is of surface order.

One crucial ingredient in all the proofs of results like Theorem **??** is an analytic stability property of the variational problem, which in our case can be reduced to a stability property in the isoperimetric problem. This property states that if there is a (nice) subset A in $\mathbb{R}^d$ which has as volume that of the ball of radius one, and a surface which is slightly larger, then there exists a ball with radius one which is close in some sense to A. There is a substantial difference between $d = 2$ and $d \geq 3$. For $d = 2$ such a statement can easily been proved in Hausdorff-distance (with the help of the Bonnesen inequality), but in higher dimension, this evidently cannot be true, as there are regions with thin spines having essentially no volume and surface. It is therefore clear that such a stability can only be true in some L_1-sense. When I wrote the first version of my paper, and when Sznitman wrote his, such a result had not been known, and later we had not been aware of it, but such a stability result has been proved by Hall [**?**].

Theorem 2.31 *Let ω_d be the volume of the ball with radius 1, and σ_d its surface. There exist $c(d)$, $\alpha(d) > 0$ such that for any Borel subset A of $\mathbb{R}^d$ with rectifiable boundary ∂A which satisfies $|A| = \omega_d$, there exists $x \in \mathbb{R}^d$ such that*

$$|A \,\Delta\, B_1(x)| \leq c(d)(|\partial A| - \sigma_d)^{\alpha(d)}.$$

This result leads to a corresponding stability result in L_1 for the variational problem appearing in the Donsker-Varadhan result

$$\psi(\beta) = \inf_{\|g\|_2=1} \{\frac{1}{2}\int |\bigtriangledown g|^2 + \beta \int 1\{g^2 > 0\}dx\}.$$

The solutions of this variational problem is the ground state eigenfunctions over the ball with optimal radius $\varrho(\beta)$, (i.e. just the usual Bessel function) and its translates. Let $\mathcal{F}$ be the set of squares of these optimal profiles. From Theorem **??** one can derive

Proposition 2.32 *There exists $\delta > 0$*

$$\inf\{\frac{1}{2}\int |\bigtriangledown g|^2 + \beta \int 1_{\{g^2>0\}}dx : \int g^2 dx = 1, \inf_{f\in\mathcal{F}} \|f^2 - g^2\| \geq a\} \geq \psi(\beta) + (\beta)a^\delta$$

for $T^{-\delta} \leq a$ (see Lemma 3.1 in [?]).

This rigidity property implies in any dimension a corresponding probabilistic property. Due to bad continuity properties of the Lebesgue measure of the support, this is not evident, but has been proved in [**?**] for the random walk case in any dimension. To state it, first a compactification is needed, which is just the usual torus compactification. Fix some multiple of the optimal radius $\varrho(\beta)$, $R = 10g(\beta)$, say. Then we perform the usual periodization on a torus of size $M_T = RT^{1/(2+d)}$, and then scaling down everything on a torus of finite size M_T. Therefore, the random walk $X_t, t \geq 0$, on the torus is replaced by

$$\eta_t = T^{-1/(d+2)} X_{tT^{2/(2+d)}},$$

living on $\mathbb{L}_T^{(R)} = T^{-1/(2+d)}\{1, \ldots, n_T\}^d$. Next we consider the "local times"

$$\ell_T(x) = \int_0^{T^{d/(2+d)}} 1_x(\eta_s) ds,$$

$x \in \mathbb{L}_T^{(R)}$. Remark that $\ell_T^{(R)}$ is normalized in the upper sense:

$$\int \ell_T^{(R)}(x) dx = 1,$$

if $\ell_T^{(R)}$ is extended to the continuous torus by putting it constant on the plaquette, of side length $T^{-1/(2+d)}$. It has been proved in [**?**] that Proposition in all dimensions implies

Proposition 2.33 *There exists $\delta > 0$ such that*

$$\lim_{T \to \infty} \hat{P}_{T,\beta}(\|\ell_T^{(R)} - \mathcal{F}\|_1 \geq T^{-\delta}) = 0.$$

There is of course no problem to define $\mathcal{F}$ on the torus, as the members of $\mathcal{F}$ (on $\mathbb{R}^d$) have as support the balls of radius $\varrho(\beta)$.

A consequence of this is that (on our torus) most of the mass of ℓ_T is concentrated inside a ball of the optimal radius. The crucial problem is however to boost that to a proof that there is *no* mass outside. This then also means that the torus is completely superfluous, and one gets the confinement statement we are interested in.

The method how to achieve this had been completely different in Sznitmann (and now in Povels) papers, and in [?]. Roughly speaking, in [?] it was done by "bare hand", whereas in [?] and [?] this came out much more elegantly, however using all the sophistication of the enlargement of obstacles technique, which I do not present here.

The basic idea in [?] was to use an iterative procedure, essentially to boost the δ in Proposition **??** This Proposition implies that the total time spent outside an optimal ball is bounded by some $T^\eta, \eta < 1$. Now pretend for a moment that this would be just one piece of an excursion from the ball. However, then by a discussion of this part separately we would now that instead of floating around, it would be better for this path to stay inside some ball of radius $T^{\eta/(1+d)}$ possibly with having "bad" excursions from this of time length, now of smaller order. But then to these, one can apply the arguing again, reducing these excursions again etc.

The trouble is of course that a priori one has no knowledge about how the excursion from the balls behave. There could be many pieces, starting and entering from remote points on the surface of the droplet, and doing all kind of pranks. The argument to get this under control was roughly as follows. One introduces a (finite) number of radii slightly lager than the optimal $\varrho(\beta) = r_0 < r_1 < r_2 < \dots r_m$. Actually the differences $r_i - r_{i-1}$ can be chosen to decay with T. Then one knows from Proposition that the total time spent outside r_1 (after the rescaling) is at most $T^\eta, \eta < 1$. This can then be boosted to prove that outside r_2 there is still less, namely $\leq T^{1/2}, \eta_2 < \eta_1$. The reasoning roughly is that one can separate what is outside r_1 from what is inside and argue as before. One can proceed in this way and prove that outside r_m, there is nothing left. The proof in [?] of this inductive cutting of the excursions was quite involving and depended on some previous knowledge which was very easy for $d = 2$, namely that the ball inside has no holes. This is still not proved (also not in Povels paper) for $d \geq 3$, but might be not so difficult. (For $d = 2$ it is a very easy consequence of Proposition **??**.)

2.5 Moderate deviations for the Wiener sausage and the critically shrinking Wiener sausage

2.5.1 Introduction and heuristics

In the classical Donsker-Varadhan result for the Wiener sausage discussed in the previous section, the main contribution to $E\left(\exp\left[-\beta V_T\right]\right)$ was coming from paths which stay inside a ball of radius $r_T = \rho(\beta)T^{1/(d+2)}$. The "strategy" the path has to follow is somewhat trivial: The ball is filled just completely. Even if this is not fully proved in all dimensions, the fact that the trivial lower bound is correct in first order tells us that this is at least up to leading order the correct picture.

Consider a much easier problem namely a Brownian motion, which is conditioned to stay inside a ball of radius T^γ. What is the effect on V_T of that? It is well known that under Wiener measure, V_T is typically of order $\kappa_a T$, for $d \geq 3$, where κ_a is the Newtonian capacity of a ball with radius a (seen [**?**], [**?**], there is a logarithmic correction for $d = 2$). If the Brownian is confined in this ball, the volume can be at most of order $T^{d\gamma}$. Therefore, this confinement has trivially a substantial effect on the volume when $0 < \gamma < 1/d$, and it is not difficult to prove that a sausage of the Brownian which is confined to stay inside such a small ball is filling it completely, except near the boundary. Therefore, the volume is (up to smaller order corrections) just the volume of the ball. Let's look now at the opposite situation where $\gamma > 1/d$. In that case, of course, the volume of the ball is much larger than the expectation of the sausage it has when not confined, although for $d \geq 3$, and $\gamma < 1/2$, confining the Brownian to stay inside $B_{T^\gamma}(0)$ is still a large deviation. It is however not difficult to see that the confinement in this case has no effect in leading order on the volume of the sausage. It is therefore clear that the critical confinement radius which should lead to a sizeable effect on the sausage is that of order $T^{1/d}$. It is also not difficult to prove that for a confinement inside a ball of radius $T^{1/d}$ $(d \geq 3)$, the expectation of the sausage is of order T but smaller than $\kappa_a T$.

Let's now go back to the (much more difficult problem) to discus $E\left(\exp\left[-\beta T^{-\alpha} V_T\right]\right)$. If we proceed with the lower bound explained in the last section, the optimal radius to choose will be the one where $T^{-\alpha} r_T^d$ is of order T/r_T^2, i.e. where r_T is of order $T^{(1+\alpha)/(2+d)}$. Now, this radius becomes of order $\geq T^{1/d}$ when $\alpha \geq 2/d$. It can therefore be expected that for $\alpha < 2/d$, the Donsker-Varadhan picture stays correct, and

$$\lim_{T\to\infty} T^{-d(1+\alpha)/(d+2)} \log E \exp\left[-\beta T^{-\alpha} V_T\right] = \psi(\beta).$$

This is indeed true, and has been proved independently in two paper ([?], [?]). The original Donsker-Varadhan approach however does not immediately extend to this situation and has to be refined. For $\alpha > 2/d$ this "Donsker-Varadhan ball picture" breaks down, which should be quite natural given the above discussion. In fact in leading order, the lower bound coming from the Jensen inequality is better than the one coming from the "ball strategy" and turns out to be sharp at least in leading order:

$$E \exp\left[-\beta T^{-\alpha} V_T\right] \overset{\log}{\sim} \exp\left[-\beta T^{-\alpha} E(V_T)\right] \sim \exp\left[-\beta \kappa_a T^{1-\alpha}\right].$$

The fact that Jensen is sharp can only mean that it is not "worth" for the Brownian to make any efforts and therefore that the corresponding path measure

$$d\hat{P}_T = \exp\left[-\beta T^{-\alpha} V_T\right] dP_T / Z_T$$

should just be close to ordinary Brownian. This has not been proved, and in view of the discussion given in section **??**, one would probably have to prove first that

$$E \exp\left[-\beta T^{-\alpha} V_T\right] \leq C \exp\left[-\beta \kappa_a T^{1-\alpha}\right],$$

which might be easy, but has not been done. Anyway, the most interesting case then is certainly $\alpha = 2/d$, where we now have two lower bound, one coming from Jensen, and the other one from the ball confinement strategy. On the background of the fact that a Brownian which is confined to stay in the ball of optimal radius does not fill the ball completely, one would certainly not expect lower bound to be sharp (in leading order). Somewhat surprisingly, it turns out that the Jensen inequality is sharp for small β, but not for large, where something more interesting is happening, and where the ball strategy is not the proper thing.

It turns out that we better don't start with discussing $E \exp\left[-\beta T^{-2/d} V_T\right]$, but rather with a problem which looks equivalent, namely with the probability that V_T is small in a range which would correspond to this critical case. It is natural to expect that the discussion of $E \exp\left[-\beta T^{-2/d} V_T\right]$ is tied to the question of discussing $P(V_T \leq bT)$, where $b < \kappa_a$. In fact, we can evaluate $E\exp\left[-\beta T^{-2/d} V_T\right]$ in leading order from the evaluation of $P(V_T \leq bT)$, but not vice versa.

Theorem 2.34 *Assume $d \geq 3$. Then for $b \in (0, \kappa_a)$*

$$\lim_{T\to\infty} T^{-(d-2)/d} \log P\left(V_T \leq bT\right) = -I(b)$$

where

$$I(b) = \inf\left\{\frac{1}{2}\left\|\nabla g\right\|_2^2 : g \in H_1(\mathbb{R}^d), \|g\|_2^2 = 1, \int\left(1 - \mathrm{e}^{-\kappa_a g(x)^2}\right) dx \leq b\right\}. \tag{2.21}$$

Remark 2.35 *a) There is also a version for $d = 2$. In that case, $E(V_T) \sim \kappa T / \log T$, where κ is the logarithmic capacity. The Theorem has then to be modified accordingly, i.e. one discusses $P\left(V_T \leq bT / \log T\right)$.*

b) It is easy to evaluate $E\exp\left[-\beta T^{-2/d} V_T\right]$ using Theorem **??***:*

$$\lim_{T\to\infty} T^{-(d-2)/d} \log E\exp\left[-\beta T^{-2/d} V_T\right] = -J(\beta), \tag{2.22}$$

where J is the Legendre transform of I :

$$J(\beta) = \inf\left\{b\beta - I(b) : b \in (0, \kappa_a]\right\},$$

but not the other way: I is not the Legendre transform of J. This is simply coming from the fact that I is not convex (whereas J is). This will become apparent below. It will also turn out that for small β, the infimum is attained at $b = \kappa_a$, so that for small β one has $J(\beta) = \kappa_a \beta$ ($I(\kappa_a)$ is of course 0), i.e. the Jensen inequality is sharp in leading order.

c) Presently, we are not able to discuss the path measures, for instance discuss the limiting behavior of the distribution of the end point β_T under $P(\cdot | V_T \leq bT)$ or under $d\hat{P}_T = \exp\left[-\beta T^{-2/d} V_T\right] dP / Z$. From the discussion in the last section it should be clear that the measures are living on scale $T^{1/d}$, i.e. one would expect that $T^{-1/d}\beta_T$ has under these measures a nontrivial limiting distribution. For $\hat{P}_T$ however, there should be a "collapse transition" from small to large β. For $d \leq 4$, we would expect that the distribution of

β_T under $P(\cdot|V_T \leq bT)$ is on scale $T^{1/d}$, but this is "almost surely" not the case for $d \geq 5$ if b is close to κ_a, for reasons which become apparent later on. Nothing of this is proved, and it may be quite difficult.

d) The result can easily been extended to more general "sausages" where the ball with radius r is replaced by an arbitrary compact set C with positive capacity, i.e. where $W_T = \bigcup_{s \leq T}(\beta_s + C)$. Remark also that the rate function I does depend (via the capacity) on this compact set.

I first will give an intuitive explanation why the above large deviation principle should hold and why the variational problem looks as being the right one. Afterwards, I will present in subsection **??** the main analytical properties of the variational problem which are quite interesting and partially surprising. I will not give detailed proofs, but some explanations which hopefully will convince everybody, that the results have to be true. It is clear that the properties of the variational problem should be reflected also in properties of the path measure, but as remarked above, we don't know how to do this. Especially, the somewhat strange behavior of the variational problem for $d \geq 5$ must be reflected in a somewhat strange behavior of the path measure. I will then give a fairly detailed proof of the interesting part of the Theorem **??**, namely the upper bound.

The proof could also be used to derive the original Donsker-Varadhan result, also in its refinements given in [**?**], and [**?**]. I believe that it is technically the best (and easiest and most transparent) approach to these Wiener sausage problems, or equivalently to annealed random trap problems. It probably would also have the best chance to give the corrections in the asymptotics in the original Wiener sausage problem of Donsker and Varadhan, but to be able to tackle that would (most probably) require to have precise information about the analytical stability in the Hall result (Theorem **??**).

I start with giving a heuristic derivation why the rate function should have the above form.

¿From the discussion previous to the statement of the Theorem, it should be apparent that the main contribution to the event $\{V_T \leq bT\}$ is coming from paths which are staying at distance of order $T^{1/d}$ from the origin. Furthermore, it should also be clear, that we no longer can expect that the „strategy" of the Brownian is as simple as to just fill a certain region completely, essentially without leaving holes. In contrast, we expect that there remains some porosity, and we have to control the degree of this porosity.

This porosity is however felt only on a very microscopic scale: It turns out that the holes which are of relevance and are responsible for the porosity are of order one. What we prove is essentially that the degree of the porosity is tied to empirical distribution at a macroscopic scale (i.e. $T^{1/d}$) deterministically, up to a superexponential estimate. In this respect, the situation resembles somehow the one encountered in hydrodynamic models.

We first rescale the Brownian motion accordingly, by introducing $\widetilde{\beta}_t = T^{-1/d}\beta_{tT^{2/d}}$, $t \leq \tau \stackrel{\text{def}}{=} T^{(d-2)/d}$. As τ is the "correct time scale", we keep this notation reserved in this way. Consider the empirical process

$$L_\tau \stackrel{\text{def}}{=} \frac{1}{\tau}\int_0^\tau \delta_{\widetilde{\beta}_s} ds.$$

By a (weak) LDP, we know that roughly speaking

$$P\left(L_\tau \sim f^2\right) \sim \exp\left[-\tau\frac{1}{2}\left\|\nabla f\right\|_2^2\right].$$

It is however not quite clear what L_τ really has to do with the volume of the Wiener sausage. Remember that $\widetilde{\beta}_t$ is scaled down by a factor $T^{-1/d} = \tau^{-1/(d-2)}$ in space, and therefore

$$V_T = T\left|\operatorname{supp}\left(\chi_{B_{a\tau^{-1/(d-2)}}} * L_\tau\right)\right|,$$

where $B_r(x)$ is the ball with radius r and center x, $B_r = B_r(0)$, χ_A is the indicator function of the set A, and $(f * \mu)(x) = \int f(x-y)\mu(dy)$. There is evidently some trouble as $\mathcal{M}_1^+(\mathbb{R}^d) \ni \mu \to \left|\operatorname{supp}\left(\chi_{B_{a\tau^{-1/(d-2)}}} * \mu\right)\right|$ is certainly (in an essential way) not depending continuously on μ, and depends on τ.

We call $\tau^{-1/(d-2)}$ the microscopic scale. Let's look at a small but macroscopic box, i.e. we consider a hypercube Q of side-length ε and center $x \in \mathbb{R}^d$: $Q = \prod_{i=1}^d [x_i - \varepsilon/2, x_i + \varepsilon/2)$. $L_\tau(Q)$ measures the relative amount of time, the rescaled Brownian $\widetilde{\beta}_t, t \leq \tau$, spends inside Q. Evidently, this total amount will usually be cut into many time pieces, the Brownian exiting and reentering the cube. We make a number of *very* simplifying (false) assumptions: First, we pretend that Q is not a cube, but a torus of the same size with periodic boundary conditions. Next we assume that these many pieces of the Brownian inside Q is just one piece of a Brownian on this torus running up to time $L_\tau(Q)\tau$. We will then make this assumption for a collection of Q's which cover the space and patch things together, but let's first discuss the problem how much of our Q, which is now a torus, is covered by the (shrinking) sausage. We might hope that the calculation of the expectation is sufficient, and this in fact will turn out to be correct. This may be somewhat surprising as, after all, we are after a large deviation phenomenon, and so we may expect that deviations from expectations will play a role. However, we will prove that the deviations of the volume of the *microscopic* sausage on small *macroscopic* boxes from its expectation can be estimated on a superexponential scale in τ if the boxes are small ("mesoscopic"). Therefore, our task is now to calculate now the volume of a small torus of side length ε which is covered by a periodic Brownian β_s of time length $\lambda\varepsilon^d\tau$ by a sausage of radius $a\tau^{-1/(d-2)}$. Lets denote this volume by X.

$$\begin{aligned} EX &= \int_Q dx\, P\left(\exists s \le \lambda\varepsilon^d\tau : \beta_s \in B_{a\tau^{-1/(d-2)}}(x)\right) \\ &= |Q|\left(1 - P\left(\beta_s \notin B_{a\tau^{-1/(d-2)}}(x), \forall s \le \lambda\varepsilon^d\tau\right)\right). \end{aligned}$$

We now chop the time interval $[0, \lambda\varepsilon^d\tau)$ into many pieces of large length K, which we assume not to grow with τ. The probability that the Brownian (with uniform starting distribution) hits $B_{a\tau^{-1/(d-2)}}(x)$ in the time slot $[0, K)$ is $\frac{K\kappa_a}{\varepsilon^d\tau} + o(\tau^{-1})$. If the Brownian does not have a hit in the first interval, it gets a next change in the second. The conditioning on non-hitting in the first, does not much influence the distribution, as the ball which has to be hit is small anyway. Therefore, we get the same chance for the second slot which is essentially independent of the first one. We therefore have

$$P\left(\beta_s \notin B_{a\tau^{-1/(d-2)}}(x), \forall s \le \lambda\varepsilon^d\tau\right) \simeq \left(1 - \frac{K\kappa_a}{\varepsilon^d\tau}\right)^{\lambda\varepsilon^d\tau/K} \simeq \exp\left[-\kappa_a\lambda\right]$$

and therefore

$$EX \simeq \varepsilon^d(1 - \exp\left[-\lambda\kappa_a\right]).$$

We now chop $\mathbb{R}^d$ into cubes Q_i of the above size, and *assume* for the moment that $L_\tau(Q_i) \simeq \lambda_i\varepsilon^d\tau$. Then the sausage fills up to superexponential estimates (if the reader believes in what was said above) the Q_i with a proportion $1 - \exp\left[-\lambda_i\kappa_a\right]$. Therefore, the total volume covered is $\sum_i \varepsilon^d\left(1 - \exp\left[-\lambda_i\kappa_a\right]\right)$.

This does all the job on the microscopic scale, and the large deviation we are after is now only a large deviation on the macroscopic scale, i.e. a standard large deviation for L_τ which is governed by the classical Donsker-Varadhan LDP. We have to sum over all possibilities for choosing the λ_i but according to standard wisdom in large deviations, only the maximum is counting, and we get

$$P(V_T \le bT) \simeq \max \left\{ P(L_\tau \sim f) : \int (1 - \exp[-\kappa_a f(x)])dx \le b \right\}$$

which according to the Donsker-Varadhan LDP for Brownian motion is

$$\simeq \sup \left\{ \frac{1}{2} \|\nabla g\|_2^2 : \int \left(1 - \mathrm{e}^{-\kappa_a g^2(x)}\right) dx \le b \right\}.$$

That's it, and there remains only to prove it.

I present the real core of the argument in details in subsection , taking however some (very plausible and not too difficult) technical Lemmas for granted. Before starting with it, I want to give some information about the variational problem which appears, which had been quite surprising (at least for us).

2.5.2 Analytical properties of the variational problem

There is of course no chance to solve the variational problem explicitly. It is not too difficult to prove (using standard techniques) that all maximizers of the variational problem are radially symmetric. In principle, one can then discuss the one-dimensional Euler equation, which is just a nonlinear equation, but this seems not be of much help. For instance, we have been unable to prove that there is a unique maximizer, and the problem does not appear to belong to a class which has been treated in the literature.

The behavior of $I(b)$ for $b \sim 0$ is easy and offers no surprise: The variational problem goes over (after a rescaling) into the variational problem for the classical Donsker-Varadhan situation. It is fairly evident what the best way is in which a normed L_2-function can achieve a small value of $\int \left(1 - \mathrm{e}^{-\kappa_a g(x)^2}\right) dx$, best in the sense of having small value of $\|\nabla g\|_2^2 : g$ just has to be essentially 0 outside some small ball, but inside very large.

Inside the ball, g then is quite large, because of the restriction $\int g^2(x)dx = 1$. Therefore $1 - e^{\kappa_a g^2(x)}$ is there essentially 1. This means that for small b we have

$$I(b) \sim \inf\{\frac{1}{2}\| \bigtriangledown g\|_2^2 : \|g\|_2 = 1, \ |\text{ supp } (g)| \le b\}.$$

After rescaling, this leads to

Proposition 2.36 *For $b \to 0$*

$$I(b) \sim \frac{1}{2}\lambda_d(\omega_d b)^{-2/d}.$$

More interesting is the behavior for $b \sim \kappa_a$. We naturally expect that the relevant functions for the variational problem are those which become flat as $b \uparrow \kappa$. Following this idea, one expects that we just may expand the exponential:

$$1 - \exp(-\kappa_a g^2) \simeq \kappa_a g^2 - \frac{1}{2}\kappa_a^2 g^4,$$

and therefore

$$\int (1 - e^{-\kappa_a g^2})dx \simeq \kappa_a - \frac{1}{2}\kappa_a^2 \int g^4(x)dx.$$

This means that for $b < \kappa_a, b \sim \kappa_a$, we should have

$$I(b) \approx \inf\{\frac{1}{2}\| \bigtriangledown g\|_2^2 : \int g^2(x)dx = 1, \frac{1}{2}\kappa_a^2 \int g^4(x)dx = \kappa_a - b\}. \qquad (2.23)$$

The trouble is however that the r.h.s. is not always different from 0, in fact it is 0 for $d \geq 5$. This is well known. Here is the sequence, which does the job: We choose the ball with radius $1/n$, and over this ball a cone of height a_n. This is g_n inside the ball. We choose a_n such that $\int\limits_{B_{1/n}} g_n^4(x)dx \approx 1$, i.e. $a_n \approx n^{d/4}$. The contribution to the L_2-norm inside the ball is then negligible, but choosing g_n very flat. It is clear that we can do that in such a way that this contributes nothing to the L_4-norm, and nothing to $\| \bigtriangledown g\|_2^2$ (asymptotically). In this way we take care of $\|g_n\|_2^2$, and $\| \bigtriangledown g_n\|_2^2$ is now determined by

$$\int\limits_{B_{1/n}} | \bigtriangledown g_n(x)|^2 dx \approx n^{-d}(na_n)^2 \approx n^{-d+2}n^{d/2}.$$

which goes to 0 for $d \geq 5$, proving that the r.h.s. of (**??**) is 0. $d \leq 4$, the r.h.s. gives however the correct behavior of $I(b)$ for $b \sim \kappa_a$, which after some rescaling leads to

Proposition 2.37 *Assume $d \leq 4$. Then as $b \uparrow \kappa_a$*

$$I(b) \sim 2^{-\frac{d-2}{2}} \kappa_a^{-4/d} (\kappa_a - b)^{2/d} \mu d,$$

where
a) for $d \leq 3$ $\quad \mu_d = \inf\{\| \bigtriangledown g\|_2^2 : g \in H^1(\mathbb{R}^d), \|g\|_2 = 1, \|g\|_4 = 1\} > 0$
b) for $d = 4$ $\quad \mu_d = \inf\{\| \bigtriangledown g\|_2^2 : g \in W^1(\mathbb{R}^4), \|g\|_4 = 1\}$.

A consequence of this Proposition is that for $d = 3, 4,$ I is concave close to κ_a (of course, the above result does not quite prove this), and has infinite tangent at κ_a.

We come now to the case $d \geq 5$. The argument above leading to the conclusion that the r.h.s. of (**??**) is zero does of course not tell that $I(b) = 0$, simply because the sequence we have chosen crucially depended on the large g-values inside a small ball, and for large g-values, the expansion we performed is certainly not appropriate. In fact $I(b) > 0$ for all $b \in (0, \kappa_a$. There is however one feature of the above considerations which are important for the behavior of $I(b), b \sim \kappa_a$, namely the possibility that L_2 is leaking to infinity (which happens for the sequence g_n). To catch this, we apply a trick. For $\int g^2 dx = 1$, we of course have $\int (1 - e^{-\kappa_a g^2}) dx = u$ if and only if

$$\int (\kappa_a g^2 - 1 + e^{-\kappa_a g^2}) dx = \kappa_a - u.$$

The integrand has the advantage that it decays with g^4 if g is small. If therefore L_2-mass of g is wandering to infinity, this is not visible in the integrand, meaning that the integrand would behave continuously, although the L_2-norm would jump. We can therefore try to look at the variational problem forgetting for the moment the $\|g\|_2 = 1$ condition, i.e. look at

$$\varrho(a) = \inf\{\frac{1}{2}\| \bigtriangledown g\|_2^3 : \int (\kappa_a g^2 - 1 + e^{-\kappa_a g^2}) dx = \varepsilon\}.$$

This problem is "well posed", one can prove that minimizers exist, and the infimum is > 0. In fact, the ε dependence is trivial, and can be obtained by a rescaling

$$\varrho(\varepsilon) = \varepsilon^{(d-2)/d} \varrho(1).$$

This follows from the following observation: If g satisfies $\int(\kappa_a g^2 - 1 + e^{-\kappa_a g^2})dx = 1$ then $g_\varepsilon(x) = g(\varepsilon^{-1/d}x)$ satisfies

$$\int (\kappa_a g_\varepsilon(x)^2 - 1 + \int e^{-\kappa_a g_\varepsilon(x)^2})dx = \varepsilon,$$

and

$$\| \bigtriangledown g_\varepsilon\|_2^2 = \varepsilon \frac{d-2}{d} \| \bigtriangledown g\|_2^2.$$

Unfortunately, we are not able to prove that the variational problem for $\varrho(1)$ has a unique minimizer, and we cannot exclude that there are several minimizers with different L_2-norm, although this does not look very plausible. One can however prove that there are minimizers, which are postive everywhere, and any minimizer has to be rotationally symmetric. Let us *pretend* that there is (modulo shifts) just one or at least that all have the same L_2-norm. If this is not the case, the statement needs some messy but not very important modifications. Let therefore ψ_1 be the minimizer for $\varrho(1)$. If we scale ψ_1 to serve for $\varrho(\varepsilon)$, i.e. take $\psi_\varepsilon(x) = \psi_1(\varepsilon^{-1/d}x)$, then

$$\|\psi_\varepsilon\|_2^2 = \varepsilon \, \|\psi_1\|_2^2 .$$

Now, our real problem is to determine

$$I(b) = \inf\{\frac{1}{2}\| \bigtriangledown g\|^2 : \|g\|_2 = 1, \int (\kappa_a g^2 - 1 + e^{-\kappa_a g^2})dx = \kappa_a - b\},$$

and it looks like that this has something to do with $\varrho(\kappa_a - b)$ only if $(\kappa_a - b)\|\psi_1\|_2^2 = 1$. However, it is easy to see that if $(\kappa_a - b)\|\psi_1\|_2^2 < 1$, on has

$$I(b) = \varrho(\kappa_a - b) = (\kappa_a - b)^{(d-2)/d}\varrho(1).$$

In fact, in this case the deficit of the L_2-mass is just leaking to infinity. This also means, that the variational problem for $I(b)$ has in this case no solution with $\|g\|_2 = 1$. This leads to the following conclusion (which is correct regardless of the uniqueness question).

Proposition 2.38 *Assume $d \geq 5$. Then there exists $b_0(d) \in (0, \kappa_a)$ such that for $b \in [b_0(d), \kappa_a]$ one has*

$$I(b) = (\kappa_a - b)^{(d-2)/d}\varrho(1).$$

In the case where $(\kappa_a - b)\|\psi_1\|_2^2 > 1$, which is true for b small, $I(b)$ has nothing to do with ϱ. The L_2-restriction then "deforms" ψ in an essential way. We also know that in this case the variational problem for $I(b)$ has solutions which have L_2-norm 1.

It might be interesting to speculate what the behavior of the variational implies for the path measure. It should be evident that for $d = 3, 4$ and for $d \geq 5$ and b small, the paths under $P(\cdot|V_T \leq bT)$ are living on scale $T^{1/d}$, meaning for instance that

$$\sup_T E(T^{-1/d}\beta_T|V_T \leq bT) < \infty.$$

On the other hand, when $d \geq 5$ and b is close to κ_a, probably the behavior is different. The fact that the variational problem looses mass to infinity can only mean that the path stays "confined on scale $T^{1/d}$" only on part of its life time. For instance, one can imagine that the path first feels the confinement on a fixed proportion of T, and afterwards floats diffusively, but one could also imagine that a mere complicated behavior emerges. All this would probably be extremely difficult to prove.

2.5.3 Proof of the upper bound in Theorem ??

I prove here the upper bound in details, except that I leave however some (fairly evident) technical Lemmas unproved, but will give some explanations for them.

It is convenient to use the usual trivial compactification procedure winding the Brownian motion on a torus. This we do however after having done the rescaling leading to $\widetilde{\beta}_s = T^{-1/d}\beta_{sT^{2/d}}$, $s \leq \tau = T^{(d-2)/d}$. We get $V_T^a = TV_\tau^{a\tau^{-1/(d-2)}}$. We wind the Brownian motion $\left\{\widetilde{\beta}_s\right\}_{s\leq\tau}$ on a torus Λ_N of fixed size side length N . By an abuse of notation, we set $V_\tau^N = V_\tau^{a\tau^{-1/(d-2)}}$, but we also often drop the index N. Evidently, we have

$$P\left(V_T^a \leq bT\right) \leq P(V_\tau^N \leq b).$$

To get an upper bound of the left hand side, we therefore have to bound the right hand side. The main result to get that is:

Proposition 2.39 *V_τ^N satisfies a τ-large deviation principle with rate function*

$$I_N(a) = \inf\left\{\frac{1}{2}\int_{\Lambda_N} |\nabla g(x)|^2\, dx : g \in H_1(\Lambda_N),\ \int_{\Lambda_N} \left[1 - \exp\left(-\kappa_a g^2(x)\right)\right] dx = a\right\}$$

where $H_1(\Lambda_N)$ is the usual Sobolev space of once weakly differentiable functions with derivative in $L_2(\Lambda_N)$.

In order to prove then the upper bound in our main Theorem, the only thing which remains (and which is easy) is

Lemma 2.40 $\lim_{N\to\infty} I_N(a) = I(a)$ *for all a.*

We will not prove this Lemma, which is not difficult.

The above Proposition also leads easily to a lower bound. On the torus, we have a full LDP, not just an upper bound. We then get a lower bound by proving first that we don't loose much by confining the original Brownian to a box (after the rescaling) of fixed but large size. For such a Brownian inside a box, there is no difference between it and a Brownian similarly restricted but living on a torus. As we have a full LDP on the torus, we therefore get the lower bound. I don't spell out the details which are not very interesting.

¿From the form of the Proposition, it is clear that we get the LDP by a kind of contraction principle. It seems however impossible to get that directly, and we use an approximation procedure. For the rest of this chapter, the torus Λ_N is fixed. We usually drop N in the notation. We also drop the tilde in $\widetilde{\beta}_s$, and just write β_s for this rescaled Brownian motion. Time is always running up to τ.

Here is an outline of the procedure:

(A) We first approximate $V_\tau (= V_\tau^N)$ by its conditional expectation $E_\varepsilon(V_\tau) = E(V_\tau | \{\beta_{i\varepsilon}\}_{0\le i\le \tau/\varepsilon})$, where ε is a parameter > 0. We prove that the difference between V_τ and $E_\varepsilon(V_\tau)$ is negligible in the $\varepsilon \to 0$ limit. This is done by an application of a concentration inequality of Talagrand.

(B) We represent $E_\varepsilon(V_\tau)$ as a functional of the empirical distribution

$$L_{\varepsilon,\tau} = \frac{\varepsilon}{\tau}\sum_{i=1}^{\tau/\varepsilon} \delta_{(\beta_{\varepsilon(i-1)},\beta_{\varepsilon i})}.$$

According to one of the very basic large deviation results of Donsker and Varadhan, $L_{\varepsilon,\tau}$ satisfies a strong LDP (on the torus). We still will have need of some further approximations to get the dependence of $E_\varepsilon(V_\tau)$ on $L_{\varepsilon,\tau}$ in a suitable form, but essentially based just on this basic LDP for $L_{\varepsilon,\tau}$, we get via a contraction principle a LDP for $E_\varepsilon(V_\tau)$.

(C) We finally have to perform the $\varepsilon \to 0$ limit. We now already know that V_τ is approximated by $E_\varepsilon(V_\tau)$. It therefore will suffice to have an appropriate transition for the variational formula.

We write $\mathbb{X}_{\tau,\varepsilon} = \{\beta_{i\varepsilon}\}_{1\le i\le \tau/\varepsilon}$. (For notational convenience, we always assume that τ/ε is an integer). We denote by P_ε and E_ε the conditional probability and expectation with respect to $\mathbb{X}_{\tau,\varepsilon}$. The first main step (A) is to prove that V_τ is well approximated by $E_\varepsilon(V_\tau)$ in the following sense:

Proposition 2.41 *For all $\delta > 0$ we have*

$$\lim_{\varepsilon\to 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\log P(|V_\tau - E_\varepsilon(V_\tau)| \ge \delta) = -\infty.$$

Proof The proof is based on Talagrand's concentration inequalities. We denote by $m_{\tau,\varepsilon}$ the median of the distribution of V_t under the conditional law P_ε . Furthermore, let W_i, $1 \le i \le \tau/\varepsilon$, be defined by

$$W_i = \cup_{s\in[(i-1)\varepsilon,i\varepsilon]} B_{a\tau^{-1/(d-2)}}(\beta_s). \tag{2.24}$$

Evidently, the W_i are, conditionally on $\mathbb{X}_{\tau,\varepsilon}$, independent random closed subsets of Λ_N, and we have

$$V_\tau = \left|\cup_{i=1}^{\tau/\varepsilon} W_i\right|.$$

Let S be the set of closed subsets of Λ_N. The mapping $d : S \times S \to [0,\infty)$, $d(A,B) = |A\Delta B|$, defines a pseudometric on S. We equip S with the Borel field $\mathfrak{S}$ generated by this pseudometric. P_ε then defines a product measure on $(S,\mathfrak{S})^{\tau/\varepsilon}$, which, by an abuse of notation, we denote by P_ε, too. We apply one of Talagrand's concentration inequality to the function $V : S^{\tau/\varepsilon} \to [0,\infty)$, defined by

$$V(C) = \left|\cup_{i=1}^{\tau/\varepsilon} C_i\right|,\ C = \{C_i\}.$$

Evidently, V is Lipshitz in the sense that

$$|V(C) - V(C')| \leq \sum_{i=1}^{\tau/\varepsilon} |C_i \Delta C'_i| .$$

Let

$$A = \left\{ C \in S^{\tau/\varepsilon} : V(C) \leq m_{\tau/\varepsilon} \right\} .$$

The distribution of V under P_ε has no atoms. Therefore, we have $P_\varepsilon(A) = \frac{1}{2}$. From Theorem 2.4.1 of [**?**], we have $E_\varepsilon(\exp[\lambda f(A, \{W_i\})] \leq 2 \prod_{i=1}^{\tau/\varepsilon} E_\varepsilon(\cosh(\lambda |W_i \Delta W'_i|))$, where $f(A, \{C_i\}) = \inf_{(D_i) \in A} \sum_i d(C_i, D_i)$ and $\{W'_i\}$ is an independent copy of $\{W_i\}$. From the Markov inequality, we therefore get

$$P_\varepsilon(f(A, \{W_i\}) \geq \delta) \leq 2 \inf_{\lambda > 0} e^{-\lambda\delta} \prod_{i=1}^{\tau/\varepsilon} E_\varepsilon(\cosh(\lambda |W_i \Delta W'_i|)) = \Phi_{\tau,\varepsilon}(\delta), \text{say.} \tag{2.25}$$

Arguing similarly with $A' = \left\{ C \in S^{\tau/\varepsilon} : V(C) \geq m_{\tau/\varepsilon} \right\}$, we get

$$P_\varepsilon(|V_\tau - m_{\tau,\varepsilon}| \geq \delta) \leq 2\Phi_{\tau,\varepsilon}(\delta).$$

Remark now that $|V_\tau|$ is bounded by $|\Lambda_N|$. Therefore

$$|E_\varepsilon(V_\tau) - m_{\tau,\varepsilon}| \leq \frac{\delta}{3} + |\Lambda_N| \, P_\varepsilon \left(|V_\tau - m_{\tau,\varepsilon}| \geq \frac{\delta}{3} \right) .$$

Using this, we have

$$\begin{aligned} P_\varepsilon \left(|V_\tau - E_\varepsilon(V_\tau)| \geq \delta \right) &\leq 2\Phi_{\tau,\varepsilon} \left(\frac{\delta}{3} \right) + I \left[P_\varepsilon \left(|V_\tau - m_{\tau,\varepsilon}| \geq \frac{\delta}{3} \right) \geq \frac{\delta}{3\,|\Lambda_N|} \right] \\ &\leq 2\Phi_{\tau,\varepsilon} \left(\frac{\delta}{3} \right) + I \left[2\Phi_{\tau,\varepsilon} \left(\frac{\delta}{3} \right) \geq \frac{\delta}{3\,|\Lambda_N|} \right] , \end{aligned}$$

where $I[.]$, denotes the indicator function of an event. Using this inequality, we get

$$P(|V_\tau - E_\varepsilon(V_\tau)| \geq \delta) \leq 2 \left(1 + \frac{\delta}{3\,|\Lambda_N|} \right) E \left(\Phi_{\tau,\varepsilon} \left(\frac{\delta}{3} \right) \right) .$$

In order to prove the Proposition, it therefore suffices to show that

$$\lim_{\varepsilon\to 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\log E\left(\Phi_{\tau,\varepsilon}(\delta)\right)=-\infty \tag{2.26}$$

holds for all $\delta>0$. We actually prove much more, namely that

$$\lim_{\varepsilon\to 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\log\|\Phi_{\tau,\varepsilon}(\delta)\|_\infty=-\infty. \tag{2.27}$$

In order to estimate $E_\varepsilon(\cosh(\lambda\,|W_i\Delta W_i'|))$, we will take $\lambda=a\varepsilon^{-1}\tau$ with $0<a\le 1$. Remark that $\cosh(ab)\le 1+a^2\exp(b)$, if $0<a\le 1$ and $b>0$. If $x\in\Lambda_N$, we write $E_{x,\varepsilon}$ for the expectation under a Brownian bridge on the Λ_N-torus, i.e. a Brownian motion $(\beta_s)_{0\le s\le\varepsilon}$ starting at 0 and conditioned to be at x at time ε. It is evident that the volume for the sausage of such a Brownian bridge on the torus is stochastically smaller than the corresponding sausage of a Brownian bridge on $\mathbb{R}^d$. We then have

$$E_\varepsilon(\cosh(a(\tau/\varepsilon)\,|W_i\Delta W_i'|))\le 1+a^2\left(E_{\beta_{\varepsilon i}-\beta_{\varepsilon(i-1)},\varepsilon}(\exp\left[(\tau/\varepsilon)\left|W^{a\tau^{-1/(d-2)}}(\varepsilon)\right|\right]\right)^2,$$

where $W^a(t)=\cup_{s\le t}B_a(\beta_s)$. As remarked above, we can replace the right hand side in the above inequality by the corresponding quantity for the standard Brownian motion, which has the advantage that we now can use the standard rescaling properties. Using these, we get

$$\begin{aligned}&E_{x,\varepsilon}\left(\exp\left[(\tau/\varepsilon)\left|W^{a\tau^{-1/(d-2)}}(\varepsilon)\right|\right]\right)\\ \le\ & E^\infty_{\tau^{1/(d-2)}x,\varepsilon\tau^{2/(d-2)}}\left(\exp\left[\varepsilon^{-1}\tau^{-2/(d-2)}\left|W^a(\varepsilon\tau^{2/(d-2)})\right|\right]\right),\end{aligned}$$

where P^∞, E^∞ as usual refer to the Brownian on $\mathbb{R}^d$. According to the Lemma **??** below, we see that there is a $\tau_o(\varepsilon,N)$ such that for all $\tau\ge\tau_o(\varepsilon,N)$, all N , and all $x\in\Lambda_N$ we have

$$E^\infty_{x,\varepsilon}\left(\exp\left[(\tau/\varepsilon)\left|W^{a\tau^{-1/(d-2)}}(\varepsilon)\right|\right]\right)\le C.$$

We therefore get

$$\prod_{i=1}^{\tau/\varepsilon}E_\varepsilon(\cosh(\lambda\,|W_i\Delta W_i'|))\le\prod_{i=1}^{\tau/\varepsilon}(1+a^2C^2)\le\exp\left((\tau/\varepsilon)a^2C^2\right).$$

Choosing now a small enough, and implementing it in the definition of Φ , we see that (**??**) follows, and therefore the Proposition **??** follows. ■

Lemma 2.42 *There exists a constant C with*

$$\sup_{t\geq 1, |x|\leq t} E^{\infty}_{x,t}\left(\exp\left[\frac{1}{t}\left|W^a(t)\right|\right]\right) \leq C.$$

I will not give a proof of this. For the unconditioned Brownian motion, this follows from estimates in [**?**]. The Lemma states that the situation does not change much if we condition the Brownian to end in a point which is away from the starting point at maximum t. Although this is a large deviation for the Brownian, it is evident that this increases the sausage at maximum to something of order t, and so the statement of the Lemma looks plausible. It is not difficult to prove if by chopping time into small pieces.

We now have finished the first part (A) of the proof, and we come to (B). During the proof of this part, we keep the parameter ε completely fixed.

We denote by p_s the transition densities for the Brownian motion (on the torus Λ_N, but as usually, we drop the N in the notation). For $y, z \in \Lambda_N$ we define

$$q^{\varepsilon}_b(y,z) = P(\exists s \leq \varepsilon \text{ with } \beta_s \in B_b(0) | \beta_0 = y, \beta_\varepsilon = z),$$

and by an abuse of notation $q^{\varepsilon}_{\tau}(y,z) = q^{\varepsilon}_{a\tau^{-1/(d-z)}(y,z)}$ where a is our radius of the original sausage. We also set for $y, z \neq 0$

$$\varphi_\varepsilon(y,z) = \frac{\int_0^\varepsilon p_s(y)p_{\varepsilon-s}(z)\,ds}{p_\varepsilon(z-y)}.$$

It is evident (see below) that $E_\varepsilon(V_\tau)$ can be expressed with the help $q^{\varepsilon}_{\tau}(y,z)$ and the empirical measure $L_{\varepsilon,\tau}$, and we therefore easily get a LDP, except for the problem that $q^{\varepsilon}_{\tau}(y,z)$ still depends on τ. We don't like this τ-dependence. The basis for being able to remove this is the following technical result.

Lemma 2.43 *a)Let $b < b_1 < N/4$. Then*

$$\sup_{x,y\notin B_{b_1}} q^{\varepsilon}_b(x,y) \leq c_1\left(\frac{b}{b_1}\right)^{d-2}\exp[-c_2(b_1-b)^2]$$

b) For any $\varepsilon, b > 0$ we have

$$\lim_{\tau\to\infty}\sup_{y,z\notin B_b(0)}\left|\tau q^{\varepsilon}_{\tau}(y,z) - \kappa_a\varphi_\varepsilon(y,z)\right| = 0,$$

where κ_a is the Newtonian capacity of the ball with radius a.

a) is rather evident and easy to prove. Remember that ε is fixed. The claim is that if the starting and the end point of the bridge are sufficiently for away from the ball to be hit, then there is only a small chance for this hitting. The exact form of the estimates comes easily from standard estimates of hitting probabilities.

b) is more delicate. From scaling, one sees that $q_\tau^\varepsilon(y,z)$ is in fact of order τ. The bridge has a chance to hit the small ball only if it already gets close to it. $\varphi_\varepsilon(y,z)$ measure the expectation of the total time, the bridge spends in the vicinity of the ball. This quantity has to be multiplied with the capacity of the ball, which is κ_a/τ. A detailed proof takes about two pages, but it is essentially straightforward.

We now perform the approximation of $E_\varepsilon(V_\tau)$. We first approximate V_τ by cutting out small holes around the points $\beta_{i\varepsilon}$: Fix $b>0$ and define

$$W_b^a = W_i \backslash \left(B_b(\beta_{(i-1)\varepsilon}) \cup B_b(\beta_{i\varepsilon})\right),$$

and set

$$V_\tau^K = \left| \bigcup_{i=1}^{\tau/\varepsilon} W_i^{K\tau^{-1/(d-2)}} \right|.$$

Evidently, we have cut out at maximum τ/ε times the volume of a ball of radius $K\tau^{-1/(d-2)}$. Therefore

$$\left|V_\tau - V_\tau^K\right| \le c\varepsilon^{-1}K^d\tau^{-2/(d-2)}, \tag{2.28}$$

and therefore the difference is negligible for our purpose. The cutting is convenient, because we can invoke then the Lemma **??** which helps to expand $\log(1-q)$ linearly in q just by $-q$.

$$\begin{aligned} E_\varepsilon(V_\tau^K) &= \int_{\Lambda_N} dx \left(1 - P_\varepsilon(x \notin \cup_{i=1}^{\tau/\varepsilon} W_i^{K\tau^{-1/(d-2)}})\right) \qquad (2.29) \\ &= \int_{\Lambda_N} dx \left(1 - \prod_{i=1}^{\tau/\varepsilon} \left[1 - P_\varepsilon(x \in W_i^{K\tau^{-1/(d-2)}})\right]\right) \\ &= \int_{\Lambda_N} dx \left(1 - \exp\left[\frac{\tau}{\varepsilon}\int \log\left(1 - q_\tau^{\varepsilon, K\tau^{-1/(d-2)}}(z-x, y-x)\right) L_{\varepsilon,\tau}(dy,dz)\right]\right), \end{aligned}$$

where $q_\tau^{\varepsilon,b}(z,y) = q_\tau^\varepsilon(z,y)$ if $z,y \notin B_b(0)$ and 0 otherwise. We are therefore naturally led to the investigation of mappings $\mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \to [0,\infty)$

$$\Phi_{\tau,\beta,b}(\mu) \stackrel{\text{def}}{=} \int_{\Lambda_N} dx \left(1 - \exp\left[-\beta\tau \int q_\tau^{\varepsilon,b}(z-x, y-x)\mu(dy,dz)\right]\right).$$

Then, we get the sandwiching

$$\Phi_{\tau,(1+\delta_K)/\varepsilon,K\tau^{-1/(d-2)}}(L_{\varepsilon,\tau}) \le E_\varepsilon(V_K) \le \Phi_{\tau,1/\varepsilon,K\tau^{-1/(d-2)}}(L_{\varepsilon,\tau}).$$

With the help of Lemma **??** a), we easily get that we can replace $K\tau^{-1/(d-2)}$ with a fixed (small) value b :

$$\left\| E_\varepsilon(V_K) - \Phi_{\tau,1/\varepsilon,b}(L_{\varepsilon,\tau}) \right\|_\infty \le \delta_1(\tau,K,b), \tag{2.30}$$

where $\lim_{b\to 0} \limsup_{K\to\infty} \limsup_{\tau\to\infty} \delta_1(\tau,K,b) = 0$. (Of course, we just estimate $|\exp[-\xi] - \exp[-\eta]| \le |\xi - \eta|$). Define now

$$\Phi_{\infty,\beta,b}(\mu) \stackrel{\text{def}}{=} \int_{\Lambda_N} dx \left(1 - \exp\left[-\beta\kappa_a \int \varphi_\varepsilon^b(y-x, z-x)\mu(dy,dz)\right]\right),$$

where $\varphi_\varepsilon^b(x,y)$ is $\varphi_\varepsilon(x,y)$ if x,y are both outside $B_b(0)$, and 0 otherwise. Lemma **??** b) now easily gives

$$\left\| \Phi_{\infty,1/\varepsilon,b}(L_{\varepsilon,\tau}) - \Phi_{\tau,1/\varepsilon,b}(L_{\varepsilon,\tau}) \right\|_\infty \le \delta_2(\tau,b), \tag{2.31}$$

where $\lim_{\tau\to\infty} \delta_2(\tau,b) = 0$ for all b. Combining now (**??**) and (**??**), we get, by letting $\tau \to \infty$, $K \to \infty$, and finally $b \to 0$ (in this order):

$$\lim_{\tau\to\infty} \left\| \Phi_{\infty,1/\varepsilon,0}(L_{\varepsilon,\tau}) - E_\varepsilon(V_\tau) \right\|_\infty = 0.$$

$\Phi_{\infty,1/\varepsilon,0}(\mu)$ is easily seen to be continuous in μ (even in total variation), and therefore, we get the following large deviation principle for $E_\varepsilon(V_T)$ (ε arbitrary > 0, but fixed)

Proposition 2.44 $\{E_\varepsilon(V_\tau)\}_{\tau>0}$ *satisfies a τ-LDP with rate function*

$$J_\varepsilon(b) \stackrel{\text{def}}{=} \inf\left\{ I_\varepsilon^{(2)}(\mu) : \mu \in \mathcal{M}_1^+(\Lambda_N \times \Lambda_N),\ \Phi_{\infty,1/\varepsilon,0}(\mu) = b \right\}.$$

Here $I_\varepsilon^{(2)}(\mu)$ is the rate function of the LDP for $L_{\varepsilon,\tau}$ on $\mathcal{M}_1^+(\Lambda_N \times \Lambda_N)$ which is just

$$I_\varepsilon^{(2)}(\mu) = \int \log\left(\frac{d\mu}{d(\mu_1 \otimes \pi_\varepsilon)}\right) d\mu,$$

if $\mu_1 = \mu_2$, μ_i being the margins of μ, and ∞ otherwise. π_ε is the transition kernel of the Brownian on the torus on a time interval ε.

We come now to the last step C) of the proof of Proposition **??**. Up to now, we have a LDP for $E_\varepsilon(V_\tau)$, and we know that this quantity approximates the one we are interested in. We therefore only have to prove now that the rate function approximates the right one. There is one delicacy. The rate function we have for fixed ε is of course a rate function of the bivariate chain. It is well known, that the rate function of the univariate discrete time approximation approximates in a certain sense the one for the Brownian motion, and the rate function of the univariate discrete chain is the projection of the bivariate one. In our case, however, the function really depends on the bivariate chain. It however turns out that for small ε, the bivariate chain is essentially determined by the univariate one, up to a superexponential decay. This is probably well known.

For $\mu \in \mathcal{M}_1^+(\Lambda_N)$, we write $I(\mu)$ for the standard large deviation rate function for the empirical distribution of the Brownian motion: $I(\mu) = \frac{1}{2}\int |\nabla g|^2 \, dx$, $g^2(x) = \mu(dx)/dx$ if μ is absolutely continuous, and its density is in H_1 and $I(\mu) = \infty$ otherwise. We also denote by $I_\varepsilon : \mathcal{M}_1^+(\Lambda_N) \to [0,\infty]$ the projection of $I_\varepsilon^{(2)}$: $I_\varepsilon(\nu) = \inf\left\{I_\varepsilon^{(2)}(\mu) : \mu_1 = \nu\right\}$. We collect some basic facts about these entropies which have been proved by Donsker and Varadhan or are simple consequences of their results:

Lemma 2.45 *Let $(\pi_t)_{t\geq 0}$ be the Brownian semigroup. Then for all $\nu, \mu \in \mathcal{M}_1^+(\Lambda_N)$ we have*
a) $I_\varepsilon(\nu) = -\inf_{u\in\mathcal{D}^+} \int \log \frac{\pi_\varepsilon u}{u} d\nu$, where $\mathcal{D}^+$ is the set of positive measurable functions which are bounded and bounded away from 0.
b) $t \to I_t(\nu)/t$ is non-increasing with $I(\nu) = \lim_{t\to 0} \frac{I_t(\nu)}{t}$.
c) $\|\nu - \nu\pi_s\|_{\mathrm{TV}} \leq 8\sqrt{I_s(\nu)}$ for $s > 0$
d) $I_s(\nu\pi_t) \leq I_s(\nu)$ for $s, t > 0$.
e) $\|\mu - \mu_1 \otimes \pi_s\|_{\mathrm{TV}} \leq 8\sqrt{I_s^{(2)}(\mu)}$

Proof a) This is Theorem 2.1 of [?], combined with Lemma 2.1 of [?].
b) Let $u \in \mathcal{D}^+$ and $s, t > 0$. Then

$$\int \log \frac{\pi_{s+t}u}{u} d\nu = \int \log \frac{\pi_s(\pi_t)u}{\pi_t u} d\nu + \int \log \frac{\pi_t u}{u} d\nu \geq -I_s(\nu) - I_t(\nu).$$

Therefore $I_{s+t}(\nu) \leq I_s(\nu) + I_t(\nu)$. Hence, $I_t(\nu)/t$ is non-decreasing. The fact that $\lim_{t\to 0} I_t(\nu)/t = I(\nu)$ is Lemma 3.1 from [?].
c) This is Lemma 4.1 of [?]. (The function ϕ used there is easily seen to be $\leq 8\sqrt{x}$).
d) follows from the convexity of I_s.
e) Let $P^\mu(x, dy)$ be a transition kernel on Λ_N with $\mu = \mu_1 \otimes \pi^\mu$. Then

$$\|\mu - \mu_1 \otimes \pi_s\|_{\mathrm{TV}} \leq \int \mu_1(dx) \, \|P^\mu(x, \cdot) - \pi_s(x, \cdot)\|_{\mathrm{TV}} \, .$$

By Theorem 4.1 of [?], we have

$$\|P^\mu(x, \cdot) - \pi_s(x, \cdot)\|_v \leq 8\sqrt{k(P^\mu(x, \cdot)|\pi_s(x, \cdot))},$$

where k is the usual Kullback-Leibler information, i.e. $k(\gamma|\sigma) = \int \log(d\gamma/d\sigma)d\gamma$. Therefore

$$\begin{aligned} \|\mu - \mu_1 \otimes \pi_s\|_{\mathrm{TV}} &\leq 8 \int \mu_1(dx) \sqrt{k(P^\mu(x, \cdot)|\pi_s(x, \cdot))} \\ &\leq 8\sqrt{\int \mu_1(dx) k(P^\mu(x, \cdot)|\pi_s(x, \cdot))} = 8\sqrt{I_s^{(2)}(\mu)}. \end{aligned}$$

■

Next, we need an approximation of our functions $\Phi_{\infty,1/\varepsilon,0}$, for which we had derived a LDP by the Proposition **??**, by the simpler functions $\Psi_\varepsilon : \mathcal{M}_1^+(\Lambda_N) \to [0, \infty)$, defined by

$$\Psi_\varepsilon(\nu) = \int dx \left[1 - \exp\left(-\frac{\kappa_a}{\varepsilon} \int_0^\varepsilon p_s(y - x)\nu(dy)\right)\right].$$

Lemma 2.46 *For any $K > 0$*

$$\lim_{\varepsilon \to 0} \sup_{\mu: \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) \leq K} \left|\Phi_{\infty,1/\varepsilon,0}(\mu) - \Psi_\varepsilon(\mu_1)\right| = 0.$$

Proof Evidently $\Psi_\varepsilon(\mu_1) = \Phi_{\infty,1/\varepsilon,0}(\mu_1 \otimes \pi_\varepsilon)$. Therefore

$$\begin{aligned} &\left|\Phi_{\infty,1/\varepsilon,0}(\mu) - \Psi_\varepsilon(\mu_1)\right| \\ &= \left|\Phi_{\infty,1/\varepsilon,0}(\mu) - \Phi_{\infty,1/\varepsilon,0}(\mu_1 \otimes \pi_\varepsilon)\right| \\ &\leq \frac{\kappa_a}{\varepsilon} \left| \int dx \int_{\Lambda_N \times \Lambda_N} \varphi_\varepsilon(y-x, z-x) \left(\mu(dy,dz) - \mu_1 \otimes \pi_\varepsilon(dy,dz)\right) \right| \\ &\leq \frac{\kappa_a}{\varepsilon} \int dx \int_{\Lambda_N \times \Lambda_N} \varphi_\varepsilon(y-x, z-x) \left|\mu - \mu_1 \otimes \pi_\varepsilon\right| (dy,dz) \\ &= \kappa_a \left\|\mu - \mu_1 \otimes \pi_\varepsilon\right\|_v . \end{aligned}$$

The Lemma follows now from Lemma **??**e). ■

Next, we define $\Gamma : L_1^+(\Lambda_N) \to [0,\infty)$ by

$$\Gamma(f) = \int dx \left[1 - \exp\left(-\kappa_a f(x)\right)\right].$$

Lemma 2.47 *For any* $K > 0$

$$\lim_{\varepsilon \to 0} \sup_{\nu : \frac{1}{\varepsilon} I_\varepsilon(\nu) \leq K} \left| \Gamma\left(\frac{d\nu}{dx}\right) - \Psi_\varepsilon(\nu)\right| = 0.$$

(Remark that if $I_\varepsilon(\nu)$ *is finite, then* $d\nu \ll dx$*)*

Proof

$$\begin{aligned} &\left| \Gamma\left(\frac{d\nu}{dx}\right) - \Psi_\varepsilon(\nu)\right| \\ &\leq \int dx \left| \exp\left(-\frac{\kappa_a}{\varepsilon} \int_0^\varepsilon ds \int p_s(y-x)\nu(dy)\right) - \exp\left(-\frac{\kappa_a}{\varepsilon}\int_0^\varepsilon ds \frac{d\nu}{dx}(x)\right)\right| \\ &\leq \int dx \frac{\kappa_a}{\varepsilon} \int_0^\varepsilon ds \left| \frac{\nu\pi_s}{dx}(x) - \frac{d\nu}{dx}(x)\right| = \frac{\kappa_a}{\varepsilon}\int_0^\varepsilon ds \left\|\nu\pi_s - \nu\right\|_{\mathrm{TV}} . \end{aligned}$$

Now, for $s \leq \varepsilon$

$$\|\nu\pi_s - \nu\|_{\mathrm{TV}} \leq \|\nu\pi_s\pi_\varepsilon - \nu\pi_s\|_{\mathrm{TV}} + \|\nu\pi_{s+\varepsilon} - \nu\|_{\mathrm{TV}} \leq 8\sqrt{I_\varepsilon(\nu\pi_s)} + 8\sqrt{I_{\varepsilon+s}(\nu)}.$$

Now $I_\varepsilon(\nu\pi_s) \leq I_\varepsilon(\nu)$ by Lemma **??**d). Furthermore, $I_{\varepsilon+s}(\nu) \leq 2\varepsilon I_{\varepsilon+s}(\nu)/(\varepsilon + s) \leq 2I_\varepsilon(\nu)$ by Lemma **??**b). Therefore, we get $\|\nu\pi_s - \nu\|_{\mathrm{TV}} \leq 8(1+\sqrt{2})\sqrt{K\varepsilon}$ if $I_\varepsilon(\nu) \leq K\varepsilon$. Using this, the Lemma follows. ■

Proof or Proposition **??**

Consider a continuous bounded function $f : \mathbb{R} \to \mathbb{R}$. Then

$$\begin{aligned}
&\lim_{\tau\to\infty} \frac{1}{\tau} \log E\left(\mathrm{e}^{\tau f(V_\tau)}\right) \\
&= \lim_{\varepsilon\to 0}\lim_{\tau\to\infty} \frac{1}{\tau} \log E\left(\exp\left[\tau f(E_\varepsilon(V_\tau))\right]\right) \ \text{(Proposition ??)} \\
&= \lim_{\varepsilon\to 0}\sup_{\mu} \left\{ f(\Phi_{\infty,1/\varepsilon,0}(\mu)) - \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) \right\} \ \text{(Cororollary ??)} \\
&= \lim_{K\to\infty}\lim_{\varepsilon\to 0} \sup_{I_\varepsilon^{(2)}(\mu)\leq\varepsilon K} \left\{ f(\Phi_{\infty,1/\varepsilon,0}(\mu)) - \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) \right\} \\
&= \lim_{K\to\infty}\lim_{\varepsilon\to 0} \sup_{I_\varepsilon^{(2)}(\mu)\leq\varepsilon K} \left\{ f(\Psi_\varepsilon(\mu_1)) - \frac{1}{\varepsilon} I_\varepsilon^{(2)}(\mu) \right\} \ \text{(Lemma ??)} \\
&= \lim_{K\to\infty}\lim_{\varepsilon\to 0} \sup_{I_\varepsilon(\nu)\leq\varepsilon K} \left\{ f(\Psi_\varepsilon(\nu)) - \frac{1}{\varepsilon} I_\varepsilon(\nu) \right\} \\
&= \lim_{K\to\infty}\lim_{\varepsilon\to 0} \sup_{I_\varepsilon(\nu)\leq\varepsilon K} \left\{ f\left(\Gamma\left(\frac{d\nu}{d\lambda}\right)\right) - \frac{1}{\varepsilon} I_\varepsilon(\nu) \right\} \ \text{(Lemma ??)} \\
&= \sup_{\nu} \left\{ f\left(\Gamma\left(\frac{d\nu}{d\lambda}\right)\right) - I(\nu) \right\}.
\end{aligned}$$

This proves now the Proposition **??**.

3 Localization-delocalization transitions at interfaces

3.1 A localization-delocalization transition for a heteropolymer

3.2 Wetting transition for a one dimensional random walk

3.3 Strong localization for the pinned harmonic and anharmonic crystal

3.4 Entropic repulsion for the free field

3.5 Absence of a wetting transition for dimension three and larger

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