LAPLACE APPROXIMATIONS FOR LARGE DEVIATIONS OF NON-REVERSIBLE MARKOV PROCESSES ON COMPACT STATE SPACES PART II: THE DEGENERATE HESSIAN CASE

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Abstract. Let L_T be the empirical measrue of a uniformly ergodic nonreversible Markov process on a compact metric space and U be a smooth functional. We give the long time asymptotic evaluation of the form of $E[\exp(TL_T)]$ in the case where the Hessian of $J - U$ may be degenerate, where J is a rate function of the large deviations of empirical measure.

0. Introduction.

In our previous article [3], we obtained the Laplace approximation based on non-reversible Markov processes on compact metric spaces in the case where the Hessian of the free energy is non-degenerate. In this paper, we investigate the same asymptotics in the case where the Hessian of free energy may be degenerate. Namely, let E be a compact metric space and $\{P_x\}_{x\in E}$ be probability measures of a Markov process on $\Omega = D([0,\infty); E)$ satisfying an appropriate condition (see Ass. 1 in §1). Let L_T be an empirical probability mersure on Ω (see (1.25)) and U be a bounded and smooth function on the probability measures on E in suitable sense (see Ass. 2 in §1). The purpose of this paper is to obtain the asymptotic behavior of $\mathrm{e}^{f_0 T} P_x[\exp(TU(L_T))]$ in general framework as T goes to ∞ where $f_0 = -\lim_{T\to\infty} T^{-1} \log P_x[\exp(TU(L_T))]$.

Laplace approximations in degenerate Hessian case were investigated by Bolthausen [1] and by Chiyonobu [4] for *i.i.d.* random variables and in [7] for symmetric Markov processes. Here we treat this asymptotics for non-symmetric Markov processes by using the same idea as in [4] and [7] which was first used in Kusuoka and Stroock [6]. Key points to obtain this asymptotics in this way are to determine a finite dimensional submanifold whose tangent spaces include the whole directions in which the Hessian is degenerate and to obtain a uniform estimate for uniform integrability.

Here we state our result briefly. Let the Markov process $\{P_x\}_{x\in E}$ satisfy a strong ergodicity condition (Ass. 1) and U satisfy a smoothness conditon (Ass.

2). Then, there exist a finite dimensional manifold $\mathcal{N}$ and bounded continuous functions $\overline{f}, g : \mathcal{N} \to [0, \infty)$ such that

$$P_x[\exp(TU(L_T))] \sim \mathrm{e}^{-f_0 T} (\frac{T}{2\pi})^{(\dim \mathcal{N})/2} \int_{\mathcal{N}} g(\xi) \mathrm{e}^{-\overline{f}(\xi)T} n_0(d\xi)$$

as $T \to \infty$, where n_0 is the Riemannian volume on $\mathcal{N}$ and $f(T) \sim g(T)$ as $T \to \infty$ means that $\lim_{T\to\infty} f(T)/g(T) = 1$. This implies that $\mathrm{e}^{f_0 T} P_x[\exp(TU(L_T))] = O(n^{\dim(\mathcal{N})/2})$ (see Th.4.1).

In §1 we explain the setting and problem exactly and in §2 we study about uniformity of some estimates and Gaussian behavior treated in [3]. We will treat a finite dimensional submanifold refrecting singularities in §3 and in §4 we state our main theorem. In §5 we give the proof of Lem.2.1.

1. Setting and preliminaries.

The setting is almost the same as in [3], but we state here again to fix the notation. Let E be a compact metric space. $C(E)$ is a Banach space of continuous functions on E with values in $\mathbf{R}$ with the supremum norm $\|\cdot\|_\infty$ and $C^+(E)$ is the set of strictly positive continuous functions on E. $\mathcal{M} = \mathcal{M}(E)$ is the set of signed measures on E with finite total variations and $\mathcal{M}_1^+ = \mathcal{M}_1^+(E)$ is the set of probability measures on E with weak topology.

Let $\{P_x, x \in E\}$ be a family of time homogeneous Markov probability measures on the path space $\Omega = D([0,\infty); E)$ with $P_x\{\omega(0) = x\} = 1$. We denote by $\{P_t\}_{t\geq 0}$ the corresponding semigroup on $(C(E), \|\cdot\|_\infty)$ and impose the following assumption.

Assumption 1 (Strong uniform ergodicity). *There exists a (P_t)-invariant probability measure μ on E with supp$(\mu) = E$ and for any $t > 0$, there exists a continuous function $p(t,\cdot,\cdot) : E \times E \longrightarrow (0,\infty)$ such that*

$$P_t\varphi(x) = \int_E p(t,x,y)\varphi(x)\mu(dy), \quad \mu\text{-}a.a\ x \in E \text{ for all } \varphi \in C(E).$$

For $T > 0$ and $\omega \in \Omega$, let $L_T(\omega)(\cdot) \in \mathcal{M}_1^+$ be the empirical measure given by

$$\int_E \varphi(x) L_T(\omega)(dx) = \frac{1}{T}\int_0^T \varphi(\omega(t))dt \quad \text{for } \varphi \in C(E). \tag{1.1}$$

It is known (see Deuschel and Stroock [5]) that under Ass. 1, $\{P_x \circ L_T^{-1}\}_{T\geq 0}$ satisfy a strong uniform large deviation principle with rate function $J : \mathcal{M}_1^+ \to [0,\infty]$,

$$J(\lambda) = \sup\{-\int_E \frac{L\varphi}{\varphi} d\lambda; \varphi \in C^+(E) \cap \mathcal{D}om(L)\}, \tag{1.2}$$

where L is the infinitesimal generator of $\{P_t\}_{t\geq 0}$ on $C(E)$ and $\mathcal{D}om(L)$ is the domain of L in $C(E)$.

Let $\{\psi_k\}_{k=1}^\infty \subset C(E)$ be a complete orthonormal basis in $L^2(E, d\mu)$ and Span$\{\psi_k, k \geq 1\}$ is dense in $C(E)$ and let $a = \{a_k\}_{k=1}^\infty$ be a sequence of positive numbers which satisfy the following:

(1) $\{a_k\}_{k=1}^\infty$ is monotone decreasing and

$$\lim_{k\to\infty} a_k = 0,$$

(2) $\sum_{k=1}^\infty a_k \|\psi_k\|_\infty^2 = 1$.

We fix $\{\psi_k\}_{k=1}^\infty$ and $\{a_k\}_{k=1}^\infty$ throughout this paper. For λ and $\nu \in \mathcal{M}$, we set

$$(\lambda, \nu)_a = \sum_{k=1}^\infty a_k (\int_E \psi_k d\lambda)(\int_E \psi_k d\nu) \quad \text{and} \tag{1.3}$$

$$\|\lambda\|_a = (\lambda, \lambda)^{1/2}.$$

We denote by $\mathcal{M}_a$ the completion of $\mathcal{M}$ with respect to $\|\cdot\|_a$. Note that

$$\|\lambda\|_a \leq \|\lambda\|_{\text{var}} \quad \text{for any } \lambda \in \mathcal{M}. \tag{1.4}$$

The following lemma is easily obtained.

Lemma 1.1. *Let* $\nu_n, \nu \in \mathcal{M}_1^+, n = 1, 2, \ldots$ *Then,* $\nu_n \to \nu, \quad n \to \infty$, *weakly if and only if* $\|\nu_n - \nu\|_a \to 0, \quad n \to \infty$.

Let U be a continuous function on $\mathcal{M}_A$ satisfying the following smoothness assumption.

Assumption 2 (Smoothness of U). *U is bounded continuous and has third bounded continuous derivatives in Frechet sense.*

Lemma 1.2. *For any* $\nu \in \mathcal{M}_1^+$, *there exist* $U_\nu^{(1)} \in C(E)$ *and* $U_\nu^{(2)} \in C(E \times E)$ *such that*

(1)

$\mathcal{M}_1^+ \ni \nu \mapsto U_\nu^{(1)} \in C(E)$ *is continuous,*
$\mathcal{M}_1^+ \ni \nu \mapsto U_\nu^{(2)} \in C(E \times E)$ *is continuous and*

(2) *for any* $\lambda \in \mathcal{M}_1^+$,

$$U(\lambda) = U(\nu) + \int_E U_\nu^{(1)}(x)(\lambda - \nu)(dx) + \frac{1}{2} \iint_{E\times E} U_\nu^{(2)}(x)(\lambda - \nu)(dx)$$
$$(\lambda - \nu)(dy) + O(\|\lambda - \nu\|_a^3).$$

Proof. By Ass. 2,

$$U(\lambda) = U(\nu) + DU(\nu)[\lambda - \nu] + \frac{1}{2} D^2 U(\nu)[\lambda - \nu, \lambda - \nu] \\ + O(\|\lambda - \nu\|_a^3).$$

Set

$$U_\nu^{(1)}(x) = DU(\nu)[\delta_x] \quad \text{and} \\ U_\nu^{(2)}(x, y) = D^2 U(\nu)[\delta_x, \delta_y].$$

Suppose $x_n \to x$ in E. Then, $\delta_{x_n} \to \delta_x$ in $\mathcal{M}_a$, hence $U_\nu^{(1)} \in C(E)$ and $U_\nu^{(2)} \in C(E \times E)$.

Suppose $\nu_n \to \nu$ weakly in $\mathcal{M}_1^+$. Then,

$$\begin{aligned} &\sup_{x \in E} |DU(\nu_n)[\delta_x] - DU(\nu)[\delta_x]| \\ &\quad \le \sup_{x \in E} \|\delta_x\|_a^{-1} |DU(\nu_n)[\delta_x] - DU(\nu)[\delta_x]| \\ &\quad \le \sup_{h \in \mathcal{M}_a, h \neq 0} \|h\|_a^{-1} |DU(\nu_n)[h] - DU(\nu)[h]| \\ &\quad \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

This means the continuity of $U_\cdot^{(1)}$ and the continuity of $U_\cdot^{(2)}$ is proved similarly. □

Since E is compact and $U_\nu^{(2)} \in C(E \times E)$, we have

Lemma 1.3. *$C(E) \ni \varphi \mapsto \int_E U_\nu^{(2)}(\cdot, y)\varphi(y)\nu(dy) \in C(E)$ is a trace class operator.*

For $V \in C(E)$, we set

$$P^V(t, x, A) = P_x[\exp(\int_0^t V(\omega(s))ds) \mathbf{1}_A(\omega(t))], \quad A \in \mathcal{B}(E), \tag{1.5}$$

$$P_t^V \varphi(x) = \int_E \varphi(y) P^V(t, x, dy), \quad \varphi \in C(E), \quad \text{and} \tag{1.6}$$

$$\alpha^V = \lim_{t \to \infty} \frac{1}{t} \log \|P_t^V\|_{\text{op}}. \tag{1.7}$$

Then the following duality relations hold:

$$\alpha^V = \sup\{\int_E V d\lambda - J(\lambda); \quad \lambda \in \mathcal{M}_1^+\}, \tag{1.8}$$

$$J(\lambda) = \sup\{\int_E V d\lambda - \alpha^V; \quad V \in C(E)\}. \tag{1.9}$$

The following is proved in Bolthausen, Deuschel and Schmock [2].

Proposition 1.1. *For $V \in C(E)$, there uniquely exist u^V and $\ell^V \in C^+(E)$ with $\|u^V\|_{L^1(d\mu)} = 1$ and $(u^V, \ell^V)_{L^2(d\mu)} = 1$ such that*

$$P_t^V u^V = \mathrm{e}^{\alpha^V t} u^V, \qquad \textit{and}$$
$$\int_E \mu(dy)\ell^V(y)P^V(t,y,dz) = \mathrm{e}^{\alpha^V t}\ell^V(z)\mu(dz).$$

Set

$$\nu^V(x) = \ell^V(x)u^V(x) \quad \in C^+(E), \tag{1.10}$$
$$\nu^V(dx) = \nu^V(x)\mu(dx) \in \mathcal{M}_1^+. \tag{1.11}$$

Furthermore we set

$$Q^V(t,x,dy) = \mathrm{e}^{-\alpha^V t}\frac{1}{u^V(x)}P^V(t,x,dy)u^V(y). \tag{1.12}$$

Then, $\{Q^V(t,x,dy)\}$ are transition probabilities of a Markov process with the generator $L^{\overline{V}}$ given by

$$L^{\overline{V}} = \frac{1}{u^V(x)}(L + V - \alpha^V)(u^V\cdot). \tag{1.13}$$

The following is also proved in Bolthausen, Deaschel and Schmock [2].

Proposition 1.2. *Let $V \in C(E)$.*

(i) *ν^V is a unique invariant probability measure of $\{Q_t^V\}_{t\geq 0}$ and for any $t > 0$ there exists a probability density $q^V(t,x,y)$ of Q_t^V with respect to $d\nu^V$ and $q^V(t,\cdot,\cdot) \in C^+(E\times E)$.*

(ii) *There exist an $\alpha_1^V > 0$ and $c_1^V < \infty$ such that*

$$\sup_{x\in E}\|q^V(t,x,\cdot) - 1\|_\infty \leq c_1^V \mathrm{e}^{-\alpha_1^V t}.$$

Set

$$g^V(x,y) = \int_0^\infty (q^V(t,x,y) - 1)dt, \tag{1.14}$$
$$g^{V\prime}(x,y) = g^V(y,x), \tag{1.15}$$
$$\overline{g^V}(x,y) = g^V(x,y) + g^{V\prime}(y,x). \tag{1.16}$$

For $\varphi \in C(E)$, set

$$G^V\varphi(x) = \int_E g^V(x,y)\varphi(y)\nu^V(dy), \tag{1.17}$$

$$G^{V\prime}\varphi(x) = \int_E g^{V\prime}(x,y)\varphi(y)\nu^V(dy), \tag{1.18}$$

$$\overline{G^V}\varphi(x) = \int_E \overline{g^V}(x,y)\varphi(y)\nu^V(dy). \tag{1.19}$$

Note that $G^V, G^{V\prime}$ and $\overline{G^V}$ are bounded linear operators on $C(E)$. Furthermore, note that for any $\varphi \in C(E)$,

$$(\varphi, \overline{G^V}\varphi)_{\nu^V} \geq 0. \tag{1.20}$$

Actually,

$$0 \leq Q^V_{\nu^V}[(\int_E \varphi d(\sqrt{T}(L_T - \nu^V)))^2] \to (\varphi, \overline{G^V}\varphi)_{\nu^V} \quad \text{as } T \to \infty.$$

For $\varphi \in C(E)$, if $\mathcal{M} \ni \lambda \mapsto \int_E \varphi d\lambda$ is continuous on $\mathcal{M}_a$, then we write $\int_E \varphi d\lambda$ in the form $(\widehat{\varphi}, \lambda)_a$ by appropriate $\widehat{\varphi} \in \mathcal{M}_a$. For ψ_n, $\widehat{\psi_n}$ is given by

$$\widehat{\psi_n} = \frac{1}{a_n}\psi_n\mu.$$

We write $C_0(E)$ the set of finite linear combinations of ψ_n. For $V \in C(E)$, we define the bounded linear operator S^V on $\mathcal{M}_a$ by

$$S^V e_k = \sum_{j=1}^{\infty} \sqrt{a_k a_j}(\psi_k, \overline{G^V}\psi_j)_{\nu^V} e_j$$

here we abbreviate $(\cdot,\cdot)_{L^2(d\nu^V)}$ as $(\cdot,\cdot)_{\nu^V}$ and put

$$e_k = \frac{1}{\sqrt{a_k}}\psi_k\mu,$$

then, $\{e_k\}_{k=1}^{\infty}$ is a complete orthonormal basis in $\mathcal{M}_a$.

Then we have the following (see Lem.2.30 in [3]).

Lemma 1.4.

(i) *S^V is a symmetric non-negative definite trace class operator on $\mathcal{M}_a$.*

(ii) *If $\varphi \in C(E)$, then $S^V\widehat{\varphi} = (\overline{G^V}\varphi)\nu^V$ and therfore*

$$(\varphi, \overline{G^V}\varphi)_{\nu^V} = (\widehat{\varphi}, S^V\widehat{\varphi})_a.$$

Set

$$\mathcal{M}_{1,0}^+ = \{\nu \in \mathcal{M}_1^+;\ \text{there exists a } V \in C(E) \text{ such that } \nu = \nu^V\}. \tag{1.24}$$

The following proposition in our previous paper [3] is essential for our argument.

Proposition 1.3. *Let $\nu \in \mathcal{M}_{1,0}^+$ and $\nu = \nu^V$, $V \in C(E)$. Then,*

$$J(\nu) = -(\ell^V, Lu^V)_\mu.$$

Proof. ¿From the definition,

$$J(\nu) \geq -\int_E \frac{Lu^V}{u^V} d\nu = -(\ell^V, Lu^V)_\mu.$$

On the other hand, letting $\mathbb{H}(Q^V|P)$ be the process level entropy of Q^V with respect to P, we have

$$\begin{aligned}\mathbb{H}(Q^V|P) &= Q^V_{\nu^V}[\log \frac{dQ^V_{\omega(0)}}{dP_{\omega(0)}}|_{\mathcal{F}_0^1}] \\ &= -(\ell^V, Lu^V)_\mu.\end{aligned}$$

By contraction principle, $J(\nu) \leq \mathbb{H}(Q^V|P)$. Then we have our assertion. □

Set

$$\begin{aligned}Z_{T,x} &= P_x[\exp(TU(L_T))], \\ f_0 &= \inf\{F(\lambda);\ \lambda \in \mathcal{M}_1^+\},\end{aligned} \tag{1.25}$$

where $F : \mathcal{M}_1^+ \to \mathbf{R} \cup \{\infty\}$ is given by

$$F(\lambda) = J(\lambda) - U(\lambda). \tag{1.26}$$

Then, by Varadhan's theorem

$$\lim_{T\to\infty} \frac{1}{T} \log Z_{T,x} = -f_0.$$

We set

$$\mathcal{V} = \{\nu \in \mathcal{M}_1^+; F(\nu) = f_0\}. \tag{1.27}$$

Our problem is to get the asymptotics of $\mathrm{e}^{f_0 T} Z_{T,x}$ as T goes to ∞.

2. Uniform Gaussian behavior near limiting point.

First we quote the following proposition from Bolthausen, Deuschel and Schmock [2].

Proposition 2.1.

(i) *Tere exists a* $c_1 \in [1, +\infty)$ *such that for any* $x \in E$,

$$\frac{1}{c_1}\exp(-6\|V\|_\infty t) \le \nu^V(x) \le c_1 \exp(6\|V\|_\infty t).$$

(ii) *For any* $a > 0$, *there exist an* $\alpha_1(a) > 0$ *and an* $M(a) < \infty$ *such that for any* $V \in C(E)$ *with* $\|V\|_\infty \le a$ *and any* $t \ge 1/2$,

$$\sup_{x,y\in E} |q^V(t,x,y) - 1| \le M(a)e^{-\alpha_1(a)t}.$$

We can prove the following lemma in a way similar to Lem. 2.8 in [3]. The proof of this lemma is given in Appendix.

Lemma 2.1. *Let* $V \in C(E)$.

(1) *For any* $\varphi \in C(E)$,$\varepsilon \in [-1,1]$ *and any* $T > 0$,

$$\begin{aligned}&|\log Q_x^V[\exp(\varepsilon\int_0^T(\varphi(\omega(s)) - (\varphi,1)_{\nu^V})ds)|\omega(T)=y] - \frac{T}{2}\varepsilon^2(\varphi,\overline{G^V}\varphi)_{\nu^V}| \\ &\quad \le B_1|\varepsilon|\|\varphi\|_\infty + B_2|\varepsilon|^2\|\varphi\|_\infty^2 + B_{3,1}|\varepsilon|^3\|\varphi\|_\infty^3 + B_{3,2}|\varepsilon|^3\|\varphi\|_\infty^3 T\end{aligned}$$

(2) *For any* $\varphi \in C(E)$, *any* $\xi \in \mathbf{C}$ *with* $|\xi| < 4\|\varphi\|_\infty/\alpha_1$ *and any* $T > 1$,

$$\begin{aligned}&|q^V(T,x,y)Q_x^V[\exp(\xi\int_0^T\varphi(\omega(s))ds)|\omega(T)=y] \\ &- \exp(\xi^2\frac{1}{2}(\varphi,\overline{G^V}\varphi)_{\nu^V})| \\ &\le |\xi|^2\frac{8\|\varphi\|_\infty^2}{\alpha_1}\exp(\frac{8\|\varphi\|_\infty^2}{\alpha_1}|\xi|^2T) + \exp(\frac{4\|\varphi\|_\infty^2}{\alpha_1}|\xi|^2T) \\ &\times\{\exp(\frac{4\|\varphi\|_\infty^2}{\alpha_1}|\xi|^2T((1-\frac{4|\xi|\|\varphi\|_\infty}{\alpha_1})^{-1}-1))-1\} \\ &+ 2|\xi|M\frac{4\|\varphi\|_\infty}{\alpha_1}(1-\frac{4|\xi|\|\varphi\|_\infty}{\alpha_1})^{-1}\exp(\frac{8\|\varphi\|_\infty^2}{\alpha_1}\frac{T|\xi|^2}{1-\frac{4|\xi|\|\varphi\|_\infty}{\alpha_1}}) \\ &+ |\xi|^2M^2(\frac{4\|\varphi\|_\infty^2}{\alpha_1})^2(1-\frac{4|\xi|\|\varphi\|_\infty}{\alpha_1})^{-2}\exp(\frac{8\|\varphi\|_\infty^2}{\alpha_1}\frac{T|\xi|^2}{1-\frac{4|\xi|\|\varphi\|_\infty}{\alpha_1}}) \\ &+ |\xi|M^2\frac{4\|\varphi\|_\infty^2}{\alpha_1}(1-\frac{4|\xi|\|\varphi\|_\infty}{\alpha_1})^{-1},\end{aligned}$$

where

$$B_1 = \frac{1}{1-M\mathrm{e}^{-\alpha_1 T}}A_1\frac{4}{\alpha_1},$$

$$B_2 = (\frac{1}{1-M\mathrm{e}^{-\alpha_1 T}}A_2 + \frac{1}{2}(\frac{1}{1-M\mathrm{e}^{-\alpha_1 T}})^2)(\frac{4}{\alpha_1})^2 + \frac{M}{2}\frac{1}{1-M\mathrm{e}^{-\alpha_1 T}}\frac{M}{\alpha_1},$$

$$B_{3,1} = (\frac{1}{1-M'\mathrm{e}^{-\alpha_1' T}}A'_{3,1} + \frac{1}{2}(\frac{1}{1-M'\mathrm{e}^{-\alpha_1' T}})^2 A'_2 A'_1 + \frac{1}{3}(\frac{1}{1-M'\mathrm{e}^{-\alpha_1' T}})^3 (A'1)^3)(\frac{4}{\alpha_1'})^3,$$

$$B_{3,2} = \frac{1}{1-M'\mathrm{e}^{-\alpha_1' T}}A'_{3,2}(\frac{4}{\alpha_1'})^3,$$

$$A_1 = M^2 + 2M, \quad A'_1 = (M')^2 + 2M',$$

$$A_2 = 3M^2 + 2M + \frac{1}{4}, \quad A'_2 = 3(M')^2 + 2M' + \frac{1}{4},$$

$$A'_{3,1} = \frac{13}{2}(M')^2 + 2M', \quad A'_{3,2} = \frac{1}{2}M'\alpha_1' + \frac{1}{8}\alpha_1',$$

$$M = M(V), \quad M' = M(V+\theta\varepsilon\varphi),$$

$$\alpha_1 = \alpha_1(V), \quad \alpha_1 = \alpha_1'(V+\theta\varepsilon\varphi),$$

$\alpha_1(\cdot)$ *and* $M(\cdot)$ *are given in Prop.2.1.*

Here we notice that B_i's and $B_{i,j}$'s are uniformly bounded on $\{V \in C(E); \|V\|_\infty \le a\}$ and $T \ge T_0$ for any fixed $a < \infty$ and $T_0 \ge 1$.

Let K be a compact metric space. Let $U : \mathcal{M}_a \times K \to \mathbf{R}$ be a continuous function which satisfies the following assumption.

Assumption 2′ (Smoothness of U).

(U-1) $\sup\{U(\lambda,\xi); \lambda \in \mathcal{M}_1^+, \xi \in K\} < \infty$.

(U-2) $U(\cdot,\xi) : \mathcal{M}_a \to \mathbf{R}$ *is three times bounded differentiable in Freche sense for any* $\xi \in K$.

(U-3)

$$U : \mathcal{M}_a \times K \to \mathbf{R} \quad \textit{is continuous,}$$
$$DU : \mathcal{M}_a \times K \to \mathcal{L}(\mathcal{M}_A;\mathbf{R}) \quad \textit{is continuous and}$$
$$D^2U : \mathcal{M}_a \times K \to \mathcal{L}(\mathcal{M}_A \times \mathcal{M}_A;\mathbf{R}) \quad \textit{is continuous.}$$

Let us define a function $F : \mathcal{M}_a \times K \to \mathbf{R} \cup \{\infty\}$ by

$$F(m,\xi) = J(m) - U(m,\xi). \tag{2.1}$$

Set

$$f_0(\xi) = \inf\{F(\lambda,\xi); \lambda \in \mathcal{M}_1^+\}. \tag{2.2}$$

¿From Ass. 2′ and Lem. 1.2, we have

Lemma 2.2. *For any* $\nu \in \mathcal{M}_1^+$, *there exist* $U_\nu^{(1)} \in C(E \times K)$ *and* $U_\nu^{(2)} \in C(E \times E \times K)$ *such that*

(i)

$$\mathcal{M}_1^+ \ni \nu \mapsto U_\nu^{(1)} \in C(E \times K) \quad \textit{is continuous,}$$
$$\mathcal{M}_1^+ \ni \nu \mapsto U_\nu^{(2)} \in C(E \times E \times K) \quad \textit{is continuous and}$$

(ii) *for any* $\lambda \in \mathcal{M}_1^+$,

$$\begin{aligned} U(\lambda, \xi) &= U(\nu, \xi) + \int_E U_\nu^{(1)}(x, \xi)(\lambda - \nu)(dx) \\ &\quad + \frac{1}{2} \iint_{E \times E} U_\nu^{(2)}(x, y, \xi)(\lambda - \nu)(dx)(\lambda - \nu)(dy) + O(\|\lambda - \nu\|_a^3). \end{aligned}$$

The following proposition was proved in [3].

Proposition 2.2. *Fix* $\xi \in K$. *Suppose that* $\nu_\xi \in \mathcal{M}_1^+$ *satisfies* $F(\nu_\xi, \xi) = f_0(\xi)$. *Then,* $\nu_\xi \in \mathcal{M}_{1,0}^+$ *and the* $V_\xi \in C(E)$ *which satisfies* $\nu_\xi = \nu^{V_\xi}$ *is given by*

$$V_\xi(x) = U_{\nu_\xi}^{(1)}(x, \xi) + const. \tag{2.3}$$

Furthermore we assume

Assumption 3 (Uniqueness of the minimum point). *For any* $\xi \in K$ *there exists a unique* $\nu_\xi \in \mathcal{M}_1^+$ *such that* $F(\nu_\xi, \xi) = f_0(\xi)$.

Lemma 2.3.

1. $K \ni \xi \mapsto f_0(\xi) \in \mathbf{R}$ *is continuous.*
2. $K \ni \xi \mapsto \nu_\xi \in \mathcal{M}_1^+$ *is continuous.*

Proof. Suppose $\xi_n \in K, n \geq 1$ and $\xi_n \to \xi_\infty$ as $n \to \infty$. By Ass.2′(1), we see that

$$\sup\{J(\nu_{\xi_n}); n \in \mathbf{N}\} < \infty.$$

Then, by taking a subsequence if necessary, we can assume that there exists a $\nu_\infty \in \mathcal{M}_1^+$ and $\nu_{\xi_n} \to \nu_\infty$ weakly. Then,

$$\begin{aligned} &J(\nu_\infty) - U(\nu_\infty, \xi_\infty) \\ &\quad = \lim_{n\to\infty} \{J(\nu_\infty) - U(\nu_\infty, \xi_n)\} \\ &\quad \geq \overline{\lim_{n\to\infty}} \{J(\nu_{\xi_n}) - U(\nu_{\xi_n}, \xi_n)\} \\ &\quad \geq \underline{\lim_{n\to\infty}} \{J(\nu_{\xi_n}) - U(\nu_{\xi_n}, \xi_n)\} \\ &\quad \geq J(\nu_\infty) - U(\nu_\infty, \xi_\infty). \end{aligned}$$

This implies that $\nu_\infty = \nu_{\xi_\infty}$ and $f_0(\xi_n) \to f_0(\xi_\infty)$ as $n \to \infty$. □

This lemma and Lem. 2.2 imply

Proposition 2.3. *$K \ni \xi \mapsto U^{(1)}_{\nu_\xi}(\cdot, \xi) \in C(E)$ is continuous.*

We write $\overline{G^{(\xi)}}, \alpha^{(\xi)}, \ldots$ instead of $\overline{G^{V_\xi}}, \alpha^{V_\xi}, \ldots$. From Lem. 2.3 and Lem. 2.19 in Bolthausen, Deuschel and Schmock [2] , we have the following.

Proposition 2.4.

(1) $K \ni \xi \mapsto \alpha^{(\xi)} \in \mathbf{R}$ *is continuous.*
(2) $K \ni \xi \mapsto u^{(\xi)} \in C^+(E)$ *is continuous.*
(3) $K \ni \xi \mapsto \ell^{(\xi)} \in C^+(E)$ *is continuous.*
(4) $K \ni \xi \mapsto \nu_\xi \in C^+(E)$ *is continuous.*
(5) $K \ni \xi \mapsto Q^{(\xi)} \in \mathcal{M}_1^+(\Omega)$ *is continuous.*
(6) $K \ni \xi \mapsto q^{(\xi)}(t, \cdot, \cdot) \in C^+(E \times E)$ *is continuous.*

For $x \in \mathcal{M}_a$ and $\xi \in K$, let

$$\Gamma^{(\xi)}(x) = \inf\{\|y\|_a^2; x = \sqrt{S^{(\xi)}}y\} \quad \text{and let}$$
$$\mathcal{M}_{\Gamma^{(\xi)}} = \{x \in \mathcal{M}_a; \Gamma^{(\xi)}(x) < \infty\}.$$

Then $\Gamma^{(\xi)} : \mathcal{M}_a \to [0, \infty]$ is convex, lower semi-continuous and $\{x \in \mathcal{M}_a; \Gamma^{(\xi)}(x) \le c\}$ is compact for any $c > 0$. Since ν_ξ attains the minimum of $F(\cdot, \xi)$, we have

$$(\varphi, \overline{G^V}\varphi)_{\nu_\xi} \ge D^2U(\nu_\xi, \xi)[(\overline{G^V})\nu_\xi, (\overline{G^V})\nu_\xi] \tag{2.4}$$

(see Prop.2.23 in [3]). Then for $x \in \mathcal{M}_a$,

$$\Gamma^{(\xi)}(x) \ge (x, D^2U(\nu_\xi, \xi)x)_a. \tag{2.5}$$

We impose the following assumption.

Assumption 4 (Non-degenerate Hessian). *For any $\xi \in K$ there exists a $c(\xi) \in (0, 1)$ such that for any $\varphi \in C(E)$ with $(\varphi, 1)_{\nu_\xi} = 0$,*

$$D^2U(\nu_\xi)[(\overline{G^{(\xi)}}\varphi)\nu_\xi, (\overline{G^{(\xi)}}\varphi)\nu_\xi] \le (1 - c(\xi))(\varphi, \overline{G^{(\xi)}}\varphi)_{\nu_\xi}.$$

Let $\alpha_1(\cdot)$ be the positive constant appeared in Prop. 2.1 (ii). Then, we have

$$\alpha_1 \equiv \inf\{\alpha_1(\|V_\xi\|_\infty); \xi \in K\} > 0. \tag{2.6}$$

Proposition 2.5. *For any $\varphi \in C(E)$, $K \ni \xi \mapsto (\varphi, \overline{G^{(\xi)}}\varphi)_{\nu_\xi} \in \mathbf{R}$ is continuous.*

Proof.

$$|(\varphi, G^{(\xi)}\varphi)_{\nu_\xi} - \int_0^T (\varphi, (Q_t^{(\xi)} - \Pi\varphi)_{\nu_\xi} dt| \le \frac{1}{\alpha_1} \mathrm{e}^{-\alpha_1 T} \|\varphi\|_\infty^2$$

and $K \ni \xi \mapsto (\varphi, (Q_t^{(\xi)} - \Pi)\varphi)_{\nu_\xi} \in \mathbf{R}$ is uniformly cotinuous in $t \in (0, T]$. □

¿From Ass. 4 and continuity of $\nu_\xi, (\varphi, \overline{G^{(\xi)}}\varphi)_{\nu_\xi}$ and $U^{(2)}_{\nu_\xi}(\cdot, \cdot, \xi)$, we have

Proposition 2.6. *There exists a $c_0 > 0$ such that for any $\xi \in K$ and any $\varphi \in C(E)$,*

$$D^2U(\nu_\xi)[(\overline{G^{(\xi)}}\varphi)\nu_\xi, (\overline{G^{(\xi)}}\varphi)\nu_\xi] \leq (1-c_0)(\varphi, \overline{G^{(\xi)}}\varphi)_{\nu_\xi}.$$

¿From Lem. 2.1 (1), we have

Proposition 2.7. *For any $a > 0$ there exists a constant $C < \infty$ such that for any $T \geq 1$, any $\varphi \in C(E)$ with $\|\varphi\|_\infty \leq a$, any $\xi \in K$ and any $\varepsilon \in (0,1]$,*

$$Q_x^{(\xi)}[\exp(\varepsilon T \int_E \varphi d(L_T - \nu_\xi))] \leq \exp(\varepsilon^2 \frac{T}{2}(\varphi, \overline{G^{(\xi)}}\varphi)_{\nu_\xi}) \times \exp(C(\varepsilon\|\varphi\|_\infty + \varepsilon^2\|\varphi\|_\infty^2 + \varepsilon^3\|\varphi\|_\infty^3 + T\varepsilon^3\|\varphi\|_\infty^3))$$

Set

$$\ell_T^{(\xi)} = \sqrt{T}(L_T - \nu_\xi). \tag{2.7}$$

Then from Lem.2.1 (2) we have

Proposition 2.8. *$(\ell_T^{(\xi)}, \omega(T))$ under $Q_x^{(\xi)}$ converges weakly to $\gamma^{(\xi)} \otimes \nu_\xi$ as $T \to \infty$ uniformly in $\xi \in K$, where $\gamma^{(\xi)}$ is a unique centered Gaussian measure on $\mathcal{M}_a$ satisfying*

$$\int_{\mathcal{M}_a} (x,z)_a (y,z)_a \gamma^{(\xi)}(dz) = (x, S^{(\xi)}y)_a.$$

The following proposition is the uniform estimate in $\xi \in K$ corresponding to Prop.3.2 in [3].

Proposition 2.9. *Let $\mathbb{K}$ be a set of all compact sets in $\mathcal{M}_a$. Suppose that the mapping $K \ni \xi \mapsto A_\xi \in \mathbb{K}$ is continuous and that $\inf_{\xi\in K}\Gamma^{(\xi)}(A_\xi) > 0$. Then,*

$$\varliminf_{c\to\infty} \inf_{\xi\in K}\{-\frac{1}{2}\Gamma^{(\xi)}(A_\xi) - \sup_{t,T}(\frac{1}{t^2}\log Q_x^{(\xi)}(\ell_T^{(\xi)} \in tA_\xi); c \leq t \leq \frac{\sqrt{T}}{c})\} \geq 0.$$

where $\Gamma^{(\xi)}(A_\xi) = \inf_{x\in A_\xi} \Gamma^{(\xi)}(x)$.

Proof. ¿From the assumptions $\cup_{\xi\in K}A_\xi$ is a compact set in $\mathcal{M}_a$. Then there exist open balls $V_i = \{y \in \mathcal{M}_a; \|y - p_i\|_a < r_i\}, i = 1,\ldots,m$, such that $\cup_{\xi\in K}A_\xi \subset \cup_{i=1}^m V_i$. If $0 < \varepsilon < \inf_{\xi\in K}\Gamma^{(\xi)}(A_\xi)$, then $V^{(\xi)} = \{x \in \mathcal{M}_a; \Gamma^{(\xi)}(x) > \Gamma^{(\xi)}(A_\xi) -\frac{\varepsilon}{2}\}$ is an open set since $\Gamma^{(\xi)}$ is lower semicontinuous. Set $V_i^{(\xi)} = V_i \cap V^{(\xi)}$. As $\Gamma^{(\xi)}$ is convex, there exist open convex sets $U_i^{(\xi)}, i = 1,\ldots,m$, with $\cup_{i=1}^m U_i^{(\xi)} \supset$

$\cup_{\xi\in K} A_\xi$ and $\overline{U_i^{(\xi)}} \subset V_i^{(\xi)}$. Let $x_i^{(\xi)} \in \partial U_i^{(\xi)}$ satisfy $\Gamma^{(\xi)}(x_i^{(\xi)}) = \Gamma^{(\xi)}(U_i^{(\xi)})$. Then, there exist $y_i^{(\xi)} \in \mathcal{M}_a, i = 1, \ldots, m$, such that

$$\overline{U_i^{(\xi)}} \subset \{x \in \mathcal{M}_a; (x, y_i^{(\xi)})_a \geq 1\}, \tag{2.8}$$

$$(y_i^{(\xi)}, S^{(\xi)} y_i^{(\xi)})_a = \frac{1}{\Gamma^{(\xi)}(x_i^{(\xi)})} = \frac{1}{\Gamma^{(\xi)}(U_i^{(\xi)})}, \tag{2.9}$$

$$\sup_{\xi\in K} \|y_i^{(\xi)}\|_a < \infty. \tag{2.10}$$

Actually we can choose $y_i^{(\xi)}$ such that $S^{(\xi)} y_i^{(\xi)} = \Gamma^{(\xi)}(x_i^{(\xi)})^{-1} x_i^{(\xi)}$. Since $C_0(E)$ is dense in $\mathcal{M}_a$, for any $\varepsilon' > 0$ there exist $\varphi_i^{(\xi)} \in C_0(E), i = 1, \ldots, m$ such that

$$\overline{U_i^{(\xi)}} \subset \{x \in \mathcal{M}_a; (x, \widehat{\varphi_i^{(\xi)}})_a \geq 1 - \varepsilon'\}, \tag{2.11}$$

$$\|y_i^{(\xi)} - \widehat{\varphi_i^{(\xi)}}\|_a < \varepsilon', \tag{2.12}$$

$$\sup_{\xi\in K} \|\varphi_i^{(\xi)}\|_\infty < \infty. \tag{2.13}$$

Set $v_i^{(\xi)} = (\widehat{\varphi_i^{(\xi)}}, S^{(\xi)} \widehat{\varphi_i^{(\xi)}})_a$. Then,

$$\begin{aligned}
&\frac{1}{t^2} \log Q_x^{(\xi)}(\ell_T^{(\xi)} \in tU_i^{(\xi)}) \\
&\quad \leq \frac{1}{t^2} \log Q_x^{(\xi)} \left(\int_E \varphi_i^{(\xi)} d\ell_T^{(\xi)} \geq t(1-\varepsilon') \right) \quad \text{(by (2.11))}
\end{aligned}$$

by using the usual exponential inequality

$$\begin{aligned}
&\leq -\frac{(1-\varepsilon')^2}{v_i^{(\xi)}} + \frac{1}{t^2} \log Q_x^{(\xi)} \left[\exp\left(\frac{(1-\varepsilon')t}{v_i^{(\xi)}} \int_E \varphi_i^{(\xi)} d\ell_T^{(\xi)} \right) \right] \\
&\leq -\frac{(1-\varepsilon')^2}{v_i^{(\xi)}} + C\Big(\frac{1-\varepsilon'}{v_i^{(\xi)}\sqrt{T}t} \|\varphi_i^{(\xi)}\|_\infty + \frac{(1-\varepsilon')^2}{{v_i^{(\xi)}}^2 T} \|\varphi_i^{(\xi)}\|_\infty^2 \\
&+ \frac{(1-\varepsilon')^3 t(1+T)}{{v_i^{(\xi)}}^3 T^{3/2}} \|\varphi_i^{(\xi)}\|_\infty^3 \Big) \quad \text{(by (2.13) and Prop.2.7)} \\
&= -\frac{(1-\varepsilon')^2}{2v_i^{(\xi)}} + o(1),
\end{aligned}$$

where $o(1)$ is a general term that tends to 0 as $c \to \infty$ uniformly in $\xi \in K$ for any t, T with $c \leq t \leq \sqrt{T}/c$.

Choosing $\varepsilon' > 0$ sufficiently small, by (2.9) we have

$$\frac{1}{t^2}\log Q_x^{(\xi)}(\ell_T^{(\xi)} \in tU_i^{(\xi)}) \le -\frac{1}{2}(\Gamma^{(\xi)}(U_i^{(\xi)}) - \frac{\varepsilon}{2}) + o(1),$$

and therefore, for sufficiently large $c > 0$ we have

$$\begin{aligned}\sup\left\{\frac{1}{t^2}\log Q_x^{(\xi)}(\ell_T^{(\xi)} \in tA_\xi),\ c \le t \le \frac{\sqrt{T}}{c}\right\}& \\ &\le \sup\left\{\frac{1}{t^2}\log\sum_{i=1}^m Q_x^{(\xi)}(\ell_T^{(\xi)} \in tU_i^{(\xi)}),\ c \le t \le \frac{\sqrt{T}}{c}\right\} \\ &\le -\frac{1}{2}\min_{1\le i\le m}(\Gamma^{(\xi)}(U_i^{(\xi)}) - \frac{\varepsilon}{2}) + o(1) \\ &\le -\frac{1}{2}(\Gamma^{(\xi)}(A_\xi) - \varepsilon) + o(1),\end{aligned}$$

which completes the proof. □

Let $b = \{b_k\}_{k=1}^\infty$ be a sequence of strictly positive real numbers satisfying

(1) $\lim_{k\to\infty} b_k = 0, \quad \lim_{k\to\infty} b_k/a_k = \infty$,
(2) $\sum_{k=1}^\infty b_k\|\psi_k\|_\infty^2 = 1$.

We define the Hilbert space $\mathcal{M}_b$ in the same way as $\mathcal{M}_a$. Then the imbedding $\mathcal{M}_b \subset \mathcal{M}_a$ is compact. We can prove the following uniform estimate in $\xi \in K$ in a way similar to Lem.3.4 in [3].

Lemma 2.4.

$$\rho(b) \equiv -\overline{\lim_{c\to\infty}}\sup_{\xi\in K}\sup_{t,T}\{\frac{1}{t^2}\log\sup_{x\in E} Q_x^{(\xi)}(\|\ell_T^{(\xi)}\|_b > t); \quad c \le t \le \frac{\sqrt{T}}{c}\} > 0.$$

Theorem 2.1. *Let us assume Ass. 1, Ass. 2′, Ass. 3, and Ass. 4. Then, for any $T_0 > 0$ and any $\mathcal{F}_0^{T_0}$-measurable function $\Phi : \Omega \to \mathbf{R}$,*

$$e^{Tf_0(\xi)} P_x[\Phi\exp(TU(L_T, \xi))] \to g(\xi)Q_x^{(\xi)}[\Phi]$$

as $T \to \infty$ uniformly in $\xi \in K$, where

$$g(\xi) = u^{(\xi)}(x)(\int_E \ell^{(\xi)}(y)\mu(dy))\{\det(I - D^2U(\nu_\xi, \xi)\circ S^{(\xi)})\}^{-1/2}.$$

Proof. We will prove the case of $\Phi \equiv 1$ for simplicity. Set

$$\widetilde{U}(\lambda, \xi) = U(\lambda, \xi) - U(\nu_\xi, \xi) - DU(\nu_\xi, \xi)[\lambda - \nu_\xi]. \tag{2.14}$$

Then from the definition, we have

$$\begin{aligned}
&\mathrm{e}^{Tf_0(\xi))} P_x[\exp(TU(L_T, \xi)] \\
&= u^{(\xi)}(x) Q_x^{(\xi)}[\exp(T\widetilde{U}(L_T, \xi)) \frac{1}{u^{(\xi)}(\omega(T))}] \\
&= u^{(\xi)}(x)(I_1^{(\xi)}(c_1, T) + I_2^{(\xi)}(c_1, c_2, T) + I_3^{(\xi)}(c_2, T)),
\end{aligned}$$

where

$$\begin{aligned}
&c_1, c_2 > 0, \\
I_1^{(\xi)}(c_1, T) &= Q_x^{(\xi)}[\exp(T\widetilde{U}(L_T, \xi)) \frac{1}{u^{(\xi)}(\omega(T))},\ \|\ell_T^{(\xi)}\|_a \le c_1], \\
I_2^{(\xi)}(c_1, c_2, T) &= Q_x^{(\xi)}[\exp(T\widetilde{U}(L_T, \xi)) \frac{1}{u^{(\xi)}(\omega(T))},\ c_1 \le \|\ell_T^{(\xi)}\|_a \le c_2\sqrt{T}], \\
I_3^{(\xi)}(c_2, T) &= Q_x^{(\xi)}[\exp(T\widetilde{U}(L_T, \xi)) \frac{1}{u^{(\xi)}(\omega(T))},\ c_2\sqrt{T} \le \|\ell_T^{(\xi)}\|_a].
\end{aligned}$$

By the following lemmas 2.5, 2.6 and 2.7, we complete the proof of Theorem 2.1. □

¿From the large deviation principle, we have

Lemma 2.5. *For any $c_2 > 0$,*

$$\lim_{T\to\infty} \sup_{\xi\in K} I_3^{(\xi)}(c_2, T) = 0.$$

¿From Prop.2.8, we see that

Lemma 2.6. *For all but countablly many $c_1 > 0$, there exists*

$$\begin{aligned}
\widehat{I}_1^{(\xi)}(c_1) &= \lim_{T\to\infty} I_1^{(\xi)}(c_1, T) \qquad \textit{and} \\
\lim_{c_1\to\infty} \widehat{I}_1^{(\xi)}(c_1) &= \{\det(I - D^2U(\nu_\xi, \xi) \circ S^{(\xi)}\}^{-1/2} (\int_E \frac{1}{u^{(\xi)}(y)} \nu_\xi(dy))
\end{aligned}$$

uniformly in $\xi \in K$

Lemma 2.7. *If $c_2 > 0$ is small enough, then*

$$\lim_{c_1\to\infty}\overline{\lim_{T\to\infty}}\sup_{\xi\in K} I_2^{(\xi)}(c_1,c_2,T)=0.$$

Proof. For $\varepsilon > 0$ set

$$C_\varepsilon^{(\xi)}=\{x\in\mathcal{M}_a;\frac{1}{2}(D^2U(\nu_\xi,\xi)x,x)_a+\frac{1}{2}\varepsilon\|x\|_a^2\geq 1\}.$$

By Prop.2.6, we see that if $\varepsilon > 0$ is sufficiently small, then $\Gamma^{(\xi)}(C_\varepsilon^{(\xi)}) > 2$ uniformly in $\xi\in K$. If $c_2>0$ is small enough and $\|\ell_T^{(\xi)}\|_a\leq c_2\sqrt{T}$, then

$$T\widetilde{U}(L_T,\xi)\leq\frac{1}{2}(D^2(\nu_\xi,\xi)\ell_T^{(\xi)},\ell_T^{(\xi)})_a+\frac{1}{2}\varepsilon\|\ell_T^{(\xi)}\|_a^2.$$

Therefore

$$\begin{aligned}
&I_2^{(\xi)}(c_1,c_2,T)\\
&\leq Q_x^{(\xi)}[\exp(\frac{1}{2}(D^2(\nu_\xi,\xi)\ell_T^{(\xi)},\ell_T^{(\xi)})_a+\frac{1}{2}\varepsilon\|\ell_T^{(\xi)}\|_a^2),c_1<\|\ell_T^{(\xi)}\|_a\leq c_2\sqrt{T}]\\
&=Q_x^{(\xi)}(c_1<\|\ell_T^{(\xi)}\|_a\leq c_2\sqrt{T})\\
&+\int_0^\infty \mathrm{e}^tQ_x^{(\xi)}(\frac{1}{2}(D^2(\nu_\xi,\xi)\ell_T^{(\xi)},\ell_T^{(\xi)})_a+\frac{1}{2}\varepsilon\|\ell_T^{(\xi)}\|_a^2)\geq t,c_1<\|\ell_T^{(\xi)}\|_a\leq c_2\sqrt{T})dt\\
&=Q_x^{(\xi)}(c_1<\|\ell_T^{(\xi)}\|_a\leq c_2\sqrt{T})\\
&+\int_0^\infty \mathrm{e}^tQ_x^{(\xi)}(\ell_T^{(\xi)}\in\sqrt{t}C_\varepsilon^{(\xi)},c_1<\|\ell_T^{(\xi)}\|_a\leq c_2\sqrt{T})dt.
\end{aligned}$$

Here we take $b=\{b_k\}_{k=1}^\infty$ as before and set

$$D_t=\{x\in\mathcal{M}_a;\|x\|_b>t\}.$$

Then D_t^c is a compact set in $\mathcal{M}_a$.

$$\begin{aligned}
&\frac{1}{t}\log Q_x^{(\xi)}(\ell_T^{(\xi)}\in\sqrt{t}C_\varepsilon^{(\xi)},c_1<\|\ell_T^{(\xi)}\|_a\leq c_2\sqrt{T})\\
&\qquad=\frac{1}{t}\log 2+\frac{1}{t}\log Q_x^{(\xi)}(\|\ell_T^{(\xi)}\|_b>r\sqrt{t},c_1<\|\ell_T^{(\xi)}\|_a\leq c_2\sqrt{T})\\
&\qquad\vee\frac{1}{t}\log Q_x^{(\xi)}(\ell_T^{(\xi)}\in\sqrt{t}(C_\varepsilon^{(\xi)}\cap D_r^c),c_1<\|\ell_T^{(\xi)}\|_a\leq c_2\sqrt{T}).
\end{aligned}$$

By Lem.2.4, for any $\varepsilon'>0$ there exists a $c_0'>0$ such that

$$\begin{aligned}
&\sup_{\xi\in K}\sup_{t,T}\{\frac{1}{t}\log\sup_{x\in E}Q_x^{(\xi)}(\|\ell_T^{(\xi)}\|_b>r\sqrt{t}),c\leq\sqrt{t}\leq\frac{\sqrt{T}}{c}\}\\
&\qquad\leq -r^2\rho(b)+\varepsilon'
\end{aligned}$$

for any $c \geq c_0$ and any $r > 0$.

By Prop.2.9, we see that for any $0 < \varepsilon' < \frac{1}{2}\Gamma^{(\xi)}(C_\varepsilon^{(\xi)}) - 1$, there exists a $c_0'' > 0$ such that for any $c \geq c_0''$ and any $\xi \in K$

$$\begin{aligned}\sup_{t,T}\{\frac{1}{t}\log Q_x^{(\xi)}(\ell_T^{(\xi)} \in \sqrt{t}(C_\varepsilon^{(\xi)} \cap D_r^c), c \leq \sqrt{t} \leq \frac{\sqrt{T}}{c}\} \\ < -\frac{1}{2}\Gamma^{(\xi)}(C_\varepsilon^{(\xi)} \cap D_r^C) + \varepsilon' \\ \leq -\frac{1}{2}\Gamma^{(\xi)}(C_\varepsilon^{(\xi)}) + \varepsilon' < -1.\end{aligned}$$

Therefore there exist an $\varepsilon_1 > 0$ and a $c_0 > 0$ such that for any $c \geq c_0$ and any $\xi \in K$

$$\sup_{t,T}\{Q_x^{(\xi)}(\ell_T^{(\xi)} \in \sqrt{t}C_\varepsilon^{(\xi)}), c \leq \sqrt{t} \leq \frac{\sqrt{T}}{c}\} \leq 2\exp(-(1+\varepsilon_1)t).$$

Let $k = \sup_{\xi\in K}\sup\{8D^2(\nu_\xi,\xi)x,x)_a;\ \|x\|_a = 1\}$. Then $\inf\{\|x\|_a;\ x \in C_\varepsilon^{(\xi)}\} \geq \sqrt{\frac{2}{k+\varepsilon}}$. Therefore for $c > 0$, if $c_2 < \frac{1}{c}\sqrt{\frac{2}{k+\varepsilon}}$, then $\sqrt{t}C_\varepsilon^{(\xi)} \cap \{x;\ \|x\|_a \leq c_2\sqrt{T}\} = \emptyset$ for $\sqrt{t} \geq \sqrt{T}/c$. Then for any $d > c$,

$$\begin{aligned}I_2^{(\xi)}(c_1,c_2,T) \leq Q_x^{(\xi)}(c_1 < \|\ell_T^{(\xi)}\|_a) + \int_0^d \mathrm{e}^t Q_x^{(\xi)}(c_1 < \|\ell_T^{(\xi)}\|_a)dt \\ + \int_d^{\sqrt{T}/c} \mathrm{e}^t Q_x^{(\xi)}(\ell_T^{(\xi)} \in \sqrt{t}C_\varepsilon^{(\xi)})dt \\ \leq \mathrm{e}^d Q_x^{(\xi)}(c_1 < \|\ell_T^{(\xi)}\|_a) + \frac{2}{\varepsilon_1}\mathrm{e}^{-\varepsilon_1 d}.\end{aligned}$$

By Prop.2.8, we complete the proof. □

3. Manifold reflecting singularities.

In [3], we proved the following proposition (Lemma 2.17. in [3]).

Proposition 3.1. *Let V,$\varphi \in C(E)$ and $\int_E \varphi d\nu^V = 0$. Then, for $\varepsilon \in \mathbf{R}$, $|\varepsilon|$ is sufficiently small*

(i) $\nu^{V+\varepsilon\varphi} = \nu^V(1 + \varepsilon\overline{G^V}\varphi + \varepsilon^2\nu_2^V + r_\nu(V,\varepsilon))$,

(ii) $J(\nu^{V+\varepsilon\varphi}) = J(\nu) + \varepsilon(\overline{V},\overline{G^V}\varphi)_{\nu^V} + \varepsilon^2\frac{1}{2}(\varphi,\overline{G^V}\varphi)_{\nu^V} + \varepsilon^2(V,\nu_2^V)_{\nu^V} + r_J(V,\varepsilon)$,

where

$$\overline{V} = V - \alpha^V,$$
$$\nu_2^V = G^V \varphi G^V \varphi + (G^V)'\varphi(G^V)'\varphi + G^V\varphi \cdot (G^V)'\varphi - (G^V\varphi \cdot (G^V)'\varphi, 1)_{\nu^V},$$
$$|r_\nu(V,\varepsilon)| \le M(\|V\|_\infty)|\varepsilon|^3, \qquad |r_J(V,\varepsilon)| \le M(\|V\|_\infty)|\varepsilon|^3$$

and $M(x)$ is increasing in x.

Here we rewrite Prop. 2.2 as follows.

Proposition 3.2. *Let $\nu \in \mathcal{V}$. Then, $\nu \in \mathcal{M}_{1,0}^+$ and V in $\nu = \nu^V$ is given by $V = U_\nu^{(1)} + const.$*
where $\mathcal{V}$ and $\mathcal{M}_{1,0}^+$ were given in (1.27) and (1.24), respectively.

Let us define C^2 mapping $\mathcal{F} : C(E) \longrightarrow \mathbf{R} \cup \{\infty\}$ and C^1mapping $\mathcal{G} : C(E) \longrightarrow C(E)$ by

$$\mathcal{F}(V) = J(\nu^V) - U(\nu^V), \tag{3.1}$$

$$\mathcal{G}(V) = U_{\nu^V}^{(1)} + J(\nu^V) - \int_E U_{\nu^V}^{(1)}(x)\nu^V(dx), \tag{3.2}$$

respectively. We define a set $\mathbb{V}$ in $C(E)$ by

$$\mathbb{V} = \{V \in C(E); \nu^V \in \mathcal{V} \text{ and } \alpha^V = 0\}. \tag{3.3}$$

Furthermore we set

$$\widehat{\mathbb{V}} = \{(V,\varphi) \in C(E) \times C(E); V \in \mathbb{V}, (\varphi,1)_{\nu^V} = 0 \text{ and } \frac{d^2}{d\varepsilon^2}\mathcal{F}(V+\varepsilon\varphi)|_{\varepsilon=0} = 0\}. \tag{3.4}$$

Then, we have

Proposition 3.3.

1. *If $V \in \mathbb{V}$, then $V = \mathcal{G}(V)$.*
2. *Let $V \in \mathbb{V}$. Then, $(V,\varphi) \in \widehat{\mathbb{V}}$ if and only if $\varphi = D\mathcal{G}(V)[\varphi], \quad (\varphi,1)_{\nu^V} = 0$.*

Here the derivative $D\mathcal{G}$ of $\mathcal{G}$ is given by

$$D\mathcal{G}(V)[\varphi](x) = \int_E U_{\nu^V}^{(2)}(x,y)\overline{G^V}\varphi(y)\nu^V(dy) - \int_E \nu^V(dx)\int_E U_{\nu^V}^{(2)}(x,y)\overline{G^V}\varphi(y)\nu^V(dy), \tag{3.5}$$

where $U^{(2)}_{\nu^V}(x,y) = D^2U(\nu^V)[\delta_x,\delta_y]$.

Proof. For $\varphi \in C(E)$ with $\int_E \varphi d\nu^V = 0$,

$$\frac{d}{d\varepsilon}\mathcal{F}(V+\varepsilon\varphi)|_{\varepsilon=0} = (\overline{V} - U^{(1)}_{\nu^V}, \overline{G^V}\varphi)_{\nu^V} = 0.$$

Then, we have (1). By Prop.3.1,

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}\mathcal{F}(V+\varepsilon\varphi)|_{\varepsilon=0} &= (\varphi, \overline{G^V}\varphi)_{\nu^V} + 2(\overline{V}, \nu_2^V)_{\nu^V} \\ &- \iint_{E\times E} U^{(2)}_{\nu^V}(x,y)\overline{G^V}\varphi(x)\overline{G^V}\varphi(y)\nu^V(dx)\nu^V(dy) \\ &- 2(U^{(1)}_{\nu^V}, \nu_2^V)_{\nu^V}. \end{aligned} \tag{3.6}$$

By using (1) and the fact that $(1, \nu_2^V)_{\nu^V} = 0$, we see that (3.6) is equal to

$$\int_E \nu^V(dx)\overline{G^V}\varphi(x)\{\varphi(x) - \int_E U^{(2)}_{\nu^V}(x,y)\overline{G^V}\varphi(y)\nu^V(dy)\}$$

Thus $\varphi = D\mathcal{G}(V)[\varphi]$ implies $\frac{d^2}{d\varepsilon^2}\mathcal{F}(V+\varepsilon\varphi)|_{\varepsilon=0} = 0$

Since $\mathcal{F}$ attains the minimum at V, for any $\varphi \in C(E)$ with $\int_E \varphi d\nu^V = 0$,

$$(\varphi, \overline{G^V}\varphi)_{\nu^V} - (U^{(2)}_{\nu^V}, \overline{G^V}\varphi \times \overline{G^V}\varphi)_{\nu^V\otimes\nu^V} \geq 0.$$

Therefore, for any $\varphi, \psi \in C(E)$ with $\int_E \varphi d\nu^V = \int_E \psi d\nu^V = 0$, and $t \in \mathbf{R}$,

$$\begin{aligned} 0 &\leq (\varphi + t\psi, \overline{G^V}(\varphi+t\psi)) - (U^{(2)}_{\nu^V}, \overline{G^V}(\varphi+t\psi) \times \overline{G^V}(\varphi+t\psi))_{\nu^V\otimes\nu^V} \\ &= \{(\varphi, \overline{G^V}\varphi)_{\nu^V} - (U^{(2)}_{\nu^V}, \overline{G^V}\varphi \times \overline{G^V}\varphi)_{\nu^V\otimes\nu^V}\} \\ &+ 2t\{(\varphi, \overline{G^V}\psi)_{\nu^V} - (U^{(2)}_{\nu^V}, \overline{G^V}\varphi \times \overline{G^V}\psi)_{\nu^V\otimes\nu^V}\} \\ &+ t^2\{(\psi, \overline{G^V}\psi)_{\nu^V} - (U^{(2)}_{\nu^V}, \overline{G^V}\psi \times \overline{G^V}\psi)_{\nu^V\otimes\nu^V}\}. \end{aligned}$$

Then we have

$$\begin{aligned} &\{(\varphi, \overline{G^V}\psi)_{\nu^V} - (U^{(2)}_{\nu^V}, \overline{G^V}\varphi \times \overline{G^V}\psi)_{\nu^V\otimes\nu^V}\}^2 \\ &- \{(\varphi, \overline{G^V}\varphi)_{\nu^V} - (U^{(2)}_{\nu^V}, \overline{G^V}\varphi \times \overline{G^V}\varphi)_{\nu^V\otimes\nu^V}\} \\ &\quad \times \{(\psi, \overline{G^V}\psi)_{\nu^V} - (U^{(2)}_{\nu^V}, \overline{G^V}\psi \times \overline{G^V}\psi)_{\nu^V\otimes\nu^V}\} \leq 0 \end{aligned}$$

Therefore, if $\frac{d^2}{d\varepsilon^2}\mathcal{F}(V+\varepsilon\varphi)|_{\varepsilon=0} = 0$, then $(\varphi, \overline{G^V}\varphi)_{\nu^V} - (U^{(2)}_{\nu^V}, \overline{G^V}\varphi \times \overline{G^V}\psi)_{\nu^V\otimes\nu^V}$ $= 0$ for any $\psi \in C(E), \int_E \psi d\nu^V = 0$. Hence $\varphi(x) = \int_E U^{(2)}_{\nu^V}(x,y)\overline{G^V}\varphi(y)\nu^V(dy)$ +const. □

Note that $D\mathcal{G} : C(E) \longrightarrow C(E)$ is a compact operator by Lem. 1.3.

Proposition 3.4. *$\mathbb{V} \subset C(E)$ is a non-void compact set.*

Proof. Let $\{V_n\}_{n=1}^{\infty} \subset \mathbb{V}$. We denote ν^{V_n} by ν_n. Then, $\{V_n\}_{n=1}^{\infty} \subset \mathcal{V}$. Since $\mathcal{V}$ is compact with respect to the weak topology in $\mathcal{M}_1^+$, there exist $\nu \in \mathcal{V}$ and a subsequence $\{V_{n'}\}_{n=1}^{\infty}$ of $\{V_n\}_{n=1}^{\infty}$ such that $\nu_{n'} \to \nu$ weakly as $n \to \infty$. This implies that $U^{(1)}_{\nu_{n'}} \to U^{(1)}_{\nu}$ in $C(E)$ by Lem. 1.2 (2). Then, by noting that $J(\nu_{n'}) = f_0 + U(\nu_{n'})$ and $J(\nu) = f_0 + U(\nu)$, we see that the weak convergence of $\nu_{n'}$ to ν implies that $J(\nu_{n'}) \to J(\nu)$ as $n' \to \infty$. From Prop. 3.3, we see that

$$V_n = U^{(1)}_{\nu_n} + J(\nu_n) - \int_E U^{(1)}_{\nu_n} d\nu_n, \quad V = U^{(1)}_{\nu} + J(\nu) - \int_E U^{(1)}_{\nu} d\nu.$$

and $V \in \mathbb{V}$. Then, $V_{n'} \to V$ in $C(E)$. □

Let X be a Banach space.
We say that X has the approximation property *if there exists a set $\{\Pi_n\}$ of operators of finite rank such that* $\lim_n \Pi_n = I$*, I is the identity map on X, and this convergence is uniform on any totally bounded set in X.*
Then, the following proposition is known.

Proposition 3.5. *If E is a compact metric space, then $(C(E), \|\cdot\|_\infty)$ has the approximation property.*

Lemma 3.1. *There exists a sequence $\{\Pi_n\}_{n=1}^{\infty}$, such that $\Pi_n : C(E) \to C(E)$ is an operator of finite rank for any n, Π_n converges to I uniformly on any totally bounded set in $C(E)$ and $\Pi_n|_{\mathbb{V}} : \mathbb{V} \to \Pi_n(\mathbb{V})$ is injective.*

Proof. Let us assume that $\Pi_n|_{\mathbb{V}}$ is not injective for any n. Then, there exist $V_n, W_n \in \mathbb{V}$ such that $\Pi_n V_n = \Pi_n W_n$ and $V_n \neq W_n$. Since $\mathbb{V}$ is a compact set, we can assume that $V_n \to V$ and $W_n \to W$, $V, W \in \mathbb{V}$ as $n \to \infty$, by taking a subsequence if necessary. Since $\mathbb{V}$ is campact, Π_n converges to I uniformly on $\mathbb{V}$. Then, we have $\|W_n - V_n\|_\infty \to 0$ as $n \to \infty$, and hence $W = V$.

On the other hand, by Prop. 3.3,

$$\begin{aligned}W_n - V_n &= \mathcal{G}(W_n) - \mathcal{G}(V_n)\\ &= D\mathcal{G}(V)[W_n - V_n] + \int_0^1 dt\{D\mathcal{G}(V_n + t(W_n - V_n))\\ &- D\mathcal{G}(V)\}[W_n - V_n]\end{aligned}$$

Let $h_n = (W_n - V_n)/\|W_n - V_n\|_\infty$. Then,

$$\begin{aligned}h_n &= D\mathcal{G}(V)[h_n] + r_n,\\ r_n &= \int_0^1 dt\{D\mathcal{G}(V_n + t(W_n - V_n)) - D\mathcal{G}(V)\}[h_n].\end{aligned}$$

¿From Lem. 1.2 and Prop. 3.3 (2), we see that $\|r_n\|_\infty \to 0$ as $n \to \infty$. Since $D\mathcal{G}(V) : C(E) \to C(E)$ is a compact operator and $\|h_n\|_\infty = 1$, there exists an $h \in C(E)$ such that $D\mathcal{G}(V)[h_n] \to h, n \to \infty$ by taking a subsequence if necessary. Then we have $\|h\|_\infty = 1$. On the other hand,

$$\begin{aligned}\|h\|_\infty &\le \|h - h_n\|_\infty + \|h_n - \Pi_n h_n\|_\infty \\ &\le \|h - h_n\|_\infty + \|D\mathcal{G}(V)[h_n] - \Pi_n D\mathcal{G}(V)[h_n]\|_\infty \\ &+ \|r_n - \Pi_n r_n\|_\infty \\ &\longrightarrow 0 \qquad \text{as } n \to \infty\end{aligned}$$

Then, $h = 0$. This is a contradiction. □

For n, we define $\Phi_n : C(E) \to C(E)$ by

$$\Phi_n(V) = V - (I - \Pi_n)\mathcal{G}(V). \tag{3.7}$$

Lemma 3.2. *There exists an $n \in \mathbf{N}$ such that $D\Phi_n(V)$ is invertible for any $V \in \mathbb{V}$.*

Proof. Let $V \in \mathbb{V}$ and let us assume that $D\Phi_n(V)$ is not invertible for any $n \in \mathbf{N}$. Then, for any n there exist a $V_n \in \mathbb{V}$ and a $\varphi_n \in C(E)$ with $\|\varphi\|_\infty = 1$ such that

$$\varphi_n = (I - \Pi_n)D\mathcal{G}(V_n)[\varphi_n]. \tag{3.8}$$

Since $\mathbb{V}$ is compact, we can assume that there exists a $V \in \mathbb{V}$ such that $V_n \to V$ by taking a subsequence if necessary. We rewrite (3.8) to

$$\varphi_n = (I - \Pi_n)D\mathcal{G}(V)[\varphi_n] - (I - \Pi_n)(D\mathcal{G}(V) - D\mathcal{G}(V_n))[\varphi_n]. \tag{3.9}$$

The second term of the right hand side of (3.9) converges to 0 as $n \to \infty$ by (3.5) and Lem. 1.2. Since $D\mathcal{G}(V)$ is a compact operator, $\{D\mathcal{G}(V)[\varphi]; \|\varphi\|_\infty = 1\}$ is a totally bounded set in $C(E)$. Then, the first term of the right hand side of (3.9) converges to 0 as $n \to \infty$. This contradicts that $\|\varphi\|_\infty = 1$. □

Since $(I - \Pi_n)D\mathcal{G}(V)$ is a compact operator, $D\Phi_n(V)^{-1}$ is a bounded operator.

Corollary. *There exists an open neighborhood $\mathbb{U}_n^{(1)}$ of $\mathbb{V}$ such that*

$$\Phi_n : \mathbb{U}_n^{(1)} \to \Phi_n(\mathbb{U}_n^{(1)}) \qquad \textit{is local diffeomorphism.}$$

Proposition 3.6. *There exists an $n \in \mathbf{N}$ such that*

(i) $\Pi_n|_{\mathbb{V}} : \mathbb{V} \to \Pi_n(\mathbb{V})$ *is injective,*
(ii) *for any* $V \in \mathbb{V}$, $D\Phi_n(V)$ *is invertible and*
(iii) *there exists an open neighborhood* $\mathbb{U}_n^{(2)}$ *of* $\mathbb{V}$ *such that* $\Phi_n|_{\mathbb{U}_n^{(2)}} : \mathbb{U}_n^{(2)} \to C(E)$ *is injective.*

Proof. ¿From the proofs of Lem.3.1 and Lem.3.2, we see that there exist infinitely many $n \in \mathbf{N}$ for which (i) and (ii) hold. We assume that (iii) does not hold for any $n \in \mathbf{N}$ for which (i) and (ii) hold. Then, there exist φ_n and $\psi_n \in C(E)$ such that $\varphi_n \neq \psi_n, \mathrm{dist}_\infty(\varphi_n, \mathbb{V}) + \mathrm{dist}_\infty(\psi_n, \mathbb{V})$ converges to 0 as $n \to \infty$ and $\Phi_n(\varphi_n) = \Phi_n(\psi_n)$. Since $\mathbb{V}$ is compact, we can assume that there exist φ and $\psi \in C(E)$ such that $\varphi_n \to \varphi$ and $\psi_n \to \psi$ as $n \to \infty$ by taking a subsequence if necessary.

Fix $n_1 \in \mathbf{N}$ such that $\Pi_{n_1}|_{\mathbb{V}} : \mathbb{V} \to \Pi_n(\mathbb{V})$ is injective. Then, for any $n \geq n_1$

$$\Pi_{n_1}\Phi_n(V) = \Pi_{n_1}V \qquad \text{for any } V \in C(E).$$

If $\Phi_n(\varphi_n) = \Phi_n(\psi_n)$ and $n \geq n_1$, then

$$\Pi_{n_1}\varphi_n = \Pi_{n_1}\psi_n.$$

Then, $\Pi_{n_1}\varphi = \Pi_{n_1}\psi$, and this implies $\varphi = \psi$.

On the other hand, if $\Phi_n(\varphi_n) = \Phi_n(\psi_n)$, then by taking $h_n = (\varphi_n - \psi_n)/\|\varphi_n - \psi_n\|_\infty$ we have

(3.10)
$$h_n = (I - \Pi_n)D\mathcal{G}(\varphi)[h_n] + \int_0^1 dt(I - \Pi_n)\{D\mathcal{G}(\psi_n + t(\varphi_n - \psi_n)) - D\mathcal{G}(\varphi)\}[h_n].$$

The right hand side of (3.10) goes to 0 as $n \to \infty$, and this contradicts $\|h_n\|_\infty = 1$ for any n. □

Set $\mathbb{U}_n = \mathbb{U}_n^{(1)} \cap \mathbb{U}_n^{(2)}$. Then we have

Corollary. $\Phi_n|_{\mathbb{U}_n} : \mathbb{U}_n \to \Phi_n(\mathbb{U}_n)$ *is a diffeomorphism.*

Definition 3.1. $\mathbb{N} \subset C(E)$ *is called a manifold reflecting singularities if*

(i) $\mathbb{N}$ *is a finite dimensional submanifold in* $C(E)$,
(ii) $\mathbb{V} \subset \mathbb{N}$ *and*
(iii) *if* $(V, \varphi) \in \widehat{\mathbb{V}}$, *then* $\varphi \in T_V(\mathbb{N})$.

Theorem 3.1. *There exists a manifold reflecting singularities in $C(E)$.*

Proof. Let n be the number which appears in Prop. 3.6. Set

$$\mathbb{W} = \Pi_n(C(E)) \cap \Phi_n(\mathbb{U}_n). \tag{3.11}$$

Then, $\mathbb{W}$ is a finite dimensional subspace in $C(E)$. Let us define $\Psi : \mathbb{W} \to C(E)$ by

$$\Psi(V) = (\Phi_n|_{\mathbb{U}_n})^{-1}(V). \tag{3.12}$$

Then, $\mathbb{V} \subset \Psi(\mathbb{W})$. Actually, for any $V \in \mathbb{V}$, $\Phi_n(V) \in \Phi_n(\mathbb{U}_n)$ and $\Phi_n(V) = V - (I - \Pi_n)\mathcal{G}(V) = \Pi_n(V)$ by Prop. 3.3 (1).

Set $\mathbb{N} = \Psi(\mathbb{W})$. Let $(V, \varphi) \in \widehat{\mathbb{V}}$. Then,

$$\frac{d}{dt}\Phi_n(V + t\varphi)|_{t=0} = \varphi - (I - \Pi_n)D\mathcal{G}(V)[\varphi] = \Pi_n\varphi$$

by Prop. 3.3 (2). This implies that $\frac{\mathrm{d}}{\mathrm{d}t}\Psi(\Pi_n V + t\Pi_n\varphi)|_{t=0} = \varphi$. Then $\varphi \in \mathrm{T}_v(\mathbb{N})$ and $\mathbb{N}$ is a manifold refrecting singularities. □

4. Main theorem.

By Th. 3.1, we get a manifold reflecting singularities $\mathbb{N}_0$ in $C(E)$. We denote $\nu^{\cdot} : C(E) \to \mathcal{M}_a$ by $\widehat{\nu}(\cdot)$. Set

$$\mathcal{N}_0 = \widehat{\nu}(\mathbb{N}_0). \tag{4.1}$$

Then, $\mathcal{N}_0 \supset \mathcal{V}$ and by Prop. 3.1, $\mathcal{N}_0$ is a finite dimensional submanifold in $\mathcal{M}_a$. We can choose a relatively compact neighborhood $\mathcal{N}$ of $\mathcal{V}$ in $\mathcal{N}_0$ and an open neighborhood $\mathcal{U}$ of $\mathcal{N}$ in $\mathcal{M}_a$ such that the following conditions are satisfied.

(1) For any $m \in \mathcal{U}$ there exists a unique $\Psi(m) \in \mathcal{N}$ such that

$$\|m - \Psi(m)\|_a = \inf\{\|m - n\|_a; n \in \mathcal{N}_0\}.$$

(2) Coose a $\chi : \mathcal{M}_a \to \mathbf{R}$ such that

$0 \le \chi \le 1$,
$\chi(m) = 0$, if $m \in \mathcal{U}^c$,
$\chi(m) = 1$, if $m \in \mathcal{N}$.

Define $W_0 : \mathcal{M}_a \times \mathcal{N} \to \mathbf{R}$ by

$$W_0(m,\xi) = \frac{1}{2}\chi(m)\|\Psi(m) - \xi\|_a^2. \tag{4.2}$$

Then,

$$(\frac{T}{2\pi})^{d/2} \int_{\mathcal{N}_0} \exp(-TW_0(m,\xi))n_0(d\xi) \to 1 \quad \text{as } T \to \infty \tag{4.3}$$

uniformly in $m \in \mathcal{U}$, where n_0 is the Riemannian volume in $\mathcal{N}_0$. Set

$$W(m,\xi) = U(m) - W_0(m,\xi), \tag{4.4}$$

$$F(m,\xi) = J(m) - W(m,\xi), \tag{4.5}$$

$$f_0(\xi) = \inf\{F(\lambda,\xi); \lambda \in \mathcal{M}_1^+\}. \tag{4.6}$$

Then, there exists a unique $\nu_\xi \in \mathcal{M}_1^+$ such that $F(\nu_\xi,\xi) = f_0(\xi)$ and Ass. 4 holds. Note that if $\xi \in \mathcal{V}$, then $\nu_\xi = \xi$.

The following is our main theorem.

Theorem 4.1. *Suppose that Ass. 1 and Ass. 2 are satisfied. Then,*

(1) $\mathcal{V} = \{\xi \in \mathcal{N}; f_0(\xi) = f_0\}$.

(2) *For any $\Phi : \Omega \to \mathbf{R}$ which is bounded continuous and $\mathcal{F}_0^{T_0}$-measurable for some $T_0 > 0$,*

$$\begin{aligned} &e^{Tf_0} P_x[\Phi \exp(TU(L_T))] \\ &\quad \sim (\frac{T}{2\pi})^{d/2} \int_{\mathcal{N}} e^{-(f_0(\xi)-f_0)T} g(\xi) Q_x^{(\xi)}[\Phi] n_0(d\xi) \end{aligned}$$

as $T \to \infty$, where n_0 is the Riemannian volume of $\mathcal{N}_0$, d is a dimension of $\mathcal{N}_0$ and

$$g(\xi) = u^{(\xi)}(x)(\int_E \ell^{(\xi)} d\mu)\{\det(I - D^2W(\nu_\xi,\xi) \circ S^{(\xi)})\}^{-1/2}.$$

Proof. Note that $W : \mathcal{M}_a \times \mathcal{N} \to \mathbf{R}$ satisfies the assumptions Ass. 2' $\sim$ Ass. 4 in §2. By the large deviation principle,

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \log P_x[\Phi \exp(TU(L_T)), L_T \in \mathcal{M}_a \cap \mathcal{U}^c] < -f_0. \tag{4.7}$$

Then,

$$\begin{aligned} &e^{f_0 T} P_x[\Phi \exp(TU(L_T))] \\ &\quad \sim e^{f_0 T} P_x[\Phi \exp(TU(L_T)), L_T \in \mathcal{U}] \end{aligned}$$

using (4.3) we get

$$\sim (\frac{T}{2\pi})^{d/2} \int_{\mathcal{N}_0} e^{f_0 T} P_x[\Phi \exp(TW(L_T,\xi))] n_0(d\xi)$$
$$= (\frac{T}{2\pi})^{d/2} \int_{\mathcal{N}} e^{-(f_0(\xi)-f_0)T} e^{f_0(\xi)T} P_x[\Phi \exp(TW(L_T,\xi))] n_0(d\xi)$$

by using Th. 2.1, we get

$$\sim (\frac{T}{2\pi})^{d/2} \int_{\mathcal{N}} e^{-(f_0(\xi)-f_0)T} g(\xi) Q_x^{(\xi)}[\Phi] n_0(d\xi).$$

□

5. Appendix.

In this section we give the proof of Lem.2.1. Let $\widetilde{\varphi}(x) = \varphi(x) - \int_E \varphi d\nu$, $\widetilde{q}(t,x,y) = q(t,x,y) - 1$, and $\widetilde{Q}_t = Q_t - \Pi_\nu$, where $\Pi_\nu \varphi = (\varphi, 1)_\nu$. Set

$$a_n(\tau_1,\cdots,\tau_n) = (\widetilde{\varphi}, \widetilde{Q}_{\tau_1}\widetilde{\varphi}\widetilde{Q}_{\tau_2}\cdots\widetilde{Q}_{\tau_n}\widetilde{\varphi})_\nu,$$
$$b_n(x;\tau_1,\cdots,\tau_n) = (\widetilde{q}(\tau_1,x,\cdot), \widetilde{\varphi}\widetilde{Q}_{\tau_2}\widetilde{\varphi}\cdots\widetilde{Q}_{\tau_n}\widetilde{\varphi})_\nu,$$
$$c_n(y;\tau_1,\cdots,\tau_n) = (\widetilde{\varphi}, \widetilde{Q}_{\tau_1}\widetilde{\varphi}\cdots\widetilde{Q}_{\tau_{n-1}}\widetilde{q}(\tau_n,\cdot,y))_\nu,$$
$$d_n(x,y;\tau_1,\cdots,\tau_n) = (\widetilde{q}(\tau_1,x,\cdot), \widetilde{\varphi}\widetilde{Q}_{\tau_2}\widetilde{\varphi}\cdots\widetilde{Q}_{\tau_{n-1}}\widetilde{q}(\tau_n,\cdot,y))_\nu.$$

Then we have the following lemma (see Prop.1.3 in [7]).

Lemma. 5.1.

(i)

$$|\iint_{\substack{0<\tau_1,\tau_2\\ \tau_1+\tau_2<t}} a_1(\tau_1)d\tau_1 d\tau_2 - \frac{t}{2}(\varphi, \overline{G_\nu}\varphi)_\nu| \le (\frac{2}{\alpha_1})^2 \|\varphi\|_\infty^2 \tag{5.1}$$

(ii)

$$|a_n(\tau_1,\cdots,\tau_n)| \le 2^{n+1}\|\varphi\|_\infty^{n+1} \exp(-\alpha_1 \sum_{i=1}^n \tau_i), \tag{5.2}$$

$$|b_n(x;\tau_1,\cdots,\tau_n)| \le M_x e^{\alpha_1} 4^n \|\varphi\|_\infty^n \exp(-\alpha_1 \sum_{i=1}^n \tau_i), \tag{5.3}$$

$$|c_n(y;\tau_1,\cdots,\tau_n)| \le N_y e^{\alpha_1} 4^n \|\varphi\|_\infty^n \exp(-\alpha_1 \sum_{i=1}^n \tau_i), \tag{5.4}$$

$$|d_n(x,y;\tau_1,\cdots,\tau_n)| \le M_x N_y e^{-2\alpha_1} 4^{n-1} \|\varphi\|_\infty^{n-1} \times \exp(-\alpha_1 \sum_{i=1}^n \tau_i), \quad \textit{for} \sum_{i=1}^n \tau_i > 2. \tag{5.5}$$

where

$$M_x(s) = \sup\{q(t,x,\cdot); t \geq s\} \ (< +\infty), \quad M_x = M_x(1),$$
$$N_y(s) = \sup\{q(t,\cdot,y); t \geq s\} \ (< +\infty), \quad N_y = N_y(1).$$

Proof. (i) Note that $\frac{t}{2}(\varphi, \overline{G_\nu}\varphi)_\nu = \int_0^\infty d\tau_1 a_1(\tau_1) \int_0^t d\tau_2$. Then we have

$$|\iint_{\substack{0<\tau_1,\tau_2 \\ \tau_1+\tau_2<t}} a_1(\tau_1) d\tau_1 d\tau_2 - \frac{t}{2}(\varphi, \overline{G_\nu}\varphi)_\nu| \leq \int_0^t d\tau_2 \int_{t-\tau_2}^\infty |a_1(\tau_1)| d\tau_1$$
$$\leq \frac{1}{\alpha_1^2} 2^2 \|\varphi\|_\infty^2.$$

(ii) Since for $h \in C(E), \|\widetilde{Q}_t h\|_\nu \leq \mathrm{e}^{-\alpha_1 t}\|h\|_\infty$ and $\|\widetilde{\varphi}\|_\infty \leq 2\|\varphi\|_\infty$, we have (5.2).

Let $h \in C(E)$. Then

$$\begin{aligned}
&|(\widetilde{q}(\tau_1,x,\cdot), \widetilde{\varphi}\widetilde{Q}_{\tau_2}\widetilde{\varphi}\cdots\widetilde{Q}_{\tau_n}h)_\nu| \\
&= \lim_{T\to\infty} |\int\cdots\int_{E\times\cdots\times E} (q(\tau_1,x,z_1) - q(T,x,z_1))\widetilde{\varphi}(z_1)(q(\tau_1,z_1,z_2) \\
&- q(T,z_1,z_2))\widetilde{\varphi}(z_2)\cdots\widetilde{\varphi}(z_{n-1})(q(\tau_n,z_{n-1},z_n) - q(T,z_{n-1},z_n))h(z_n) \\
&\quad \nu(dz_1)\cdots\nu(dz_n) \\
&\leq \|\widetilde{\varphi}\|_\infty^{n-1} \lim_{T\to\infty} \int\cdots\int_{E\times\cdots\times E} (q(\tau_1,x,z_1) + q(T,x,z_1))(q(\tau_1,z_1,z_2) \\
&+ q(T,z_1,z_2))\cdots(q(\tau_n,z_{n-1},z_n) + q(T,z_{n-1},z_n))|h(z_n)|\nu(dz_1)\cdots\nu(dz_n) \\
&= 2^{n-1}\|\varphi\|_\infty^{n-1} \lim_{T\to\infty} \sum_{\substack{i=1,\cdots,n \\ \sigma_i=\tau_i \text{ or } T}} \int_E q(\sigma_1+\cdots+\sigma_n,x,z_n)|h(z_n)|\nu(dz_n) \\
&\leq 2^{n-1}\|\varphi\|_\infty^{n-1} A,
\end{aligned}$$

where $A = \sup\{\int_E q(t,x,z)|h(z)|\nu(dz);\ t \geq \sum_{i=1}^n \tau_i\}$. Since $A \leq \|h\|_\infty$ we have that if $\sum_{i=1}^n \tau_i \leq 1$ then $|b_n(x,\tau_1,\cdots,\tau_n) \leq 4^n\|\varphi\|_\infty^n$. It is also easy to see that

$$A \leq M_x(\sum_{i=1}^n \tau_i)\|h\|_\nu. \tag{5.6}$$

Let $\sum_{i=1}^n \tau_i > 1$. Then there exists an m such that $\sum_{i=1}^{m-1} \tau_i < 1 \leq \sum_{i=1}^m \tau_i$. Set$\sigma = 1 - \sum_{i=1}^{m-1} \tau_i$ and $h = \widetilde{Q}_{\tau_m-\sigma}\widetilde{\varphi}\widetilde{Q}_{\tau_{m+1}}\widetilde{\varphi}\cdots\widetilde{Q}_{\tau_n}\widetilde{\varphi}$. Then $b_n(x,\tau_1,\cdots,\tau_n) = (\widetilde{q}(\tau_1,x,\cdot), \widetilde{\varphi}\widetilde{Q}_{\tau_1}\cdots\widetilde{Q}_{\tau_{m-1}}\widetilde{\varphi}\widetilde{Q}_\sigma h)_\nu$. By (5.6) and

$$\|h\|_\nu \leq 2^{n-m+1}\|\varphi\|_\infty^{n-m+1}\exp(-\alpha_1(\sum_{i=1}^n \tau_i - 1)),$$

we have

$$|b_n(x;\tau_1,\cdots,\tau_n)| \le 2^n\|\varphi\|_\infty^n 2^m M_x(1)\mathrm{e}^{\alpha_1}\exp(-\alpha_1\sum_{i=1}^n\tau_i)$$
$$\le M_x\mathrm{e}^{\alpha_1}4^n\|\varphi\|_\infty^n\exp(-\alpha_1\sum_{i=1}^n\tau_i).$$

(5.4) and (5.5) can be shown in a similar way. □

Let $\varphi\in C(E)$ and $\Phi=\int_0^T\widetilde{\varphi}(\omega(s))ds$. Then we see that for $A\in\mathcal{F}_0^T$,

$$Q_x^{\varepsilon\varphi}(A|\omega(T)=y)=\frac{Q_x[\mathrm{e}^{\varepsilon\Phi}\mathbf{1}_A|\omega(T)=y]}{Q_x[\mathrm{e}^{\varepsilon\Phi}|\omega(T)=y]}. \tag{5.7}$$

Set

$$f(\varepsilon)=\log Q_x[\mathrm{e}^{\varepsilon\Phi}|\omega(T)=y]. \tag{5.8}$$

Then by (5.7), we have

$$f'(\varepsilon)=Q_x^{\varepsilon\Phi}[\Phi|\omega(T)=y],$$
$$f''(\varepsilon)=Q_x^{\varepsilon\Phi}[(\Phi-(Q_x^{\varepsilon\Phi}[\Phi|\omega(T)=y])^2|\omega(T)=y],$$
$$f'''(\varepsilon)=Q_x^{\varepsilon\Phi}[(\Phi-(Q_x^{\varepsilon\Phi}[\Phi|\omega(T)=y])^3|\omega(T)=y].$$

There exists a $\theta\in(0,1)$ such that

$$\begin{aligned} &f(\varepsilon)-\frac{T}{2}\varepsilon^2(\varphi,\overline{G_\nu}\varphi)_\nu \\ &\quad=f'(0)\varepsilon+\frac{1}{2}(f''(0)-\frac{T}{2}\varepsilon^2(\varphi,\overline{G_\nu}\varphi)_\nu)\varepsilon^2+\frac{1}{3!}f'''(\theta\varepsilon)\varepsilon^3\end{aligned} \tag{5.9}$$

By Prop.2.1.(ii),$q(T,x,y)^{-1}\le(1-M\mathrm{e}^{-\alpha_1 T})^{-1}$ and $T(q(T,x,y)^{-1}-1)\le M/(1-M\mathrm{e}^{-\alpha_1 T})\alpha_1\mathrm{e}$. These and (5.8) and the following Lem.5.2 imply (i) of Lem.2.1.

Lemma 5.2. *Let f be as in (5.8). Then,*

$$|f'(0)|\le\frac{1}{q(T,x,y)}A_1\frac{4}{\alpha_1}\|\varphi\|_\infty, \tag{5.10}$$

$$\begin{aligned}|f''(0)-\frac{T}{2}(\varphi,\overline{G_\nu}\varphi)_\nu|&\le|\frac{1}{q(T,x,y)}-1|\frac{T}{2}(\varphi,\overline{G_\nu}\varphi)_\nu \\ &\quad+(\frac{1}{q(T,x,y)}A_2+\frac{1}{2}(\frac{1}{q(T,x,y)})^2)(\frac{4}{\alpha_1})^2\|\varphi\|_\infty^2\end{aligned} \tag{5.11}$$

$$
\begin{aligned}
(5.12) \qquad |f'''(\varepsilon)| \leq & \frac{1}{q^{\varepsilon\varphi}(T,x,y)}(A_{3,1}^{\varepsilon\varphi} + A_{3,2}^{\varepsilon\varphi}T)(\frac{4}{\alpha_1^{\varepsilon\varphi}})^3 \|\varphi\|_\infty^3 \\
& + \frac{1}{2}(\frac{1}{q^{\varepsilon\varphi}(T,x,y)})^2 A_2^{\varepsilon\varphi} A_1^{\varepsilon\varphi}(\frac{4}{\alpha_1^{\varepsilon\varphi}})^3 \|\varphi\|_\infty^3 \\
& + (\frac{1}{q^{\varepsilon\varphi}(T,x,y)})^3 (A_1^{\varepsilon\varphi})^3 (\frac{4}{\alpha_1^{\varepsilon\varphi}})^3 \|\varphi\|_\infty^3,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1^\varphi &= (M_x^\varphi + N_y^\varphi)\exp(-\alpha_1^\varphi) + \mathrm{e}^{-1} M_x^\varphi N_y^\varphi \exp(-2\alpha_1^\varphi), \\
A_2^\varphi &= \frac{1}{4} + (M_x^\varphi + N_y^\varphi)\exp(-\alpha_1^\varphi) + (2\mathrm{e}^{-2}+1) M_x^\varphi N_y^\varphi \exp(-2\alpha_1^\varphi), \\
A_{3,1}^\varphi &= (M_x^\varphi + N_y^\varphi)\exp(-\alpha_1^\varphi) + (\frac{9}{2}\mathrm{e}^{-3}+2) M_x^\varphi N_y^\varphi \exp(-2\alpha_1^\varphi), \\
A_{3,2}^\varphi &= \frac{\alpha_1^\varphi}{4}(M_x^\varphi + N_y^\varphi)\exp(-\alpha_1^\varphi) + \frac{1}{8}\alpha_1^\varphi.
\end{aligned}
$$

Proof. Since

$$
\begin{aligned}
|f'(0)| &= \frac{1}{q(T,x,y)} \Big| \int_0^T (q(s,x,\cdot), \widetilde{\varphi} q(t-s,\cdot,y))_\nu ds \Big| \\
&= \frac{1}{q(T,x,y)} \Big| \int_0^T \{b_1(x;s) + c_1(y;t-s) + d_2(x,y;s,t-s)\} ds \Big|,
\end{aligned}
$$

we have (5.10) by Lem.5.1 (ii).

Since

$$
\begin{aligned}
\frac{1}{2}f''(0) = & \frac{1}{2}Q_x[(\int_0^T \widetilde{\varphi}(\omega(s))ds)^2 | \omega(T)=y] - \frac{1}{2}(Q_x[\int_0^T \widetilde{\varphi}(\omega(s))ds | \omega(T)=y])^2 \\
= & \frac{1}{q(T,x,y)} \iint_{\substack{0<\tau_1,\tau_2 \\ \tau_1+\tau_2<T}} (q(\tau_1,x,\cdot), \widetilde{\varphi} Q_{\tau_2} \widetilde{\varphi} q(\tau_3,\cdot,y))_\nu d\tau_1 d\tau_2 \\
& - \frac{1}{2}(Q_x[\int_0^T \widetilde{\varphi}(\omega(s))ds | \omega(T)=y])^2 \\
= & \frac{1}{q(T,x,y)} \iint_{\substack{0<\tau_1,\tau_2 \\ \tau_1+\tau_2<T}} d\tau_1 d\tau_2 \{a_1(\tau_2) + b_2(x;\tau_1,\tau_2) + c_2(y;\tau_2, \\
& T-(\tau_1+\tau_2)) + d_3(x,y;\tau_1,\tau_2,T-(\tau_1+\tau_2)) + b_1(x,\tau_2)c_1(y,T-(\tau_1 \\
& + \tau_2))\} - \frac{1}{2}(\frac{1}{q(T,x,y)} \int_0^T \{b_1(x;s) + c_1(y;t-s) + d_2(x,y; \\
& s,t-s)\} ds)^2,
\end{aligned}
$$

we have (5.11) by Lem.5.1 (i) and (ii).

$$\frac{1}{3!}f'''(\varepsilon) = I + II + III,$$
$$I = \frac{1}{3!}Q_x^{\varepsilon\varphi}[\int_0^T (\widetilde{\varphi}(\omega(s))ds)^3|\omega(T) = y],$$
$$II = \frac{1}{2}Q_x^{\varepsilon\varphi}[\int_0^T (\widetilde{\varphi}(\omega(s))ds)^2|\omega(T) = y]Q_x^{\varepsilon\varphi}[\int_0^T \widetilde{\varphi}(\omega(s))ds|\omega(T) = y],$$
$$III = \frac{1}{3}(Q_x^{\varepsilon\varphi}[\int_0^T \widetilde{\varphi}(\omega(s))ds|\omega(T) = y])^3.$$

Then by Lem 5.1 (ii), we have

$$|I| = \frac{1}{q^{\varepsilon\varphi}(T,x,y)}|\iiint_{\substack{\tau_1,\tau_2,\tau_3>0\\ \tau_1+\tau_2+\tau_3<T}} d\tau_1 d\tau_2 d\tau_3\{d_4(x,y;\tau_1,\tau_2,\tau_3,T-(\tau_1+\tau_2$$
$$+\tau_3)) + b_3(x;\tau_1,\tau_2,\tau_3) + c_3(y;\tau_2,\tau_3,T-(\tau_1+\tau_2+\tau_3)) + b_1(x;\tau_1)c_2(y;\tau_3,$$
$$T-(\tau_1+\tau_2+\tau_3)) + b_2(x;\tau_1,\tau_2)c_1(y;T-(\tau_1+\tau_2+\tau_3)) + a_1(\tau_3)b_1(x;,\tau_1)$$
$$+ a_1(\tau_2)c_1(y;T-(\tau_1+\tau_2+\tau_3)) + a_2(\tau_2,\tau_3)\}|$$
$$\leq \frac{1}{q^{\varepsilon\varphi}(T,x,y)}\{M_x^{\varphi}N_y^{\varphi}e^{2\alpha_1^{\varphi}}4^3\|\varphi\|_\infty^3 e{-}\alpha_1^{\varphi}T\frac{T^3}{3!} + (M_x^{\varphi}+N_y^{\varphi})e^{alp}(\frac{4}{\alpha_1^{\varepsilon\varphi}})$$
$$\times 3\|\varphi\|_\infty^3 + 2M_x^{\varphi}e^{\alpha_1^{\varphi}}\frac{4}{\alpha_1^{\varepsilon\varphi}}\|\varphi\|_\infty N_y^{\varphi}e^{\alpha_1^{\varphi}}(\frac{4}{\alpha_1^{\varepsilon\varphi}})^2\|\varphi\|_\infty^2 + 2^2\|\varphi\|_\infty^2(M_x^{\varphi}+N_y^{\varphi})$$
$$\times 4\|\varphi\|_\infty\frac{T}{\alpha_1^{\varphi}} + 2^3\|\varphi\|_\infty^3\frac{T}{\alpha_1^{\varphi}}\},$$
$$|II| \leq \frac{1}{2}(\frac{1}{q^{\varepsilon\varphi}(T,x,y)})^2 A_2^{\varepsilon\varphi}A_1^{\varepsilon\varphi}(\frac{4}{\alpha_1^{\varepsilon\varphi}})^3\|\varphi\|_\infty^3,$$
$$|III| \leq \frac{1}{3}(\frac{1}{q^{\varepsilon\varphi}(T,x,y)})^3(A_1^{\varepsilon\varphi})^3(\frac{4}{\alpha_1^{\varepsilon\varphi}})^3\|\varphi\|_\infty^3.$$

These imply Lem.5.2. □

(ii) of Lem.2.1 can be proved in the following way.

$$Q_x[e^{\xi\Phi}|\omega(T) = y]$$
$$= Q_x[\sum_{n=0}^{\infty}\frac{1}{n!}(\xi\Phi)^n|\omega(T) = y]$$
$$= \frac{1}{q(T,x,y)}\{1 + \sum_{n=1}^{\infty}\xi^n\int\cdots\int_{\substack{\tau_1,\cdots,\tau_n\\ \tau_1+\cdots+\tau_n<T}} d\tau_1\cdots d\tau_n(I+II+III+IV)\},$$

where

$$\begin{aligned}
I &= (\widetilde{\varphi}, (\widetilde{Q}_{\tau_2} + \Pi_\nu)\widetilde{\varphi} \cdots \widetilde{\varphi}(\widetilde{Q}_{\tau_n} + \Pi_\nu \widetilde{\varphi})_\nu, \\
II &= (\widetilde{q}(\tau_1, x, \cdot), \widetilde{\varphi}(\widetilde{Q}_{\tau_2} + \Pi_\nu)\widetilde{\varphi} \cdots \widetilde{\varphi}(\widetilde{Q}_{\tau_n} + \Pi_\nu \widetilde{\varphi})_\nu, \\
III &= (\widetilde{\varphi}, (\widetilde{Q}_{\tau_2} + \Pi_\nu)\widetilde{\varphi} \cdots \widetilde{\varphi}(\widetilde{Q}_{\tau_n} + \Pi_\nu \widetilde{\varphi}\widetilde{q}(T - (\tau_1 + \cdots + \tau_n), \cdot, y))_\nu, \\
IV &= (\widetilde{q}(\tau_1, x, \cdot), \widetilde{\varphi}(\widetilde{Q}_{\tau_2} + \Pi_\nu)\widetilde{\varphi} \cdots \widetilde{\varphi}(\widetilde{Q}_{\tau_n} + \Pi_\nu \widetilde{\varphi}\widetilde{q}(T - (\tau_1 + \cdots + \tau_n), \cdot, y))_\nu
\end{aligned}$$

Using Lem.5.1 (ii), we can estimate II, III and IV in the same way as in the proof of Lem1.1 in [7] to get

$$\begin{aligned}
&|\sum_{n=1}^\infty \xi^n \int \cdots \int_{\substack{\tau_1, \cdots, \tau_n \\ \tau_1 + \cdots + \tau_n < T}} d\tau_1 \cdots d\tau_n II| \\
&\quad \le |\xi| M_x \mathrm{e}^{\alpha_1} \frac{\frac{4}{\alpha_1}\|\varphi\|_\infty}{1 - 4\|\varphi\|_\infty |\xi|/\alpha_1} \exp\left(\frac{T|\xi|^2 8\|\varphi\|_\infty^2}{\alpha_1} \frac{1}{1 - 4\|\varphi\|_\infty |\xi|/\alpha_1}\right), \\
&|\sum_{n=1}^\infty \xi^n \int \cdots \int_{\substack{\tau_1, \cdots, \tau_n \\ \tau_1 + \cdots + \tau_n < T}} d\tau_1 \cdots d\tau_n III| \\
&\quad \le |\xi| N_y \mathrm{e}^{\alpha_1} \frac{\frac{4}{\alpha_1}\|\varphi\|_\infty}{1 - 4\|\varphi\|_\infty |\xi|/\alpha_1} \exp\left(\frac{T|\xi|^2 8\|\varphi\|_\infty^2}{\alpha_1} \frac{1}{1 - 4\|\varphi\|_\infty |\xi|/\alpha_1}\right), \\
&|\sum_{n=1}^\infty \xi^n \int \cdots \int_{\substack{\tau_1, \cdots, \tau_n \\ \tau_1 + \cdots + \tau_n < T}} d\tau_1 \cdots d\tau_n IV| \\
&\quad \le |\xi|^2 M_x N_y \mathrm{e}^{2\alpha_1} \left(\frac{\frac{4}{\alpha_1}\|\varphi\|_\infty}{1 - 4\|\varphi\|_\infty |\xi|/\alpha_1}\right)^2 \exp\left(\frac{T|\xi|^2 8\|\varphi\|_\infty^2}{\alpha_1} \frac{1}{1 - 4\|\varphi\|_\infty |\xi|/\alpha_1}\right) \\
&\quad + |\xi| M_x N_y \mathrm{e}^{2\alpha_1} \frac{4\|\varphi\|_\infty/\alpha_1}{1 - |\xi| 4\|\varphi\|_\infty/\alpha_1}.
\end{aligned}$$

On the other hand,

$$1 + \sum_{n=1}^\infty \xi^n \int \cdots \int_{\substack{\tau_1, \cdots, \tau_n \\ \tau_1 + \cdots + \tau_n < T}} d\tau_1 \cdots d\tau_n I = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= 1 + \sum_{\ell=1}^\infty \xi^{2\ell} \int \cdots \int_{\substack{\tau_1, \cdots, \tau_{2\ell} \\ \tau_1 + \cdots + \tau_{2\ell} < T}} d\tau_1 \cdots d\tau_{2\ell} a_1(\tau_2) a_2(\tau_4) \cdots a_1(\tau_{2\ell}), \\
I_2 &= \sum_{\ell=1}^\infty \sum_{n=2\ell+1}^\infty \xi^n \sum_{\substack{k_1, \cdots, k_\ell \ge 1 \\ k_1 + \cdots + k_\ell = n - \ell}} \int \cdots \int_{\substack{\tau_1, \cdots, \tau_n \\ \tau_1 + \cdots + \tau_n < T}} d\tau_1 \cdots d\tau_n a_{k_1}(\tau_2, \cdots, \tau_{1+k_1}) \\
&\quad \cdots a_{k_\ell}(\tau_{\ell+1+k_1+\cdots+k_{\ell-1}}, \cdots, \tau_n).
\end{aligned}$$

By Lem.5.1 (i) and (ii), we see that

$$|I_1 - \exp(\xi^2 \frac{T}{2}(\varphi, \overline{G_\nu}\varphi)_\nu)| \leq |\xi|^2 2\frac{4}{\alpha_1}\|\varphi\|_\infty^2 \exp(|\xi|^2 T 2\frac{4}{\alpha_1}\|\varphi\|_\infty) \quad \text{and}$$

$$|I_2| \leq \exp(|\xi|^2 T\frac{4}{\alpha_1}\|\varphi\|_\infty)\{\exp(|\xi|^2 T\frac{4}{\alpha_1}(\frac{1}{1-4\|\varphi\|_\infty|\xi|/\alpha_1} - 1)) - 1\},$$

and this completes the proof of Lem.2.1 (ii). □

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