Brownian survival among Poissonian traps with random shapes at critical intensity

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Abstract

In this paper we consider a standard Brownian motion in $\mathbb{R}^d$, starting at 0 and observed until time t. The Brownian motion takes place in the presence of a Poisson random field of traps, whose centers have intensity ν_t and whose shapes are drawn randomly and independently according to a probability distribution Π on a certain class of compact subsets of $\mathbb{R}^d$. The Brownian motion is killed as soon as it hits one of the traps. With the help of a large deviation technique developed in an earlier paper, we find the tail of the probability S_t that the Brownian motion survives up to time t when

$$\nu_t = \begin{cases} ct^{-2/d}, & d \geq 3, \\ ct^{-1}\log^2 t, & d = 2, \end{cases}$$

where $c \in (0,\infty)$ is a parameter. This choice of intensity corresponds to a critical scaling. We give a detailed analysis of the rate constant in the tail of S_t as a function of c, including its limiting behaviour as $c \to \infty$ or $c \downarrow 0$. For $d \geq 3$, we find that there are three different regimes, depending on the choice of Π. In one of the regimes there is a collapse transition at a critical value $c^* \in (0,\infty)$, where the optimal survival strategy changes from being diffusive to being subdiffusive. This transition comes with a slope discontinuity. For $d = 2$, the rate constant is independent of Π, and the collapse transition has a continuous slope.

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1 Introduction and main results

1.1 Motivation

The model studied in this paper has two random ingredients:

1. Let $\beta = \{\beta(s)\colon\ s \geq 0\}$ be the standard Brownian motion in $\mathbb{R}^d$ – the Markov process with generator $\Delta/2$ – starting at 0. We write P, E to denote probability and expectation with respect to β.

2. For $t \geq 0$, let
$$K_t = \bigcup_{x \in \omega_t} [x + A_x], \tag{1.1.1}$$
where ω_t is a Poisson point process with intensity
$$\nu_t = \begin{cases} ct^{-2/d}, & d \geq 3, \\ ct^{-1}\log^2 t, & d = 2, \end{cases} \tag{1.1.2}$$
$c \in (0, \infty)$ is a parameter and, given ω_t,
$$A_x, \quad x \in \omega_t, \tag{1.1.3}$$
are i.i.d. random sets drawn according to a probability distribution Π on the class $\mathcal{Q}$ of subsets of $\mathbb{R}^d$ given by
$$\mathcal{Q} = \left\{ A \subset \mathbb{R}^d\colon\ A \text{ compact},\ A = \mathrm{cl}(\mathrm{int}(A)),\ \mathrm{int}(A) \ni 0 \right\}, \tag{1.1.4}$$
where $\mathrm{cl}(A)$ denotes the closure of A and $\mathrm{int}(A)$ the interior of A. We write $\mathbb{P}_t, \mathbb{E}_t$ to denote probability and expectation with respect to K_t.

The construction of a probability distribution Π on the class of closed subsets of $\mathbb{R}^d$: $\mathcal{C} = \{A \subset \mathbb{R}^d\colon\ A \text{ closed}\}$ is given in Molchanov [10, Chapter 1]. This class is endowed with the topology generated by the Hausdorff metric $\rho_H\colon \mathcal{C} \times \mathcal{C} \to [0, \infty]$ given by
$$\rho_H(A_1, A_2) = \inf\{\epsilon > 0\colon\ A_1 \subset A_2^\epsilon,\ A_2 \subset A_1^\epsilon\}, \tag{1.1.5}$$
where $A^\epsilon = \cup_{x \in A} B_\epsilon(x)$ is the ϵ-environment of A (with $B_\epsilon(x)$ the closed ball of radius ϵ centred at x). The probability distribution Π lives on the Borel sigma-algebra generated by ρ_H. In the present paper we assume that Π is such that $\Pi(\mathcal{Q}) = 1$ and
$$\int_{\mathcal{Q}} |A|\, \Pi(dA) < \infty. \tag{1.1.6}$$

Let
$$\tau_{K_t} = \inf\{s \geq 0\colon\ \beta(s) \in K_t\} \tag{1.1.7}$$
and
$$S_t = (\mathbb{E}_t \times P)(\tau_{K_t} > t). \tag{1.1.8}$$
In other words, we view K_t as a collection of randomly located and randomly shaped traps, τ_{K_t} as the trapping time for the Brownian motion, and S_t as the probability of survival up to time t. The goal of the present paper is to identify the asymptotic behaviour of S_t for large t.

As will become clear later on, the choice of intensity in (1.1.2) corresponds to a critical scaling. Our main results show that the tail of S_t has an interesting dependence on the parameter c, with three different regimes for $d \geq 3$, depending on the choice of Π, and one regime for $d = 2$. The proof of these results relies on a large deviation technique developed in van den Berg, Bolthausen and den Hollander [2]. For each of the regimes we provide a detailed analysis of the rate constant controlling the tail behaviour of S_t, including its scaling as $c \to \infty$ or $c \downarrow 0$. We show that for $d \geq 3$, in one of the regimes, the rate constant exhibits a collapse transition in the optimal survival strategy at a critical value $c^* \in (0, \infty)$. We analyse the behaviour of the rate constant near c^* and show that a slope discontinuity occurs. For $d = 2$ there is a collapse transition with a continuous slope.

1.2 Representation in terms of Wiener sausages

The starting point of our analysis is a representation formula expressing S_t as an exponential functional of a family of Wiener sausages with varying shape. This formula is the analogue of the well-known formula for the fixed shape case.

The Wiener sausage with shape $A \in \mathcal{Q}$ is the random process defined by

$$W^A(t) = \bigcup_{0 \leq s \leq t} [\beta(s) + A], \qquad t \geq 0. \tag{1.2.1}$$

Proposition 1.2.1 *For any $d \geq 1$, $\Pi \in \mathcal{M}_1^+(\mathcal{Q})$ and $t \geq 0$,*

$$S_t = E\left(\exp\left[-\nu_t \int_{\mathcal{Q}} \Pi(dA)\ |W^A(t)|\right]\right). \tag{1.2.2}$$

Proof. Let $\Pi^-, \Pi^+ \in \mathcal{M}_1^+(\mathcal{Q})$ be discrete probability distributions such that

$$\Pi^- \preceq \Pi \preceq \Pi^+ \tag{1.2.3}$$

in the sense of stochastic ordering by inclusion. Then

$$S_t(\Pi^+) \leq S_t(\Pi) \leq S_t(\Pi^-), \qquad t \geq 0. \tag{1.2.4}$$

For Π a discrete probability distribution, say,

$$\widehat{\Pi} = \sum_{n \in \mathbb{N}} a_n \delta_{A_n}, \qquad \sum_{n \in \mathbb{N}} a_n = 1,\ a_n \geq 0,\ A_n \in \mathcal{Q}, \tag{1.2.5}$$

the Poisson random field of traps with random shapes drawn according to $\widehat{\Pi}$ and intensity ν_t is an independent superposition of Poisson random fields of traps with shape A_n and intensity $a_n \nu_t$. The probability, under the law $\mathbb{P}_t$, that up to time t the traps labelled by n avoid a given Brownian path β equals $\exp[-\nu_t(a_n |W^{A_n}(t)|)]$. The probability that up to time t all the traps avoid the given β is the product of the latter quantity over n, which is the same as the right-hand side of (1.2.2) for $\Pi = \widehat{\Pi}$.

Any $\Pi \in \mathcal{M}_1^+(\mathcal{Q})$ can be sandwiched, as in (1.2.3), by sequences $(\widehat{\Pi}_j^-)$ and $(\widehat{\Pi}_j^+)$ of discrete probability distributions, of the type (1.2.5), such that

$$\widehat{\Pi}_j^{\pm} \Rightarrow \Pi \qquad \text{as } j \to \infty \tag{1.2.6}$$

($\Rightarrow$ denotes weak convergence). One way to do this is by approximating $A \in \mathcal{Q}$ from the inside and from the outside by a union of hypercubes of size ϵ and letting $\epsilon \downarrow 0$. Since $A \mapsto |W^A(t)|$ is continuous on $\mathcal{Q}$ in the Hausdorff metric ρ_H, we get from (1.2.6) that

$$\lim_{j\to\infty} \int_{\mathcal{Q}} \widehat{\Pi}_j^{\pm}(dA)\ |W^A(t)| = \int_{\mathcal{Q}} \Pi(dA)\ |W^A(t)|. \tag{1.2.7}$$

Therefore (1.2.2) holds for all $\Pi \in \mathcal{M}_1^+(\mathcal{Q})$.

Note that (1.1.6) guarantees that the integral in the right-hand side of (1.2.2) is finite P-a.s. ∎

1.3 Survival theorems

This section contains our main results for the tail behaviour of S_t as $t \to \infty$.

For $d \geq 3$, let $\kappa(A)$ be the Newtonian capacity of A associated with the Green function of $(-\Delta/2)^{-1}$.

Theorem 1.3.1 *Let $d \geq 3$ and fix $\Pi \in \mathcal{M}_1^+(\mathcal{Q})$. For every $c > 0$,*

$$\lim_{t\to\infty} \frac{1}{t^{(d-2)/d}} \log S_t = -J_d^{\Pi}(c) \tag{1.3.1}$$

with

$$J_d^{\Pi}(c) = \inf\left\{\tfrac{1}{2}\|\nabla\phi\|_2^2 + cF_d^{\Pi}(\phi^2)\colon\ \phi \in H^1(\mathbb{R}^d),\ \|\phi\|_2^2 = 1\right\}, \tag{1.3.2}$$

where

$$F_d^{\Pi}(\phi^2) = \int_{\mathbb{R}^d} dx \int_{\mathcal{Q}} \Pi(dA)\ \left(1 - e^{-\kappa(A)\phi^2(x)}\right). \tag{1.3.3}$$

Theorem 1.3.1 identifies the tail of S_t for $d \geq 3$ in terms of a variational problem involving Π. Since the dependence on Π enters only via the capacity of the random set A, we may rewrite (1.3.3) as

$$F_d^{\Pi}(\phi^2) = \int_{\mathbb{R}^d} dx \int_0^{\infty} \Theta(d\kappa)\ \left(1 - e^{-\kappa\phi^2(x)}\right) \tag{1.3.4}$$

with $\Theta = \Pi \circ \kappa^{-1}$ the probability distribution on $(0,\infty)$ induced from Π by κ. Note that $\kappa(A) \in (0,\infty)$ for all $A \in \mathcal{Q}$.

A similar result holds for $d = 2$, but without a role for Π.

Theorem 1.3.2 *Let $d = 2$ and fix $\Pi \in \mathcal{M}_1^+(\mathcal{Q})$. For every $c > 0$,*

$$\lim_{t\to\infty} \frac{1}{\log t} \log S_t = -J_2(c) \tag{1.3.5}$$

with

$$J_2(c) = \inf\left\{\tfrac{1}{2}\|\nabla\phi\|_2^2 + cF_2(\phi^2)\colon\ \phi \in H^1(\mathbb{R}^2),\ \|\phi\|_2^2 = 1\right\}, \tag{1.3.6}$$

where

$$F_2(\phi^2) = \int_{\mathbb{R}^2} dx\ \left(1 - e^{-2\pi\phi^2(x)}\right). \tag{1.3.7}$$

The scale of the large deviation in Theorem 1.3.2 is different from that in Theorem 1.3.1. This is due to the different choice of intensity in (1.1.2). However, the variational formula has the same structure. The difference is that $\kappa(A)$ is replaced by 2π, so that the dependence on Π drops out. This fact turns out to be related to the recurrence of planar Brownian motion.

1.4 Analysis of the variational problems

In this section we give a detailed analysis of $c \mapsto J_d^\Pi(c)$ in (1.3.2) and $c \mapsto J_2(c)$ in (1.3.6).

Let $\langle\cdot\rangle$ denote expectation with respect to Θ. For $d \geq 3$ there are three regimes:

(I) $\langle\kappa\rangle < \infty$, and

$$\begin{array}{rcll} J_d^\Pi(c) & = & c\langle\kappa\rangle & \text{for } 0 \leq c \leq c^*, \\ & < & c\langle\kappa\rangle & \text{for } c > c^*, \end{array} \tag{1.4.1}$$

for some $c^* \in (0,\infty)$.

(II) $\langle\kappa\rangle < \infty$, and (1.4.1) with $c^* = 0$.

(III) $\langle\kappa\rangle = \infty$.

We consider two subclasses for Θ with $\langle\kappa\rangle < \infty$:

$$\begin{aligned} \mathcal{S}_I &= \Big\{\Theta\colon \text{there exist}\, \kappa_0 \in (0,\infty) \,\text{and}\, K \in (0,\infty) \text{ such that for all } \kappa \geq \kappa_0 \\ &\qquad \Theta(d\kappa) \leq K\kappa^{-1-\frac{d+2}{d}} d\kappa \Big\}, \\ \mathcal{S}_{II} &= \Big\{\Theta\colon \text{there exist}\, \kappa_1 \in (0,\infty) \,\text{and}\, L\colon [\kappa_1,\infty) \to (0,\infty) \\ &\qquad \text{non-decreasing with } \lim_{\kappa\to\infty} L(\kappa) = \infty \text{ such that for all } \kappa \geq \kappa_1 \\ &\qquad \Theta(d\kappa) \geq L(\kappa)\kappa^{-1-\frac{d+2}{d}} d\kappa, \text{ and } \langle\kappa\rangle < \infty\Big\}. \end{aligned} \tag{1.4.2}$$

Note that the separation between the classes $\mathcal{S}_I$ and $\mathcal{S}_{II}$ is thin, and is very close to where $\langle\kappa^{(d+2)/d}\rangle$ diverges.

Theorem 1.4.1 *Let $d \geq 3$.*
(i) For every $\Pi \in \mathcal{M}_1^+(\mathcal{Q})$, $c \mapsto J_d^\Pi(c)$ is continuous, strictly increasing and concave on $(0,\infty)$, with $J_d^\Pi(0) = 0$.
(ii) If $\Theta \in \mathcal{S}_I$, then J_d^Π falls in regime (I). *If, in addition, $\langle\kappa^\eta\rangle < \infty$ for some $\eta > \frac{d+2}{d}$, then the variational problem in* (1.3.2) *has a minimiser with full support for $c = c^*$, and*

$$[J_d^\Pi]'(c^*+) < \langle\kappa\rangle. \tag{1.4.3}$$

(iii) If $\Theta \in \mathcal{S}_{II}$, then J_d^Π falls in regime (II), *and*

$$[J_d^\Pi]'(0+) = \langle\kappa\rangle. \tag{1.4.4}$$

(iv) In regime (III),

$$[J_d^\Pi]'(0+) = \infty. \tag{1.4.5}$$

(v) The variational problem in (1.3.2) *has a minimiser with full support for*

$$\begin{array}{ll} c > c^* & \textit{when } \Theta \in \mathcal{S}_I, \\ c > 0 & \textit{when } \Theta \in \mathcal{S}_{II} \textit{ or } \langle\kappa\rangle = \infty. \end{array} \tag{1.4.6}$$

Theorem 1.4.2 *Let $d \geq 3$.*
(i) For every $\Pi \in \mathcal{M}_1^+(\mathcal{Q})$,

$$\lim_{c\to\infty} c^{-2/(d+2)} J_d^\Pi(c) = \frac{d+2}{2}\left(\frac{\lambda_d}{d}\right)^{d/(d+2)}, \tag{1.4.7}$$

where λ_d is the principal Dirichlet eigenvalue of $-\Delta$ on the ball of unit volume.
(ii) For $\Theta \in \mathcal{S}_{II}$, let $\Theta(d\kappa) = \theta(\kappa)d\kappa$ with $\theta(\kappa) = K\kappa^{-1-\gamma}[1+o(1)]$ as $\kappa \to \infty$ and $1 < \gamma < \frac{d+2}{d}$, $0 < K < \infty$. Then

$$\lim_{c\downarrow 0} \{2K\Gamma(-\gamma)c\}^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^\Pi(c)\right] = \tfrac{1}{2}M_d(\gamma), \tag{1.4.8}$$

where

$$M_d(\gamma) = -\inf\left\{\|\nabla\psi\|_2^2 - \int|\psi|^{2\gamma}\colon\ \psi\in H^1(\mathbb{R}^d),\ \|\psi\|_2^2 = 1\right\} \in (0,\infty). \tag{1.4.9}$$

(iii) In regime (III), let $\Theta(d\kappa) = \theta(\kappa)d\kappa$ with $\theta(\kappa) = K\kappa^{-1-\gamma}[1+o(1)]$ as $\kappa\to\infty$ and $0 < \gamma < 1$, $0 < K < \infty$. Then

$$\lim_{c\downarrow 0} \{2K[-\Gamma(-\gamma)]c\}^{-2/(2+d(1-\gamma))} J_d^\Pi(c) = \tfrac{1}{2}\widetilde{M}_d(\gamma), \tag{1.4.10}$$

where

$$\widetilde{M}_d(\gamma) = \inf\left\{\|\nabla\psi\|_2^2 + \int|\psi|^{2\gamma}\colon\ \psi\in H^1(\mathbb{R}^d),\ \|\psi\|_2^2 = 1\right\} \in (0,\infty). \tag{1.4.11}$$

The qualitative behaviour of $c \mapsto J_d^\Pi(c)$ found in Theorems 1.4.1 and 1.4.2 is summarized below.

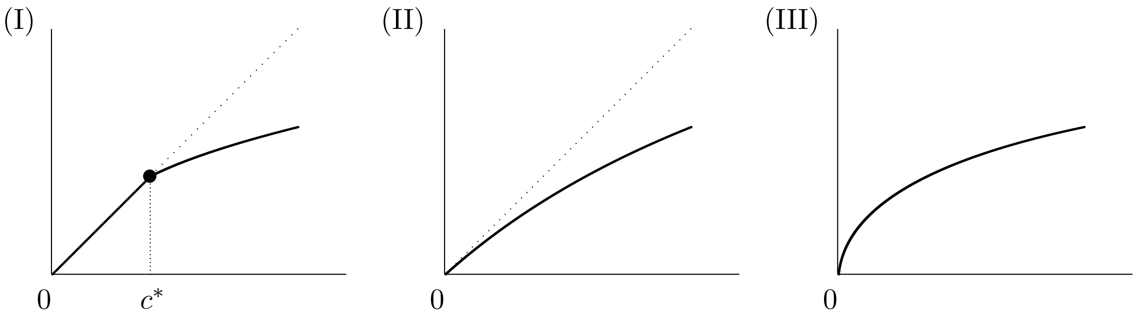

Fig. Qualitative behaviour of $c \mapsto J_d^\Pi(c)$ in the three regimes for $d \geq 3$.

Theorem 1.4.3 *Let $d = 2$.*
(i) $c \mapsto J_2(c)$ is continuous, strictly increasing and concave on $(0,\infty)$, with $J_2(0) = 0$.
(ii) There exists a number $c^* \in (0,\infty)$, given by

$$c^* = \frac{1}{4\pi^2}\inf\left\{\frac{\|\nabla\phi\|_2^2}{\|\phi\|_4^4}\colon\ \phi\in H^1(\mathbb{R}^2),\ \|\phi\|_2^2 = 1\right\}, \tag{1.4.12}$$

such that

$$J_2(c) \begin{array}{ll} = & 2\pi c \quad \text{for } 0 \le c \le c^*, \\ < & 2\pi c \quad \text{for } c > c^*, \end{array} \tag{1.4.13}$$

and

$$[J_2]'(c^*+) = 2\pi. \tag{1.4.14}$$

(iii) Formula (1.4.7) *holds with* $d = 2$.
(iv) The variational problem in (1.3.6) *has a minimiser if and only if* $c > c^*$. *This minimiser has full support.*

1.5 Discussion

The idea behind Theorem 1.3.1 is that for $d \ge 3$ the optimal strategy for the Brownian motion to survive the traps is to behave like a Brownian motion in a drift field $xt^{1/d} \mapsto (\nabla\phi/\phi)(x)$ for some smooth $\phi\colon \mathbb{R}^d \mapsto [0,\infty)$. The cost, under the law P, of adopting this drift during a time t is

$$\exp\left[-t \times t^{-2/d}\,\frac{1}{2}\int_{\mathbb{R}^d} dx\; |\nabla\phi(x)|^2\right]. \tag{1.5.1}$$

The effect of the drift is to push the Brownian motion towards the origin, so that it lives on space scale $t^{1/d}$, which is well below the diffusive scale. Conditioned on adopting the drift, the Brownian motion spends time $\phi^2(x)$ per unit volume in the neighbourhood of $xt^{1/d}$, and it turns out that, for each A, the Wiener sausage with shape A associated with the Brownian motion covers a fraction $1 - \exp[-\kappa(A)\phi^2(x)]$ of that unit volume. The cost, under the law $\mathbb{P}_t$, of the traps avoiding the Brownian motion is

$$\exp\left[-ct^{-2/d} \times t \int_{\mathbb{R}^d} dx \int_{\mathcal{Q}} \Pi(dA)\,\left(1 - e^{-\kappa(A)\phi^2(x)}\right)\right] \tag{1.5.2}$$

(recall the superposition argument in the proof of Proposition 1.2.1). Combining (1.5.1) and (1.5.2), we see that the best choice of the drift field is therefore given by a minimiser of the variational problem in (1.3.2), or by a minimising sequence.

Theorem 1.3.2 shows that for $d = 2$ the survival probability decays polynomially rather than exponentially fast. The optimal survival strategy is of the same type as for $d \ge 3$, but now the Brownian motion lives on space scale $\sqrt{t/\log t}$, which is only slightly below the diffusive scale. Apparently, the limiting behaviour does not depend on Π, which says that the Brownian motion manages to stay far away from the traps.

Theorems 1.4.1 and 1.4.2 show that for $d \ge 3$ there are three regimes: [1]

(I) There is a critical threshold ($c^* > 0$). For $c < c^*$, the Brownian motion prefers to ignore the survival strategies parametrised by ϕ and to move on space scale $\sqrt{t}$. In doing so, it behaves like a typical Brownian motion and sees the average trap capacity, i.e., also the traps field is typical . For $c > c^*$, on the other hand, the Brownian motion prefers to follow the survival strategy parametrised by a minimiser $\bar\phi$ and to move on space scale $t^{1/d}$. In doing so, it does a large deviation and sees less than the average trap capacity. Also the trap field does a large deviation, because it keeps traps out of the

[1] Even though we interpret our results in terms of an optimal survival strategy, we do not have pathwise statements. More work would be needed to prove that, conditional on survival, the Brownian motion and the trap field behave as suggested.

"spongy structure" that is formed by the Brownian motion. Since $\bar{\phi}$ has full support, the Brownian motion "sneeks around the traps and moves about" rather than "finds a large trap free hole and stays there". At $c = c^*$ there is a *collapse transition* from diffusive behavior to subdiffusive behavior. This collapse transition is *discontinuous* because a minimiser persists at the critical threshold, which leads to a slope discontinuity of J_d^Π at $c = c^*$.

(II) There is no critical threshold ($c^* = 0$). There is a minimiser $\bar{\phi}$ for all $c > 0$, meaning that the optimal survival strategy is always subdiffusive. As $c \downarrow 0$, this minimiser flattens out, the Brownian motion gradually covers more space and gradually sees the average trap capacity. The thinner the tail of Π (i.e., the closer Π to the boundary with regime (I)), the faster J_d^Π approaches the line with slope $\langle\kappa\rangle$.

(III) The behaviour is similar as in regime (II), but with an infinite slope at $c = 0$. The thicker the tail of Π (i.e., the farther Π from the boundary with regime (II)), the steeper J_d^Π approaches the vertical axis.

Theorem 1.4.3 shows that for $d = 2$ the behaviour is similar to that for $d \geq 3$ in regime (I). There is again a collapse transition, associated with a crossover in the optimal strategy. Hoever, this collapse transition is continuous because no minimiser persists as $c \downarrow c^*$.

The high intensity limit $c \to \infty$ corresponds to the minimiser contracting to a high and narrow peak. This corresponds to the optimal survival strategy looking more and more like "find a large trap free hole and stay there". This is the optimal survival strategy for all intensities that are larger than the one in (1.1.2), which is why the choice in (1.1.2) is critical.

Finally, the results in the present paper belong to a regime of critical scaling. Results of a similar nature appear in van den Berg, Bolthausen and den Hollander [3], where the large deviation behaviour of the volume of the intersection of two Wiener sausages is identified, and in a sequence of papers by Merkl and Wüthrich [7], [8], [9], which look at the principal eigenvalue of the Schrödinger operator $-\Delta + V_t$ with V_t a potential consisting of a Poisson field of obstacles with a height that shrinks to zero in a critical manner with t and with Dirichlet boundary conditions on a box of size t.

1.6 Outline

Theorems 1.3.1 and 1.3.2 are proved in Section 2. The proof closely follows Sections 2, 3 and 4 in van den Berg, Bolthausen and den Hollander [2] (henceforth referred to as vdBBdH). We sketch the main line of the argument, so that the present paper can be read almost independently. Theorems 1.4.1, 1.4.2 and 1.4.3 are proved in Section 3 and rely on variational calculus and Sobolev inequalities.

2 Proof of Theorems 1.3.1 and 1.3.2

2.1 Scaling, compactifying and coarse-graining

Recalling (1.1.2), we have from Proposition 1.2.1 that

$$S_t = \begin{cases} \exp\left[-t^{(d-2)/d} \times cV^\Pi(t)\right], & d \geq 3, \\ \exp\left[-\log t \times cV^\Pi(t)\right], & d = 2, \end{cases} \tag{2.1.1}$$

where we define

$$V^{\Pi}(t) = \begin{cases} \dfrac{1}{t} \displaystyle\int_{\mathcal{Q}} \Pi(dA)\ |W^A(t)|, & d \geq 3, \\ \\ \dfrac{\log t}{t} \displaystyle\int_{\mathcal{Q}} \Pi(dA)\ |W^A(t)|, & d = 2. \end{cases} \tag{2.1.2}$$

It follows from Spitzer [11] that, for every $A \in \mathcal{Q}$,

$$E|W^A(t)| = [1 + o(1)] \times \begin{cases} \kappa(A)t, & d \geq 3, \\ 2\pi t/\log t, & d = 2, \end{cases} \qquad t \to \infty. \tag{2.1.3}$$

Hence

$$\lim_{t\to\infty} EV^{\Pi}(t) = \begin{cases} \langle\kappa\rangle, & d \geq 3, \\ 2\pi, & d = 2. \end{cases} \tag{2.1.4}$$

Thus, by (2.1.1), the large deviations of $V^{\Pi}(t)$ driving Theorems 1.3.1 and 1.3.2 take place *on the scale of its mean*, which is order 1 for $d \geq 3$ when $\langle\kappa\rangle < \infty$ and order 1 for $d = 2$.

2.1.1 Scaling

By Brownian scaling,

$$\begin{aligned} \frac{1}{t}|W^A(t)| &\doteq |W^{At^{-1/d}}(t^{(d-2)/d})|, \quad d \geq 3, \\ \frac{\log t}{t}|W^A(t)| &\doteq |W^{A\sqrt{\log t/t}}(\log t)|, \quad d = 2, \end{aligned} \tag{2.1.5}$$

where $\doteq$ denotes equality in distribution. Hence, abbreviating

$$\tau = \begin{cases} t^{(d-2)/d}, & d \geq 3, \\ \log t, & d = 2, \end{cases} \qquad T_\tau = \begin{cases} \tau^{2/(d-2)}, & d \geq 3, \\ e^\tau/\tau, & d = 2, \end{cases} \tag{2.1.6}$$

we find from (2.1.1) and (2.1.2) that

$$S_t = E\left(\exp\left[-c\tau V_\tau^{\Pi}(\tau)\right]\right) \tag{2.1.7}$$

with

$$V_\tau^{\Pi}(\tau) = \int_{\mathcal{Q}} \Pi(dA)\ |W^{A/\sqrt{T_\tau}}(\tau)|. \tag{2.1.8}$$

The right-hand side involves Wiener sausages at time τ with a shape that shrinks with $1/\sqrt{T_\tau}$. We aim for the large deviations of $V_\tau^{\Pi}(\tau)$.

2.1.2 Compactifying

We will obtain upper and lower bounds on S_t by *wrapping the scaled Brownian motion around a finite torus*, respectively, by *killing it at the boundary of this torus.* This compactification will be exploited in Sections 2.2 and 2.3, where we prove a large deviation principle (LDP) for $(V_\tau^{\Pi}(\tau))_{\tau>0}$ restricted to the torus and use it to compute asymptotics of exponential moments. This LDP will lead to a lower, respectively, upper bound on the variational characterisation of the rate functions J_d^{Π} and J_2 in Theorems 1.3.1 and 1.3.2. By letting the torus tend to $\mathbb{R}^d$ afterwards, we will obtain the variational characterisation as claimed.

2.1.3 Coarse-graining

The proof of the LDP for the Brownian motion on the torus consists of three steps, taken from vdBBdH:

- Step 1: For $\epsilon > 0$, we chop the Brownian motion into excursions of length ϵ, and define

$$\mathbb{X}_{\tau,\epsilon} = \{\beta_N(i\epsilon)\}_{1\le i\le \tau/\epsilon}, \tag{2.1.9}$$

  which is the collection of the endpoints of the excursions. The lower index N refers to the restriction to the torus of size N, and for notational convenience we assume that τ/ϵ is integer. Let $V^{\Pi}_{\tau,N}(\tau)$ be the analogue of (2.1.8) for the Brownian motion on the torus of size N. We approximate $V^{\Pi}_{\tau,N}(\tau)$ by $\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau))$, where $\mathbb{E}_{\tau,\epsilon}$ denotes the *conditional expectation given* $\mathbb{X}_{\tau,\epsilon}$. We prove that the difference between $V^{\Pi}_{\tau,N}(\tau)$ and $\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau))$ is negligible in the limit as $\tau\to\infty$ followed by $\epsilon\downarrow 0$. This is done by an application of a concentration inequality of Talagrand.

- Step 2: We represent $\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau))$ as a functional of the *bivariate empirical measure*

$$L_{\tau,\epsilon} = \frac{\epsilon}{\tau}\sum_{i=1}^{\tau/\epsilon}\delta_{\left(\beta_N((i-1)\epsilon),\beta_N(i\epsilon)\right)}. \tag{2.1.10}$$

  According to Donsker and Varadhan, $(L_{\tau,\epsilon})_{\tau>0}$ satisfies an LDP. We need some further approximations to get the dependence of $\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau))$ on $L_{\tau,\epsilon}$ in a suitable form, but based on just this LDP we get an LDP for $(\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau)))_{\tau>0}$ via a contraction principle.

- Step 3: We take the limit $\epsilon\downarrow 0$. By Step 2 we already know that $V^{\Pi}_{\tau,N}(\tau)$ is well approximated by $\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau))$. It therefore suffices to have an appropriate approximation for the variational formula in the LDP for $(\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau)))_{\tau>0}$.

These steps were used in vdBBdH to derive an LDP for the quantity in (2.1.8) when $\Pi = \delta_{B_a(0)}$, the point measure on the ball of radius $a\in(0,\infty)$ centred at 0. All we therefore have to do is to see how the integral over Π can be incorporated and carried along. A priori, this is not difficult. However, the argument in vdBBdH is rather delicate, involving various estimates on Brownian motion and hitting times of shrinking balls. We need to check that these estimates can be handled when the balls are replaced by sets with a random shape. Therefore we provide a sketch of the main ingredients of the argument and guide the reader along.

In Sections 2.2 and 2.3 we give the proof for $d\ge 3$ when Π has *finite support*, i.e.,

$$\Pi = \sum_{m=1}^{n} a_m\delta_{A_m}, \qquad \sum_{m=1}^{n} a_m = 1,\ a_m\ge 0,\ A_m\in\mathcal{Q},\ n\in\mathbb{N}. \tag{2.1.11}$$

In Section 2.4 we explain why the proof for arbitrary Π follows via sandwiching, as in the proof of Proposition 1.2.1. In Section 2.5 we briefly indicate how to amend the proof for the case $d=2$, taking (2.1.6)–(2.1.8) into account.

2.2 Upper bound in $d \geq 3$

Write Λ_N to denote the torus of size N, i.e., $[-N/2, N/2)^d$ with periodic boundary conditions. Let $\beta_N(s)$, $s \geq 0$, be the Brownian motion wrapped around Λ_N. Let

$$W_N^{A/\sqrt{T_\tau}}(s), \qquad s \geq 0, \tag{2.2.1}$$

denote its Wiener sausage with shape A scaled down by $\sqrt{T_\tau}$, and let

$$V_{\tau,N}^{\Pi}(\tau) = \int_{\mathcal{Q}} \Pi(dA)\ |W_N^{A/\sqrt{T_\tau}}(\tau)|. \tag{2.2.2}$$

The wrapping lowers the volume of the Wiener sausages, and so we have, recalling (2.1.7) and (2.1.8),

$$S_t \leq E\left(\exp\left[-c\tau V_{\tau,N}^{\Pi}(\tau)\right]\right). \tag{2.2.3}$$

The desired upper bound on S_t will therefore come out of the following LDP:

Theorem 2.2.1 *$(V_{\tau,N}^{\Pi}(\tau))_{\tau>0}$ satisfies the LDP on $(0,\infty)$ with rate τ and with rate function $I_{d,N}^{\Pi}$ given by*

$$I_{d,N}^{\Pi}(b) = \inf\left\{\tfrac{1}{2}\|\nabla\phi\|_2^2\colon\ \phi \in H^1(\Lambda_N),\ \|\phi\|_2^2 = 1,\ F_d^{\Pi}(\phi^2) = b\right\} \tag{2.2.4}$$

with F_d^{Π} given by (1.3.3).

Proof. Any $\Pi \in \mathcal{M}_1^+(\mathcal{Q})$ can be approximated from *below*, in the sense of (1.2.3), by a sequence (Π_n) in $\mathcal{M}_1^+(\mathcal{Q})$ with *finite support*, as in (2.1.11). Such an approximation provides an approximating sequence of upper bounds on S_t. So we may assume that Π has the form (2.1.11). In Section 2.4 we will take care of the continuum limit and show why this carries through.

We follow the three steps indicated in Section 2.1.3.

Step 1:

Proposition 2.2.2 *For any $\delta > 0$,*

$$\lim_{\epsilon\downarrow 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\log P\left(|V_{\tau,N}^{\Pi}(\tau) - \mathbb{E}_{\tau,\epsilon}(V_{\tau,N}^{\Pi}(\tau))| \geq \delta\right) = -\infty. \tag{2.2.5}$$

Proof. For Π of the form (2.1.11), we decompose (2.2.2) as

$$V_{\tau,N}^{\Pi}(\tau) = \sum_{m=1}^{n} a_m |W_N^{A_m/\sqrt{T_\tau}}(\tau)|. \tag{2.2.6}$$

The proof of Proposition 4 in vdBBdH can be copied to show that, for any $\delta > 0$ and $1 \leq m \leq n$,

$$\lim_{\epsilon\downarrow 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\log P\left(\Big|\,|W_N^{A_m/\sqrt{T_\tau}}(\tau)| - \mathbb{E}_{\tau,\epsilon}\left(|W_N^{A_m/\sqrt{T_\tau}}(\tau)|\right)\Big| \geq \delta\right) = -\infty, \tag{2.2.7}$$

which yields the claim. The only property we need to check for (2.2.7) is the analogue of (2.23) in vdBBdH, which plays a pivotal role in the proof and which here reads

$$\sup_{T\geq 1} E\left(\exp\left[\frac{1}{T}|W^{A_m}(T)|\right]\right) < \infty. \tag{2.2.8}$$

Now, the left-hand side is bounded above by the same expression with A_m replaced by $B_R(0)$, where $R = \max_{1\leq m\leq n} R(A_m)$ with $R(A_m)$ the radius of the smallest ball containing A_m centred at 0. But for a ball with an arbitrary finite radius the bound in (2.2.8) is known to be true (see van den Berg and Bolthausen [1]). ∎

Step 2: Let $I_\epsilon^{(2)}\colon \mathcal{M}_1^+(\Lambda_N\times\Lambda_N)\mapsto[0,\infty]$ be the entropy function

$$I_\epsilon^{(2)}(\mu) = \begin{cases} h(\mu|\mu_1\otimes\pi_\epsilon) & \text{if } \mu_1=\mu_2,\\ \infty & \text{otherwise,}\end{cases} \tag{2.2.9}$$

where $h(\cdot\,|\,\cdot)$ denotes relative entropy between measures, μ_1 and μ_2 are the two marginals of μ, and $\pi_\epsilon(x,dy)=p_\epsilon(y-x)dy$ is the Brownian transition kernel on Λ_N associated with an ϵ-excursion. Furthermore, let $\Phi_{1/\epsilon}^\Pi\colon \mathcal{M}_1^+(\Lambda_N\times\Lambda_N)\mapsto[0,\infty)$ be the function

$$\Phi_{1/\epsilon}^\Pi(\mu) = \int_Q \Pi(dA)\int_{\Lambda_N} dx\,\left(1-\exp\left[-\frac{\kappa(A)}{\epsilon}\int_{\Lambda_N\times\Lambda_N}\varphi_\epsilon(y-x,z-x)\mu(dy,dz)\right]\right) \tag{2.2.10}$$

with

$$\varphi_\epsilon(y,z) = \frac{\int_0^\epsilon ds\, p_s(-y)p_{\epsilon-s}(z)}{p_\epsilon(z-y)}. \tag{2.2.11}$$

Proposition 2.2.3 $(\mathbb{E}_{\tau,\epsilon}(V_{\tau,N}^\Pi(\tau)))_{\tau>0}$ *satisfies the LDP on* $(0,\infty)$ *with rate* τ *and with rate function*

$$b\mapsto \inf\left\{\frac{1}{\epsilon}I_\epsilon^{(2)}(\mu)\colon\ \mu\in\mathcal{M}_1^+(\Lambda_N\times\Lambda_N),\ \Phi_{1/\epsilon}^\Pi(\mu)=b\right\}. \tag{2.2.12}$$

Proof. The claim is the analogue of Proposition 5 in vdBBdH. We indicate how the proof is adapted.

First, we fix $K>0$ and cut out holes of radius $K/\sqrt{T_\tau}$ around the endpoints of the ϵ-excursions. To that end, we define

$$W_{i,N}^{A,K} = W_{i,N}^A\setminus\left[B_{K/\sqrt{T_\tau}}(\beta_N((i-1)\epsilon))\cup B_{K/\sqrt{T_\tau}}(\beta_N(i\epsilon))\right] \tag{2.2.13}$$

with

$$W_{i,N}^A = \bigcup_{(i-1)\epsilon\leq s\leq i\epsilon}\left[\beta(s)+A/\sqrt{T_\tau}\right], \tag{2.2.14}$$

and we put

$$V_{\tau,N}^{\Pi,K}(\tau) = \int\Pi(dA)\left|\bigcup_{i=1}^{\tau/\epsilon}W_{i,N}^{A,K}\right|, \tag{2.2.15}$$

which is $V_{\tau,N}^\Pi(\tau)$ in (2.2.2) but with the holes cut out. Note that

$$0\leq V_{\tau,N}^\Pi(\tau)-V_{\tau,N}^{\Pi,K}(\tau)\leq(\tau/\epsilon+1)\,\omega_d(K/\sqrt{T_\tau})^d\leq 2K^d\omega_d/\epsilon T_\tau \tag{2.2.16}$$

(recall (2.1.6); ω_d is the volume of the ball with unit radius). The right-hand side tends to zero as $\tau \to \infty$ for any $K < \infty$, so the cutting is harmless when we let $K \to \infty$ afterwards.

Next, we express $\mathbb{E}_{\tau,\epsilon}(V^{\Pi,K}_{\tau,N}(\tau))$ in terms of the empirical measure $L_{\tau,\epsilon}$ defined in (2.1.10):

$$
\begin{aligned}
&\mathbb{E}_{\tau,\epsilon}\left(V^{\Pi,K}_{\tau,N}(\tau)\right)\\
&\quad= \int_{\mathcal{Q}} \Pi(dA) \int_{\Lambda_N} dx \left(1 - \mathbb{P}_{\tau,\epsilon}\left(x \notin \bigcup_{i=1}^{\tau/\epsilon} W^{A,K}_{i,N}\right)\right)\\
&\quad= \int_{\mathcal{Q}} \Pi(dA) \int_{\Lambda_N} dx \left(1 - \prod_{i=1}^{\tau/\epsilon}\left\{1 - \mathbb{P}_{\tau,\epsilon}\left(x \in W^{A,K}_{i,N}\right)\right\}\right)\\
&\quad= \int_{\mathcal{Q}} \Pi(dA) \int_{\Lambda_N} dx\\
&\qquad \left(1 - \exp\left[\frac{\tau}{\epsilon}\int_{\Lambda_N\times\Lambda_N} \log\left(1 - q^A_{\tau,\epsilon}(y-x,z-x)\,1_{\{y-x,z-x\notin B_{K/\sqrt{T_\tau}}(0)\}}\right) L_{\tau,\epsilon}(dy,dz)\right]\right).
\end{aligned}
\tag{2.2.17}
$$

Here,

$$
q^A_{\tau,\epsilon}(y,z) = P_{\epsilon,y,z}\left(\sigma_{A/\sqrt{T_\tau}} \le \epsilon\right) \tag{2.2.18}
$$

with $\sigma_{A/\sqrt{T_\tau}}$ the first time the Brownian motion enters $A/\sqrt{T_\tau}$, and

$$
P_{\epsilon,y,z}(\cdot) = P\big(\beta_N([0,\epsilon]) \in \cdot \mid \beta_N(0) = y, \beta_N(\epsilon) = z\big) \tag{2.2.19}
$$

the probability law of the Brownian bridge of length ϵ from y to z.

The key property of the quantity in (2.2.18) needed in the proof is the following analogue of Lemma 2 in vdBBdH:

$$
\begin{aligned}
&\text{(a)} \quad \lim_{K\to\infty} \limsup_{\tau\to\infty} \sup_{y,z\notin B_{K/\sqrt{T_\tau}}(0)} q^A_{\tau,\epsilon}(y,z) = 0 \text{ for all } A \in \mathcal{Q},\ \epsilon > 0,\\
&\text{(b)} \quad \lim_{\tau\to\infty} \sup_{y,z\notin B_\rho(0)} |\tau q^A_{\tau,\epsilon}(y,z) - \kappa(A)\varphi_\epsilon(y,z)| = 0 \text{ for all } 0 < \rho < N/4 \text{ and } A \in \mathcal{Q},\ \epsilon > 0.
\end{aligned}
\tag{2.2.20}
$$

Property (2.2.20)(a) is immediate, since $q^A_{\tau,\epsilon}(y,z)$ is non-decreasing in A and for $A = B_R(0)$ the proof is in vdBBdH. For property (2.2.20)(b) the key ingredient is the analogue of (2.64) in vdBBdH, which reads

$$
\lim_{b\downarrow 0} \frac{1}{\kappa(bA)} P_y\left(\sigma_{bA} \le t\right) = \int_0^t p_s(-y)ds \quad \text{for all } y \in \mathbb{R}^d,\ t \ge 0,\ A \in \mathcal{Q} \tag{2.2.21}
$$

with $P_y(\cdot) = P_y(\beta([0,\infty)) \in \cdot \mid \beta(0) = y)$ (see Le Gall [5]). It is through this relation that $\kappa(A)$ appears on the stage.

Next, (2.2.20) allows us to linearise the logarithm in the last line of (2.2.17) and to replace it by $-\kappa(A)\varphi_\epsilon(y-x,z-x)/\tau$, which brings us to (2.2.10) with $\mu = L_{\tau\epsilon}$. To do this properly we need some continuity properties, which are the analogues of Lemmas 3 and 4 in vdBBdH and which rely on (2.2.20)(b). Since Π has finite support, this part of the extension is again straightforward.

The combination of (2.2.16), (2.2.17) and (2.2.20) leads us to the conclusion that

$$
\lim_{\tau\to\infty} \left\|\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau)) - \Phi^{\Pi}_{1/\epsilon}(L_{\tau,\epsilon})\right\|_\infty = 0 \quad \text{for all } \epsilon > 0. \tag{2.2.22}
$$

Finally, we note the following:

(1) $\mu \mapsto \Phi^{\Pi}_{1/\epsilon}(\mu)$ is continuous in the total variation norm.

(2) $(L_{\tau,\epsilon})_{\tau>0}$ satisfies the LDP on $\mathcal{M}_1^+(\Lambda_N \times \Lambda_N)$ with rate τ and with rate function $\frac{1}{\epsilon}I^{(2)}_\epsilon$.

Therefore the claim in Proposition 2.2.3 now follows by using the contraction principle in combination with (2.2.22). ∎

Step 3: This step consists of two approximation lemmas.

• Let $\Psi^{\Pi}_{1/\epsilon}\colon \mathcal{M}_1^+(\Lambda_N) \mapsto [0,\infty)$ be the function

$$\Psi^{\Pi}_{1/\epsilon}(\nu) = \int_Q \Pi(dA) \int_{\Lambda_N} dx \left(1 - \exp\left[-\frac{\kappa(A)}{\epsilon}\int_0^\epsilon ds \int_{\Lambda_N} p_s(x-y)\nu(dy)\right]\right). \tag{2.2.23}$$

Lemma 2.2.4 *For any $K>0$,*

$$\lim_{\epsilon\downarrow 0} \sup_{\mu\colon\, \frac{1}{\epsilon}I^{(2)}_\epsilon(\mu)\le K} \left|\Phi^{\Pi}_{1/\epsilon}(\mu) - \Psi^{\Pi}_{1/\epsilon}(\mu_1)\right| = 0. \tag{2.2.24}$$

Proof. For Π of the form (2.1.11), we decompose (2.2.10) and (2.2.23) as

$$\Phi^{\Pi}_{1/\epsilon}(\mu) = \sum_{m=1}^n a_m \Phi^{\delta_{A_m}}_{1/\epsilon}(\mu), \qquad \Psi^{\Pi}_{1/\epsilon}(\nu) = \sum_{m=1}^n a_m \Psi^{\delta_{A_m}}_{1/\epsilon}(\nu). \tag{2.2.25}$$

The proof of Lemma 6 in vdBBdH can be copied to show that, for any $K>0$ and $1\le m\le n$,

$$\lim_{\epsilon\downarrow 0} \sup_{\mu\colon\, \frac{1}{\epsilon}I^{(2)}_\epsilon(\mu)\le K} \left|\Phi^{\delta_{A_m}}_{1/\epsilon}(\mu) - \Psi^{\delta_{A_m}}_{1/\epsilon}(\mu_1)\right| = 0, \tag{2.2.26}$$

which yields the claim. The only property needed for the proof of (2.2.26) is $\kappa(A_m)<\infty$. ∎

• Let $I\colon \mathcal{M}_1^+(\Lambda_N) \mapsto [0,\infty]$ be the standard large deviation rate function for the empirical distribution of the Brownian motion, given by

$$\begin{aligned} I(\nu) &= \tfrac{1}{2}\textstyle\int_{\Lambda_N} |\nabla\phi|^2(x)dx && \text{if } \tfrac{d\nu}{dx} = \phi^2 \text{ with } \phi\in H^1(\Lambda_N),\\ &= \infty && \text{otherwise.}\end{aligned} \tag{2.2.27}$$

Let $I_\epsilon\colon \mathcal{M}_1^+(\Lambda_N) \mapsto [0,\infty]$ be the projection of $I^{(2)}_\epsilon$ onto $\mathcal{M}_1^+(\Lambda_N)$, given by

$$I_\epsilon(\nu) = \inf\left\{I^{(2)}_\epsilon(\mu)\colon\ \mu_1 = \nu\right\}. \tag{2.2.28}$$

Then

$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} I_\epsilon(\nu) = I(\nu) \qquad \text{for all } \nu\in\mathcal{M}_1^+(\Lambda_N) \tag{2.2.29}$$

(see vdBBdH Lemma 5).

Lemma 2.2.5 *For any $K>0$,*

$$\lim_{\epsilon\downarrow 0} \sup_{\nu\colon\, \frac{1}{\epsilon}I_\epsilon(\nu)\le K} \left|\Psi^{\Pi}_{1/\epsilon}(\nu) - F^{\Pi}_d\left(\tfrac{d\nu}{dx}\right)\right| = 0 \tag{2.2.30}$$

with F^{Π}_d given by (1.3.3)*. (Note that if $I_\epsilon(\nu)<\infty$, then $d\nu \ll dx$ because $\nu\otimes\pi_\epsilon \ll dx\otimes dy$ by* (2.2.9) *and* (2.2.28)*).*

Proof. For Π of the form (2.1.11), we decompose (2.2.23) and (1.3.3) as

$$\Psi^{\Pi}_{1/\epsilon}(\nu) = \sum_{m=1}^{n} a_m \Psi^{\delta_{A_m}}_{1/\epsilon}(\nu), \qquad F^{\Pi}_d(\phi^2) = \sum_{m=1}^{n} a_m F^{\delta_{A_m}}_d(\phi^2). \tag{2.2.31}$$

The proof of Lemma 7 in vdBBdH can be copied to show that, for any $K > 0$ and $1 \leq m \leq n$,

$$\lim_{\epsilon\downarrow 0} \sup_{\nu:\ \frac{1}{\epsilon}I_\epsilon(\nu)\leq K} \left| \Psi^{\delta_{A_m}}_{1/\epsilon}(\nu) - F^{\delta_{A_m}}_d\left(\tfrac{d\nu}{dx}\right)\right| = 0, \tag{2.2.32}$$

which yields the claim. The only property needed for the proof of (2.2.32) is $\kappa(A_m) < \infty$. ∎

Having completed Steps 1–3, the proof of Theorem 2.2.1 now follows easily. Indeed, for any $f\colon (0,\infty) \mapsto \mathbb{R}$ bounded and continuous we have

$$\begin{aligned}
&\lim_{\tau\to\infty} \frac{1}{\tau} \log E\Big(\exp\left[\tau f(V^{\Pi}_{\tau,N}(\tau))\right]\Big) \\
&\quad = \lim_{\epsilon\downarrow 0} \lim_{\tau\to\infty} \frac{1}{\tau} \log E\left(\exp\left[\tau f(\mathbb{E}_{\tau,\epsilon}(V^{\Pi}_{\tau,N}(\tau)))\right]\right) \\
&\quad = \lim_{\epsilon\downarrow 0} \sup_{\mu} \left\{ f(\Phi^{\Pi}_{1/\epsilon}(\mu)) - \frac{1}{\epsilon} I^{(2)}_\epsilon(\mu)\right\} \\
&\quad = \lim_{K\to\infty} \lim_{\epsilon\downarrow 0} \sup_{\mu:\ \frac{1}{\epsilon}I^{(2)}_\epsilon(\mu)\leq K} \left\{ f(\Phi^{\Pi}_{1/\epsilon}(\mu)) - \frac{1}{\epsilon} I^{(2)}_\epsilon(\mu)\right\} \\
&\quad = \lim_{K\to\infty} \lim_{\epsilon\downarrow 0} \sup_{\mu:\ \frac{1}{\epsilon}I^{(2)}_\epsilon(\mu)\leq K} \left\{ f(\Psi^{\Pi}_{1/\epsilon}(\mu_1)) - \frac{1}{\epsilon} I^{(2)}_\epsilon(\mu)\right\} \\
&\quad = \lim_{K\to\infty} \lim_{\epsilon\downarrow 0} \sup_{\nu:\ \frac{1}{\epsilon}I_\epsilon(\nu)\leq K} \left\{ f(\Psi^{\Pi}_{1/\epsilon}(\nu)) - \frac{1}{\epsilon} I_\epsilon(\nu)\right\} \\
&\quad = \lim_{K\to\infty} \lim_{\epsilon\downarrow 0} \sup_{\nu:\ \frac{1}{\epsilon}I_\epsilon(\nu)\leq K} \left\{ f\left(F^{\Pi}_d\left(\frac{d\nu}{dx}\right)\right) - \frac{1}{\epsilon} I_\epsilon(\nu)\right\} \\
&\quad = \sup_{\nu} \left\{ f\left(F^{\Pi}_d\left(\frac{d\nu}{dx}\right)\right) - I(\nu)\right\} \\
&\quad = \sup_{\phi\in H^1(\Lambda_N):\ \|\phi\|_2^2=1} \left\{ f(F^{\Pi}_d(\phi^2)) - \frac{1}{2}\|\nabla\phi\|_2^2\right\}.
\end{aligned} \tag{2.2.33}$$

Here, the first equality uses Proposition 2.2.2, the second equality Proposition 2.2.3, the fourth equality Lemma 2.2.4, the fifth equality (2.2.29), the sixth equality Lemma 2.2.5, while the last equality comes from (2.2.27). The claim in Theorem 2.2.1 follows by applying to (2.2.33) the inverse of Varadhan's lemma due to Bryc [4]. ∎

It follows from (2.2.3) and Theorem 2.2.1 that

$$\limsup_{\tau\to\infty} \frac{1}{\tau} \log S_t \leq -J^{\Pi}_{d,N}(c) \tag{2.2.34}$$

with

$$\begin{aligned}
J^{\Pi}_{d,N}(c) &= \inf\Big\{ cb + I^{\Pi}_{d,N}(b)\colon\ b\in(0,\infty)\Big\} \\
&= \inf\left\{ \tfrac{1}{2}\|\nabla\phi\|_2^2 + cF^{\Pi}_d(\phi^2)\colon\ \phi\in H^1(\Lambda_N),\ \|\phi\|_2^2 = 1\right\}.
\end{aligned} \tag{2.2.35}$$

This is the same as (1.3.2), but with $\mathbb{R}^d$ replaced by Λ_N. Thus, to complete the proof of the upper bound for Π with finite support it suffices to show that

$$\lim_{N\to\infty} J^{\Pi}_{d,N}(c) = J^{\Pi}_d(c). \tag{2.2.36}$$

The latter is a standard exercise, for which the reader is referred to vdBBdH Section 2.6.

2.3 Lower bound in $d \geq 3$

If Π has unbounded support, then it cannot be approximated from *above*, in the sense of (1.2.3), by a sequence (Π_n) with finite support. However, we can truncate and estimate (2.2.2) by

$$V^{\Pi}_{\tau,N}(\tau) \leq V^{\Pi}_{\tau,N,R}(\tau) + \delta_R\,|\Lambda_N|, \qquad R > 0, \tag{2.3.1}$$

where

$$V^{\Pi}_{\tau,N,R} = \int_{\mathcal{Q}} \Pi(dA)\ 1_{\{R(A)\leq R\}}\ |W_N^{A/\sqrt{T_\tau}}(\tau)|, \tag{2.3.2}$$

with $R(A)$ the radius of the smallest ball containing A centred at 0, and $\delta_R = \Pi(\{A \in \mathcal{Q}\colon\ R(A) > R\})$. Since $\lim_{R\to\infty}\delta_R = 0$, it suffices to give the proof for $V^{\Pi}_{\tau,N,R}(\tau)$. The point is that Π restricted to $\{R(A) \leq R\}$ can be approximated by a sequence with finite support. Thus, we may again assume that Π has the form (2.1.11).

A lower bound on the survival probability is obtained by killing the Brownian motion at $\partial\Lambda_{N-R/\sqrt{T_\tau}}$. Therefore we have from (2.3.1), recalling (2.1.7) and (2.1.8), that

$$S_t \geq e^{-c\tau\delta_R|\Lambda_N|}\, E\left(\exp\left[-c\tau V^{\Pi}_{\tau,N,R}(\tau)\right]\ 1_{C_{N,R}(\tau)}\right), \tag{2.3.3}$$

where $C_{N,R}(\tau)$ is the event that the Brownian motion does not hit $\partial\Lambda_{N-R/\sqrt{T_\tau}}$ until time τ. On the event $C_{N,R}(\tau)$ we can use Theorem 2.2.1, which leads us to

$$\begin{aligned}\liminf_{\tau\to\infty}\frac{1}{\tau}\log S_t &\geq -c\delta_R|\Lambda_N| - \lambda_N + \lim_{\tau\to\infty}\frac{1}{\tau}\log E\left(\exp\left[-c\tau V^{\Pi}_{\tau,N}(\tau)\right] \mid C_{N,R}(\tau)\right)\\ &= -c\delta_R|\Lambda_N| - \lambda_N - J^{\Pi}_{d,N,*}(c).\end{aligned} \tag{2.3.4}$$

Here, λ_N is the principal Dirichlet eigenvalue of $-\Delta/2$ on Λ_N, while $J^{\Pi}_{d,N,*}(c)$ is given by (2.2.4), except that ϕ has the additional restriction $\mathrm{supp}(\phi)\cap\partial\Lambda_N = \emptyset$. First let $R \to \infty$ to get rid of the first term in the right-hand side of (2.3.4). Next let $N \to \infty$ and use that $\lim_{N\to\infty}\lambda_N = 0$ to see that it suffices to show that

$$\lim_{N\to\infty} J^{\Pi}_{d,N,*}(c) = J^{\Pi}_d(c). \tag{2.3.5}$$

The latter is again a standard exercise, for which the reader is referred to vdBBdH Section 2.6.

2.4 Continuum limit of Π

The bounds in (2.2.34) and (2.3.4) in combination with the limits in (2.2.36) and (2.3.5) yield Theorem 1.3.1 for all Π with finite support, as assumed in (2.1.11). It therefore remains to show that the variational characterisation of J_d^Π in (1.3.2) is stable under the continuum limit on Π. In order to do so, we note that

$$\{A_n, A \in \mathcal{Q},\ \rho_H(A_n, A) \to 0\} \quad \Longrightarrow \quad \{\kappa(A_n) \to \kappa(A)\}. \tag{2.4.1}$$

Consequently,

$$\{\Pi_n \Rightarrow \Pi \text{ in } \mathcal{M}_1^+(\mathcal{Q})\} \quad \Longrightarrow \quad \{\Theta_n \Rightarrow \Theta \text{ in } \mathcal{M}_1^+((0,\infty))\} \tag{2.4.2}$$

with $\Theta = \Pi \circ \kappa^{-1}$. Hence it suffices to show stability under the continuum limit on Θ. This is again a standard exercise. We refer the reader to the analysis in Section 3.

2.5 Extension to $d = 2$

The extension to $d = 2$ is minor and follows vdBBdH Section 4. The ingredients (2.2.8), (2.2.20) and (2.2.21) need to be properly modified, for which we refer to (4.8), Lemma 8 and (4.14) in vdBBdH, respectively. The rest of the argument is the same, with the notation introduced in (2.1.6).

3 Proof of Theorems 1.4.1, 1.4.2 and 1.4.3

3.1 Proof of Theorem 1.4.1(i) and Theorem 1.4.3(i)

According to (1.3.2) and (1.3.6), $c \mapsto J_d^\Pi(c)$ and $c \mapsto J_2(c)$ are infima over functions that are linear. Consequently, both are concave, and therefore also continuous except possibly at the boundary point $c = 0$. It is obvious that $J_d^\Pi(0) = J_2(0) = 0$. From the general upper bound proved in (3.6.5) below, it follows that $\lim_{c\downarrow 0} J_d^\Pi(c) = \lim_{c\downarrow 0} J_2(c) = 0$. Therefore continuity extends to the boundary. It is further obvious from (1.3.2) and (1.3.6) that $J_d^\Pi(c)$ and $J_2(c)$ are non-decreasing in c. By concavity, both are strictly increasing in c unless they are constant from some finite c onwards. But this is ruled out by the asymptotics for $c \to \infty$ in Theorem 1.4.2(i) and Theorem 1.4.3(iii) respectively.

3.2 Proof of Theorem 1.4.1(ii)

Lemma 3.2.1 *Let $d \geq 3$. Then $J_d^\Pi(c) \leq c\langle\kappa\rangle$ for all $c \geq 0$.*

Proof. Since $1 - e^{-x} \leq x$, $x \geq 0$, we have from (1.3.3) that $F_d^\Pi(\phi^2) \leq \langle\kappa\rangle\|\phi\|_2^2$. Hence the claim follows from (1.3.2), since $\inf\{\|\nabla\phi\|_2^2\colon\ \|\phi\|_2^2 = 1\} = 0$. ∎

The critical value c^* is the unique threshold such that $J_d^\Pi(c) < c\langle\kappa\rangle$ if and only if $c > c^*$. In Lemma 3.2.2 below we derive a lower bound on c^* in regime (I). To do so, we first rewrite (1.3.2) as

$$c\langle\kappa\rangle - J_d^\Pi(c) = -\inf\left\{\tfrac{1}{2}\|\nabla\phi\|_2^2 - c\,G_d^\Pi(\phi^2)\colon\ \|\phi\|_2^2 = 1,\ \phi \text{ RSNI}\right\}, \tag{3.2.1}$$

where RSNI means radially symmetric and non-increasing (see vdBBdH Lemma 10), and

$$G_d^\Pi(\phi^2) = \int_{\mathbb{R}^d} dx \int_0^\infty \Theta(d\kappa)\, \left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right). \tag{3.2.2}$$

From (3.2.1) we see that

$$c^* = \inf\left\{\frac{\frac{1}{2}\|\nabla\phi\|_2^2}{G_d^{\Pi}(\phi^2)}\colon\ \|\phi\|_2^2 = 1, \phi \text{ RSNI}\right\}. \tag{3.2.3}$$

Lemma 3.2.2 *Let $d \geq 3$. If $\Theta \in \mathcal{S}_I$, then*

$$c^* \geq S_d \left(4\kappa_0^{d/(d-1)} + \frac{d^2 K}{d-2}\right)^{-1} \tag{3.2.4}$$

where S_d is the Sobolev constant given in (3.2.15)*.*

Proof. We estimate the contribution to the double integral in (3.2.2) as follows.

First let $A < \infty$. The contribution of the rectangle $(0, \kappa_0) \times \{x \in \mathbb{R}^d\colon\ \phi^2(x) < A\}$ is bounded from above by

$$\int_0^{\kappa_0} \Theta(d\kappa) \int_{\{\phi^2 < A\}} dx\, \kappa^2 \phi^4(x) \leq \kappa_0^2 A^{2(d-2)/d} \int_{\mathbb{R}^d} dx\, \phi^{2(d+2)/d}(x), \tag{3.2.5}$$

where we have used that $e^{-x} - 1 + x \leq x^2$, $x \geq 0$. On the other hand, the contribution of the rectangle $(0, \kappa_0) \times \{x \in \mathbb{R}^d\colon\ \phi^2(x) \geq A\}$ is bounded from above by

$$\begin{aligned}\int_0^{\kappa_0} \Theta(d\kappa) \int_{\{\phi^2 \geq A\}} dx\, \kappa\phi^2(x) &\leq \kappa_0 \int_{\{\phi^2 \geq A\}} dx\, \phi^2(x) \left(\frac{\phi^2(x)}{A}\right)^{2/d} \\ &\leq \kappa_0 A^{-2/d} \int_{\mathbb{R}^d} dx\, \phi^{2(d+2)/d}(x).\end{aligned} \tag{3.2.6}$$

We choose $A = \kappa_0^{-d/2(d-1)}$ to get from (3.2.5) and (3.2.6) that the contribution of $(0, \kappa_0)$ is bounded from above by

$$2\kappa_0^{d/(d-1)} \int_{\mathbb{R}^d} dx\, \phi^{2(d+2)/d}(x). \tag{3.2.7}$$

Next, the contribution of the rectangle $[\kappa_0, \infty) \times \{x \in \mathbb{R}^d\colon\ \phi^2(x) < 1/\kappa_0\}$ is bounded from above by

$$\begin{aligned}&\int_{\{\phi^2 < 1/\kappa_0\}} dx \int_{\kappa_0}^{1/\phi^2(x)} \Theta(d\kappa)\, \kappa^2 \phi^4(x) + \int_{\{\phi^2 < 1/\kappa_0\}} dx \int_{1/\phi^2(x)}^{\infty} \Theta(d\kappa)\, \kappa\phi^2(x) \\ &\leq K \int_{\{\phi^2 < 1/\kappa_0\}} dx \int_{\kappa_0}^{1/\phi^2(x)} d\kappa\, \kappa^{-2/d} \phi^4(x) + K \int_{\{\phi^2 < 1/\kappa_0\}} dx \int_{1/\phi^2(x)}^{\infty} d\kappa\, \kappa^{-(d+2)/d} \phi^2(x) \\ &\leq K \int_{\{\phi^2 < 1/\kappa_0\}} dx\, \phi^4(x) \int_0^{1/\phi^2(x)} d\kappa\, \kappa^{-2/d} + K \int_{\{\phi^2 < 1/\kappa_0\}} dx\, \phi^2(x) \int_{1/\phi^2(x)}^{\infty} d\kappa\, \kappa^{-(d+2)/d} \\ &= \frac{d^2 K}{2(d-2)} \int_{\{\phi^2 < 1/\kappa_0\}} dx\, \phi^{2(d+2)/d}(x),\end{aligned} \tag{3.2.8}$$

where we have used the upper bound on $\Theta(d\kappa)$ that defines $\mathcal{S}_I$. On the other hand, the contribution of the rectangle $[\kappa_0,\infty)\times\{x\in\mathbb{R}^d\colon\ \phi^2(x)\geq 1/\kappa_0\}$ is bounded from above by

$$\begin{aligned}\int_{\{\phi^2\geq 1/\kappa_0\}} dx \int_{\kappa_0}^{\infty}\Theta(d\kappa)\,\kappa\phi^2(x) &\leq K\int_{\{\phi^2\geq 1/\kappa_0\}} dx\,\phi^2(x)\int_{\kappa_0}^{\infty} d\kappa\,\kappa^{-(d+2)/d}\\ &= \frac{dK}{2}\int_{\{\phi^2\geq 1/\kappa_0\}} dx\,\phi^2(x)\,\kappa_0^{-2/d}\\ &\leq \frac{dK}{2}\int_{\{\phi^2\geq 1/\kappa_0\}} dx\,\phi^{2(d+2)/d}(x).\end{aligned}\tag{3.2.9}$$

Combining (3.2.7), (3.2.8) and (3.2.9), we arrive at

$$G_d^{\Pi}(\phi^2)\leq\left(2\kappa_0^{d/(d-1)}+\frac{d^2K}{2(d-2)}\right)\int_{\mathbb{R}^d} dx\,\phi^{2(d+2)/d}(x).\tag{3.2.10}$$

Next, for any $0<\alpha<1$ and conjugate exponents $p,q>1$, we estimate

$$\int\phi^{2(d+2)/d}\leq\left(\int\phi^{[2(d+2)/d]\alpha p}\right)^{1/p}\left(\int\phi^{[2(d+2)/d](1-\alpha)q}\right)^{1/q}.\tag{3.2.11}$$

Choosing α,p,q such that

$$[2(d+2)/d]\alpha p=2d/(d-2),\quad [2(d+2)/d](1-\alpha)q=2,\tag{3.2.12}$$

i.e.,

$$p=d/(d-2),\quad q=d/2,\quad \alpha=d/(d+2),\tag{3.2.13}$$

and using that $\|\phi\|_2^2=1$, we obtain

$$\int\phi^{2(d+2)/d}\leq\left(\int\phi^{2d/(d-2)}\right)^{(d-2)/d}=\|\phi\|_{2d/(d-2)}^2.\tag{3.2.14}$$

By the Sobolev inequality (see Lieb and Loss [6] page 190)

$$\|\nabla\phi\|_2^2\geq S_d\,\|\phi\|_{2d/(d-2)}^2\tag{3.2.15}$$

we obtain that

$$\int\phi^{2(d+2)/d}\leq\frac{1}{S_d}\|\nabla\phi\|_2^2.\tag{3.2.16}$$

We obtain the claim by (3.2.3), (3.2.10) and (3.2.16). ∎

We proceed by proving the slope discontinuity at c^*.

Lemma 3.2.3 *Let $d\geq 3$. For $\Theta\in\mathcal{S}_I$, if $\langle\kappa^\eta\rangle<\infty$ for some $\eta>\frac{d+2}{d}$, then $\lim_{c\downarrow c^*}[J_d^{\Pi}(c)-J_d^{\Pi}(c^*)]/(c-c^*)<\langle\kappa\rangle$.*

Proof. Let ψ_{c^*} be any minimiser for (1.3.2) at $c=c^*$, the existence of which we prove in Lemma 3.5.3 below under the condition stated. Then

$$J_d^{\Pi}(c^*)=\tfrac12\|\nabla\psi_{c^*}\|_2^2+c^*\,F_d^{\Pi}(\psi_{c^*}^2).\tag{3.2.17}$$

But, for any $\delta>0$, we have

$$J_d^{\Pi}(c^*+\delta)\leq\tfrac12\|\nabla\psi_{c^*}\|_2^2+(c^*+\delta)F_d^{\Pi}(\psi_{c^*}^2).\tag{3.2.18}$$

Combining this with (3.2.17), we get

$$\frac{1}{\delta}\left[J_d^{\Pi}(c^*+\delta)-J_d^{\Pi}(c^*)\right]\leq F_d^{\Pi}(\psi_{c^*}^2)<\frac{1}{c^*}J_d^{\Pi}(c^*)=\langle\kappa\rangle.\tag{3.2.19}$$

∎

3.3 Proof of Theorem 1.4.1(iii)

Lemma 3.3.1 *Let $d \geq 3$. If $\Theta \in \mathcal{S}_{II}$, then $\lim_{c\downarrow 0} \frac{1}{c} J_d^{\Pi}(c) = \langle \kappa \rangle$.*

Proof. As shown in Lemma 3.5.2 below, for all $c > 0$ we have the existence of a minimiser for (1.3.2), say ψ_c. Hence

$$\frac{1}{c} J_d^{\Pi}(c) = \frac{1}{c} \|\nabla \psi_c\|_2^2 + \int_{\mathbb{R}^d} dx \int_0^\infty \Theta(d\kappa) \left(1 - e^{-\kappa \psi_c^2(x)}\right). \tag{3.3.1}$$

Let $\epsilon > 0$ and $R < \infty$. Then, since $e^{-x} - 1 + x \leq \frac{1}{2}x^2$, $x \geq 0$, we have

$$\begin{aligned}
\frac{1}{c} J_d^{\Pi}(c) &\geq \int_{\{\psi_c^2 \leq \epsilon\}} dx \int_{\{\kappa \leq R\}} \Theta(d\kappa) \left(1 - e^{-\kappa \psi_c^2(x)}\right) \\
&\geq \int_{\{\psi_c^2 \leq \epsilon\}} dx \int_{\{\kappa \leq R\}} \Theta(d\kappa) \left(\kappa \psi_c^2(x) - \frac{1}{2}\kappa^2 \psi_c^4(x)\right) \\
&\geq \int_{\{\psi_c^2 \leq \epsilon\}} dx\, \psi_c^2(x) \int_{\{\kappa \leq R\}} \Theta(d\kappa)\, \kappa - \frac{1}{2} R^2 \epsilon,
\end{aligned} \tag{3.3.2}$$

where we have used that $\|\psi_c\|_2^2 = 1$. We will show that, for any $\epsilon > 0$,

$$\lim_{c \downarrow 0} \int_{\{\psi_c^2 > \epsilon\}} dx\, \psi_c^2(x) = 0. \tag{3.3.3}$$

Combining this with (3.3.2) and again using that $\|\psi_c\|_2^2 = 1$, we obtain

$$\liminf_{c \downarrow 0} \frac{1}{c} J_d^{\Pi}(c) \geq \int_{\{\kappa \leq R\}} \Theta(d\kappa)\, \kappa - \frac{1}{2} R^2 \epsilon. \tag{3.3.4}$$

By letting $\epsilon \downarrow 0$ and then letting $R \to \infty$, we arrive at

$$\liminf_{c \downarrow 0} \frac{1}{c} J_d^{\Pi}(c) \geq \langle \kappa \rangle. \tag{3.3.5}$$

This proves the claim, since we already know from Lemma 3.2.1 that $\frac{1}{c} J_d^{\Pi}(c) \leq \langle \kappa \rangle$.

It remains to prove (3.3.3). We have

$$\begin{aligned}
\int_{\{\psi_c^2 > \epsilon\}} dx\, \psi_c^2(x) &\leq \int_{\{\psi_c^2 > \epsilon\}} dx\, \psi_c^2(x) \left(\frac{\psi_c^2(x)}{\epsilon}\right)^{2/(d-2)} \\
&\leq \epsilon^{-2/(d-2)} \int_{\mathbb{R}^d} dx\, \psi_c^{2d/(d-2)}(x) \\
&\leq \epsilon^{-2/(d-2)} S_d^{-d/(d-2)} \|\nabla \psi_c\|_2^{2d/(d-2)},
\end{aligned} \tag{3.3.6}$$

where we have used the Sobolev inequality (3.2.15). But $\lim_{c\downarrow 0} J_d^{\Pi}(c) = 0$ by Lemma 3.2.1, and therefore $\lim_{c \downarrow 0} \|\nabla \psi_c\|_2^2 = 0$. Consequently, (3.3.6) implies (3.3.3). ∎

Lemma 3.3.2 *Let $d \geq 3$. If $\Theta \in \mathcal{S}_{II}$, then $c^* = 0$.*

Proof. By (3.2.2) and the lower bound on $\Theta(d\kappa)$ that defines $\mathcal{S}_{II}$, we have

$$G_d^\Pi(\phi^2) \geq \int_{\mathbb{R}^d} dx \int_{\kappa_1}^\infty d\kappa\, L(\kappa)\kappa^{-1-(d+2)/d}\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right). \tag{3.3.7}$$

Hence we get, for all $\phi \in H^1(\mathbb{R}^d)$ that are RSNI with $\|\phi\|_2^2 = 1$ and $\kappa_1\phi^2(0) \leq 1$,

$$\begin{aligned} G_d^\Pi(\phi^2) &\geq \int_{\mathbb{R}^d} dx \int_{1/\phi^2(x)}^\infty d\kappa\, L(\kappa)\kappa^{-1-(d+2)/d}\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right)\\ &\geq L\left(\frac{1}{\phi^2(0)}\right)\int_{\mathbb{R}^d} dx \int_{1/\phi^2(x)}^\infty d\kappa\, \kappa^{-1-(d+2)/d}\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right)\\ &\geq L\left(\frac{1}{\phi^2(0)}\right)\int_{\mathbb{R}^d} dx \int_{1/\phi^2(x)}^\infty d\kappa\, \kappa^{-1-(d+2)/d}\left(\frac{\kappa\phi^2(x)}{e}\right)\\ &= \frac{d}{2e}L\left(\frac{1}{\phi^2(0)}\right)\int_{\mathbb{R}^d} dx\, \phi^{2(d+2)/d}(x), \end{aligned} \tag{3.3.8}$$

where we have used that $e^{-x} - 1 + x \geq x/e$, $x \geq 1$. Inserting (3.3.8) into (3.2.3), we find

$$c^* \leq \frac{e}{d}\inf\left\{\frac{1}{L(1/\phi^2(0))}\frac{\|\nabla\phi\|_2^2}{\int\phi^{2(d+2)/d}}\colon\ \|\phi\|_2^2 = 1,\ \phi \text{ RSNI},\ \kappa_1\phi^2(0) \leq 1\right\}. \tag{3.3.9}$$

The choice

$$\phi(x) = \epsilon^{d/2}e^{-\pi\epsilon^2|x|^2/2}, \qquad \epsilon > 0, \tag{3.3.10}$$

yields that, for all $0 < \epsilon \leq \kappa_1^{-1/d}$,

$$c^* \leq \frac{e\pi}{2}[(d+2)/d]^{d/2}\frac{1}{L(\epsilon^{-d})}. \tag{3.3.11}$$

We obtain the claim by letting $\epsilon \downarrow 0$ and using that $\lim_{\kappa\to\infty} L(\kappa) = \infty$. ∎

3.4 Proof of Theorem 1.4.1(iv)

Lemma 3.4.1 *Let* $d \geq 3$. *In regime* (III), $\lim_{c\downarrow 0}\frac{1}{c}J_d^\Pi(c) = \infty$.

Proof. The limit in (3.3.3) and the lower bound in (3.3.4) are valid also in regime (III). Therefore the claim is immediate from $\langle\kappa\rangle = \infty$. ∎

3.5 Proof of Theorem 1.4.1(v)

Lemma 3.5.1 *Let* $d \geq 3$. *In regime* (III), (1.3.2) *has a minimiser for all* $c > 0$, *which is* RSNI.

Proof. Fix $c > 0$, and let (ψ_j) be a minimising sequence for the variational problem in (1.3.2). We can extract a subsequence, also denoted by (ψ_j), such that $\psi_j \to \psi_c$ as $j \to \infty$ for some ψ_c almost everywhere and in $D^1(\mathbb{R}^d)$. It follows that ψ_c is RSNI, and that

$$J_d^\Pi(c) \geq \frac{1}{2}\|\nabla\psi_c\|_2^2 + cF_d^\Pi(\psi_c^2). \tag{3.5.1}$$

If we manage to show that $\|\psi_c\|_2^2 = 1$, then

$$J_d^\Pi(c) \le \frac{1}{2}\|\nabla\psi_c\|_2^2 + cF_d^\Pi(\psi_c^2), \tag{3.5.2}$$

and we may conclude that ψ_c is a minimiser.

To prove that $\|\psi_c\|_2^2 = 1$, we let $\epsilon \in (0,1)$ be arbitrary and use that

$$J_d^\Pi(c) \le \frac{d+2}{2}\left(\frac{\lambda_d}{d}\right)^{d/(d+2)} c^{2/(d+2)}, \tag{3.5.3}$$

which will be proved in (3.6.5) below. Since $\langle\kappa\rangle = \infty$, there exists a $\kappa_\epsilon < \infty$ such that

$$\frac{4}{c}\left(\int_0^{\kappa_\epsilon} \kappa\Theta(d\kappa)\right)^{-1} \frac{d+2}{2}\left(\frac{\lambda_d}{d}\right)^{d/(d+2)} c^{2/(d+2)} < \epsilon. \tag{3.5.4}$$

We put

$$R_\epsilon = \left(\frac{\kappa_\epsilon}{\omega_d}\right)^{1/d}, \tag{3.5.5}$$

and estimate that, for j large enough,

$$\begin{aligned} 2J_d^\Pi(c) &\ge \frac{1}{2}\|\nabla\psi_j\|_2^2 + cF_d^\Pi(\psi_j^2) \\ &\ge c\int_0^{\kappa_\epsilon} \Theta(d\kappa)\int_{\mathbb{R}^d} dx\, \left(1 - e^{-\kappa\psi_j^2(x)}\right) \\ &\ge c\int_0^{\kappa_\epsilon} \Theta(d\kappa)\int_{\{|x|>R_\epsilon\}} dx\, \left(1 - e^{-\kappa\psi_j^2(x)}\right). \end{aligned} \tag{3.5.6}$$

Since ψ_j is RSNI and $\|\psi_j\|_2^2 = 1$, for any $R > 0$ we have that

$$\omega_d R^d \psi_j^2(x) \le 1, \qquad |x| > R. \tag{3.5.7}$$

It follows from (3.5.5) and (3.5.7) that

$$\kappa\psi_j^2(x) \le 1 \tag{3.5.8}$$

on the set $(0,\kappa_\epsilon) \times \{x \in \mathbb{R}^d\colon\ |x| > R_\epsilon\}$. Hence, by (3.5.6) and the inequality $1 - e^{-x} \ge x/2$, $0 \le x \le 1$, we have

$$2J_d^\Pi(c) \ge \frac{c}{2}\int_0^{\kappa_\epsilon} \Theta(d\kappa)\,\kappa\int_{\{|x|>R_\epsilon\}} dx\, \psi_j^2(x). \tag{3.5.9}$$

By the choice of κ_ϵ in (3.5.4), the bounds in (3.5.3) and (3.5.9) combine to yield

$$\int_{\{|x|>R_\epsilon\}} dx\, \psi_j^2(x) < \epsilon. \tag{3.5.10}$$

Since $\psi_j \to \psi_c$ as $j \to \infty$ almost everywhere, we get

$$\int_{\mathbb{R}^d} dx\, \psi_c^2(x) \ge \int_{\{|x|<R_\epsilon\}} dx\, \psi_c^2(x) \ge 1 - \epsilon. \tag{3.5.11}$$

Since $\epsilon \in (0,1)$ was arbitrary, we conclude that $\|\psi_c\|_2^2 = 1$. ∎

Lemma 3.5.2 *Let $d \geq 3$. If $\langle \kappa \rangle < \infty$, then* (1.3.2) *has a minimiser in regimes I and II for all $c > c^*$ (with $c^* = 0$ in regime II).*

Proof. By definition of c^* we have $J_d^\Pi(c) < c\langle\kappa\rangle$ for $c > c^*$. For $c > 0$, define

$$\begin{aligned} &K_d^\Pi(c) \\ &= \inf\left\{\frac{1}{2}\|\nabla\psi\|_2^2 - c\int_0^\infty \Theta(d\kappa)\int_{\mathbb{R}^d} dx\,\left(e^{-\kappa\psi^2(x)} - 1 + \kappa\psi^2(x)\right) :\ \|\psi\|_2^2 = 1,\, \psi \text{ RSNI}\right\}, \\ &\widehat{K}_d^\Pi(c) \\ &= \inf\left\{\frac{1}{2}\|\nabla\psi\|_2^2 - c\int_0^\infty \Theta(d\kappa)\int_{\mathbb{R}^d} dx\,\left(e^{-\kappa\psi^2(x)} - 1 + \kappa\psi^2(x)\right) :\ \|\psi\|_2^2 \leq 1,\, \psi \text{ RSNI}\right\}. \end{aligned} \tag{3.5.12}$$

Then, for $c > c^*$,

$$\widehat{K}_d^\Pi(c) \leq K_d^\Pi(c) < 0. \tag{3.5.13}$$

Let (ψ_j) be a minimising sequence of the variational problem for $\widehat{K}_d^\Pi(c)$. As in the proof of Lemma 3.5.1, we extract a subsequence, also denoted by (ψ_j), such that $\psi_j \to \psi_c$ as $j \to \infty$ for some ψ_c almost everywhere and in $D^1(\mathbb{R}^d)$. It follows that ψ_c is RSNI and that ψ_c is a minimiser for $\widehat{K}_d^\Pi(c)$. Moreover, $\|\psi_c\|_2^2 > 0$ (because $\|\psi_c\|_2^2 = 0$ would imply $\psi_c = 0$ almost everywhere, which in turn would imply $\widehat{K}_d^\Pi(c) = 0$, which contradicts (3.5.13)). Suppose that $\|\psi_c\|_2^2 = 1 - \rho$ with $0 \leq \rho < 1$. Define

$$\phi(x) = \frac{1}{q}\psi_c(qx), \tag{3.5.14}$$

where we choose $q > 0$ such that $\|\phi\|_2^2 = 1$, i.e.,

$$q = (1-\rho)^{1/(d+2)}. \tag{3.5.15}$$

Then

$$\|\nabla\phi\|_2^2 = (1-\rho)^{-d/(d+2)}\|\nabla\psi_c\|_2^2 \tag{3.5.16}$$

and

$$\begin{aligned} &c\int_0^\infty \Theta(d\kappa)\int_{\mathbb{R}^d}\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right) \\ &\quad= c(1-\rho)^{-d/(d+2)}\int_0^\infty \Theta(d\kappa)\int_{\mathbb{R}^d}\left(e^{-\kappa(1-\rho)^{-2/(d+2)}\psi_c^2(x)} - 1 + \kappa(1-\rho)^{-2/(d+2)}\psi_c^2(x)\right) \\ &\quad\geq c(1-\rho)^{-d/(d+2)}\int_0^\infty \Theta(d\kappa)\int_{\mathbb{R}^d}\left(e^{-\kappa\psi_c^2(x)} - 1 + \kappa\psi_c^2(x)\right). \end{aligned} \tag{3.5.17}$$

Inserting (3.5.16) and (3.5.17) into the definition of $K_d^\Pi(c)$, and using the definition of $\widehat{K}_d^\Pi(c)$, we get

$$K_d^\Pi(c) \leq (1-\rho)^{-d/(d+2)}\widehat{K}_d^\Pi(c). \tag{3.5.18}$$

By (3.5.13) and (3.5.18) we conclude that $\rho = 0$. Hence $\|\psi_c\|_2^2 = 1$, and ψ_c is a minimiser for $K_d^\Pi(c)$. ∎

Lemma 3.5.3 *Let $d \geq 3$. For $\Theta \in \mathcal{S}_I$, if $\langle \kappa^\eta \rangle < \infty$ for some $\eta > \frac{d+2}{d}$, then* (1.3.2) *has a minimiser for $c = c^*$.*

Proof. Define

$$\widehat{c} = \inf \left\{ \frac{\frac{1}{2}\|\nabla\phi\|_2^2}{G_d^\Pi(\phi^2)} \colon\ 0 < \|\phi\|_2^2 \le 1,\ \phi \text{ RSNI} \right\}. \tag{3.5.19}$$

We begin by showing that $\widehat{c} = c^*$. Trivially, by comparing (3.2.3) and (3.5.19), we get $\widehat{c} \le c^*$. To prove the converse, let $(\widehat{\phi}_j)$ be a minimising sequence for (3.5.19). Put $0 < a_j = \|\widehat{\phi}_j\|_2^2 \le 1$, and

$$\phi_j(x) = a_j^{-1/(d+2)} \widehat{\phi}_j \left(a_j^{1/(d+2)} x \right). \tag{3.5.20}$$

Then $\|\phi_j\|_2^2 = 1$, and

$$c^* \le \frac{\frac{1}{2}\|\nabla\phi_j\|_2^2}{G_d^\Pi(\phi_j^2)} = \frac{\frac{1}{2}\|\nabla\widehat{\phi}_j\|_2^2}{G_d^\Pi(a_j^{-2/(d+2)}\widehat{\phi}_j^2)} \le \frac{\frac{1}{2}\|\nabla\widehat{\phi}_j\|_2^2}{G_d^\Pi(\widehat{\phi}_j^2)}. \tag{3.5.21}$$

But the right-hand side of (3.5.21) converges to $\widehat{c}$ as $j \to \infty$. Hence, $c^* \le \widehat{c}$.

By extracting a subsequence, also denoted by $(\widehat{\phi}_j)$, we may assume that $\widehat{\phi}_j \to \widehat{\phi}$ as $j \to \infty$ for some $\widehat{\phi}$ almost everywhere and weakly in $D^1(\mathbb{R}^d)$. It follows that $\widehat{\phi}$ is RSNI. Below we will show that $\|\widehat{\phi}\|_2^2 > 0$. If $\|\widehat{\phi}\|_2^2 = 1$, then $\widehat{\phi}$ is a minimiser of (3.2.3). If, on the other hand, $0 < \|\widehat{\phi}\|_2^2 = 1 - \rho < 1$, then define, as in (3.5.14),

$$\phi^*(x) = \frac{1}{q}\widehat{\phi}(qx), \tag{3.5.22}$$

where q is given by (3.5.15). Then $\|\phi^*\|_2^2 = 1$ and, as in (3.5.21),

$$c^* \le \frac{\frac{1}{2}\|\nabla\phi^*\|_2^2}{G_d^\Pi(\phi^{*\,2})} \le \frac{\frac{1}{2}\|\nabla\widehat{\phi}\|_2^2}{G_d^\Pi(\widehat{\phi}^2)} = \widehat{c} = c^*. \tag{3.5.23}$$

It follows that ϕ^* is a minimiser of (3.2.3). It then obviously also is a minimiser of (1.3.2) for $c = c^*$ (recall (3.2.1), (3.2.2) and (3.2.3)).

It remains to prove that $\|\widehat{\phi}\|_2^2 > 0$. For this it suffices to show that there exist $\delta, \epsilon > 0$ such that, for any minimising sequence $(\widehat{\phi}_j)$ of (3.5.19),

$$|\{x \in \mathbb{R}^d \colon\ \widehat{\phi}_j^2(x) \ge \epsilon\}| \ge \delta \qquad \text{for all } j. \tag{3.5.24}$$

Indeed, (3.5.24) implies that $\|\widehat{\phi}_j\|_2^2 \ge \epsilon\delta$ for all j, and hence that

$$\|\widehat{\phi}\|_2^2 \ge \epsilon\delta. \tag{3.5.25}$$

To prove (3.5.24), we argue by contradiction. Suppose that there exists a minimising sequence $(\widehat{\phi}_j)$ of (3.5.19) with the property that, for all $\epsilon > 0$,

$$\lim_{j\to\infty} |\{x \in \mathbb{R}^d \colon\ \widehat{\phi}_j^2(x) \ge \epsilon\}| = 0. \tag{3.5.26}$$

Then, for all $\epsilon > 0$, there exists an $L_1(\epsilon) \in \mathbb{N}$ such that, for all $j \ge L_1(\epsilon)$,

$$|\{x \in \mathbb{R}^d \colon\ \widehat{\phi}_j^2(x) \ge \epsilon\}| < \epsilon^{d[\eta-(d+2)/d]/2}. \tag{3.5.27}$$

We already know that there exists an $L_2 \in \mathbb{N}$ such that, for all $j \geq L_2$,

$$\frac{\frac{1}{2}\|\nabla \widehat{\phi}_j\|_2^2}{G_d^{\Pi}(\widehat{\phi}_j^2)} \leq 2\,\widehat{c}. \tag{3.5.28}$$

To arrive at a contradiction, we will show that the left-hand side of (3.5.28) is at least $5\,\widehat{c}/2$ for $j \geq L_1(\epsilon_0) \vee L_2$ for some $\epsilon_0 > 0$.

By the Sobolev inequality in (3.2.15), we have

$$\|\nabla \widehat{\phi}_j\|_2^2 \geq S_d \|\widehat{\phi}_j\|_{2d/(d-2)}^2. \tag{3.5.29}$$

Since $\langle \kappa^{\eta'} \rangle < \infty$ implies that $\langle \kappa^{\eta} \rangle < \infty$ for $\eta \leq \eta'$, we may assume that $\frac{d+2}{d} < \eta \leq 2$. To estimate the contribution of the strip $\{\widehat{\phi}_j^2 < \epsilon\}$ to the integral for $G_d^{\Pi}(\widehat{\phi}_j^2)$, we use that $e^{-x} + 1 - x \leq x^{\eta}$, $x \geq 0$, to obtain, via (3.2.14),

$$\begin{aligned}
\int_0^\infty \Theta(d\kappa) \int_{\{\widehat{\phi}_j^2 < \epsilon\}} dx\, (\kappa \widehat{\phi}_j^2(x))^{\eta} &= \langle \kappa^{\eta} \rangle \int_{\{\widehat{\phi}_j^2 < \epsilon\}} dx\, \widehat{\phi}_j^{2\eta}(x) \\
&\leq \langle \kappa^{\eta} \rangle \epsilon^{\eta - (d+2)/d} \int_{\{\widehat{\phi}_j^2 < \epsilon\}} dx\, \widehat{\phi}_j^{2(d+2)/d}(x) \\
&\leq \langle \kappa^{\eta} \rangle \epsilon^{\eta-(d+2)/d} \|\widehat{\phi}_j\|_{2d/(d-2)}^2.
\end{aligned} \tag{3.5.30}$$

Furthermore, by Hölder's inequality and (3.5.27) we have, for $j \geq L_1(\epsilon)$,

$$\begin{aligned}
\int_0^\infty \Theta(d\kappa) \int_{\{\widehat{\phi}_j^2 \geq \epsilon\}} dx\, \kappa \widehat{\phi}_j^2(x) &= \langle \kappa \rangle \int_{\mathbb{R}^d} dx\, \widehat{\phi}_j^2(x) 1_{\{\widehat{\phi}_j^2(x) \geq \epsilon\}} \\
&\leq \langle \kappa \rangle \left(\int_{\mathbb{R}^d} dx\, \widehat{\phi}_j^{2d/(d-2)}(x) \right)^{(d-2)/d} \left(\int_{\mathbb{R}^d} dx\, 1_{\{\widehat{\phi}_j^2(x) \geq \epsilon\}} \right)^{2/d} \\
&\leq \langle \kappa \rangle \|\widehat{\phi}_j\|_{2d/(d-2)}^2 \epsilon^{\eta-(d+2)/d}.
\end{aligned} \tag{3.5.31}$$

Combining (3.5.29), (3.5.30) and (3.5.31) we have, for $j \geq L_1(\epsilon)$,

$$G_d^{\Pi}(\widehat{\phi}_j^2) \leq (\langle \kappa \rangle + \langle \kappa^{\eta} \rangle)\, \frac{1}{S_d} \epsilon^{\eta-(d+2)/d} \|\nabla \widehat{\phi}_j\|_2^2, \tag{3.5.32}$$

or

$$\frac{\frac{1}{2}\|\nabla \widehat{\phi}_j\|_2^2}{G_d^{\Pi}(\widehat{\phi}_j^2)} \geq \frac{1}{2} S_d \epsilon^{-[\eta-(d+2)/d]} \left(\langle \kappa \rangle + \langle \kappa^{\eta} \rangle \right)^{-1}. \tag{3.5.33}$$

Now choose $\epsilon = \epsilon_0$ with ϵ_0 the root of

$$\frac{1}{2} S_d \epsilon_0^{-[\eta-(d+2)/d]} \left(\langle \kappa \rangle + \langle \kappa^{\eta} \rangle \right)^{-1} = \frac{5}{2} \widehat{c}, \tag{3.5.34}$$

to get that (3.5.33) contradicts (3.5.28) for all $j \geq L_1(\epsilon_0) \vee L_2$. ∎

3.6 Proof of Theorem 1.4.2(i) and Theorem 1.4.3(iii)

We give the proof for $d \geq 3$. The proof for $d = 2$ is the same but uses (1.3.7) instead of (1.3.4).

From (1.3.4) we have

$$F_d^{\Pi}(\phi^2) \leq |\mathrm{supp}(\phi)|, \tag{3.6.1}$$

and so (1.3.2) gives

$$J_d^{\Pi}(c) \leq \inf\left\{\tfrac{1}{2}\,\|\nabla\phi\|_2^2 + c\,|\mathrm{supp}(\phi)|\colon\ \|\phi\|_2^2 = 1\right\}. \tag{3.6.2}$$

We get an upper bound on the infimum by restricting $\mathrm{supp}(\phi)$ to a ball B with volume $|B|$. Therefore

$$J_d^{\Pi}(c) \leq \inf\left\{\frac{1}{2}\,\frac{\|\nabla\phi\|_2^2}{\|\phi\|_2^2} + c\,|B|\colon\ \mathrm{supp}(\phi) \subset B\right\} = \tfrac{1}{2}\lambda_d(B) + c\,|B|, \tag{3.6.3}$$

with $\lambda_d(B)$ the principal Dirichlet eigenvalue of $-\Delta$ on B. By scaling B, we have

$$\lambda_d(B) = |B|^{-2/d}\lambda_d, \tag{3.6.4}$$

Substituting this into (3.6.3) and taking the infimum over $|B|$, we arrive at

$$J_d^{\Pi}(c) \leq \inf_{|B|}\left\{\frac{1}{2}\lambda_d|B|^{-2/d} + c|B|\right\} = \frac{d+2}{2}\left(\frac{\lambda_d}{d}\right)^{d/(d+2)} c^{2/(d+2)}. \tag{3.6.5}$$

This proves the upper bound in (1.4.7).

To prove the lower bound we first scale ϕ to obtain

$$c^{-2/(d+2)} J_d^{\Pi}(c)$$
$$= \inf\left\{\frac{1}{2}\,\|\nabla\phi\|_2^2 + \int_{\mathbb{R}^d} dx \int_0^\infty \Theta(d\kappa)\left(1 - e^{-\kappa c^{d/(d+2)}\phi^2(x)}\right)\colon\ \|\phi\|_2^2 = 1,\ \phi \text{ RSNI}\right\}. \tag{3.6.6}$$

We know that this variational problem has a minimiser when $c > c^*$. Call this minimiser ψ. Pick $0 < \delta < 1/(2 \vee \lambda_d)$, and let

$$B_\delta = \left\{x \in \mathbb{R}^d\colon\ \psi(x) \geq \delta\right\}. \tag{3.6.7}$$

Restricting the x-integration to B_δ, we get

$$\mathrm{rhs}\,(3.6.6) \geq \frac{1}{2}\int_{B_\delta} dx\,|\nabla\psi(x)|^2 + |B_\delta| - \int_{B_\delta} dx \int_0^\infty \Theta(d\kappa)\,e^{-\kappa c^{d/(d+2)}\delta^2}. \tag{3.6.8}$$

By Lebesgue's dominated convergence theorem, for every $\epsilon > 0$ there exists a $C = C(\delta, \epsilon, \Theta)$ such that

$$\int_{B_\delta} dx \int_0^\infty \Theta(d\kappa)\,e^{-\kappa c^{d/(d+2)}\delta^2} \leq \epsilon \qquad \forall\, c \geq C. \tag{3.6.9}$$

Hence

$$\mathrm{rhs}\,(3.6.8) \geq \tfrac{1}{2}\int_{B_\delta} dx\,|\nabla\psi(x)|^2 + |B_\delta| - \epsilon. \tag{3.6.10}$$

Next, define ϕ by

$$\phi(x) = \begin{cases} \psi(x) - \delta & x \in B_\delta, \\ 0 & x \in \mathbb{R}^d \setminus B_\delta. \end{cases} \tag{3.6.11}$$

Then ϕ is RSNI and satisfies the Dirichlet boundary condition on ∂B_δ. Since $\|\psi\|_2^2 = 1$, we have

$$\int_{B_\delta} \phi = \int_{B_\delta} \psi - \delta|B_\delta| \le |B_\delta|^{1/2} - \delta|B_\delta|. \tag{3.6.12}$$

Hence

$$\begin{aligned} 1 = \int_{\mathbb{R}^d} \psi^2 &= \int_{B_\delta} (\phi+\delta)^2 = \delta^2|B_\delta| + 2\delta \int_{B_\delta} \phi + \int_{B_\delta} \phi^2 \\ &\le -\delta^2|B_\delta| + 2\delta|B_\delta|^{1/2} + \int_{B_\delta} \phi^2 \le 2\delta|B_\delta|^{1/2} + \|\phi\|_2^2. \end{aligned} \tag{3.6.13}$$

By (3.6.11) and the Rayleigh-Ritz variational characterisation of $\lambda_d(B_\delta)$, we have

$$\int_{B_\delta} |\nabla\psi|^2 = \int_{B_\delta} |\nabla\phi|^2 \ge \lambda_d(B_\delta)\, \|\phi\|_2^2. \tag{3.6.14}$$

Combining (3.6.6), (3.6.8), (3.6.10), (3.6.13) and (3.6.14), we obtain for $c \ge C$,

$$\begin{aligned} c^{-2/(d+2)} J_d^{\Pi}(c) &\ge \frac{1}{2}\lambda_d(B_\delta)\left(1 - 2\delta|B_\delta|^{1/2}\right) + |B_\delta| - \epsilon \\ &= \frac{1}{2}\lambda_d|B_\delta|^{-2/d}\left(1 - 2\delta|B_\delta|^{1/2}\right) + |B_\delta| - \epsilon \\ &\ge \frac{1}{2}\lambda_d|B_\delta|^{-2/d}(1-2\delta) + |B_\delta|(1-\delta\lambda_d) - \epsilon \\ &\ge \left(\frac{1}{2}\lambda_d|B_\delta|^{-2/d} + |B_\delta|\right)[1 - \delta(2\vee\lambda_d)] - \epsilon \\ &\ge \frac{d+2}{2}\left(\frac{\lambda_d}{d}\right)^{d/(d+2)}[1-\delta(2\vee\lambda_d)] - \epsilon, \end{aligned} \tag{3.6.15}$$

where the second line uses (3.6.4) and the fifth line uses (3.6.5). Now let $c \to \infty$, and subsequently let $\delta, \epsilon \downarrow 0$, to get the lower bound in (1.4.7).

3.7 Proof of Theorems 1.4.2(ii) and 1.4.2(iii)

Fix $\epsilon \in (0, K/2)$. Then there exists an $R_\epsilon \in (0,\infty)$ such that

$$(K-\epsilon)\kappa^{-1-\gamma} \le \theta(\kappa) \le (K+\epsilon)\kappa^{-1-\gamma}, \qquad \kappa \ge R_\epsilon. \tag{3.7.1}$$

By (3.2.1) and (3.2.2),

$$\begin{aligned} &c\langle\kappa\rangle - J_d^{\Pi}(c) \\ &= -\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 - c\, G_d^{\Pi}(\phi^2)\colon\ \|\phi\|_2^2 = 1\right\} \\ &\ge -\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 - c\int_{\mathbb{R}^d} dx \int_{R_\epsilon}^{\infty} d\kappa\, \theta(\kappa)\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right)\colon\ \|\phi\|_2^2 = 1\right\} \\ &\ge -\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 - c\int_{\mathbb{R}^d} dx \int_{R_\epsilon}^{\infty} d\kappa\, (K-\epsilon)\kappa^{-1-\gamma}\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right)\colon\ \|\phi\|_2^2 = 1\right\} \\ &= -\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 - c(K-\epsilon)\int_{\mathbb{R}^d} dx\, |\phi(x)|^{2\gamma} \int_{R_\epsilon\phi^2(x)}^{\infty} d\kappa\, \kappa^{-1-\gamma}\left(e^{-\kappa} - 1 + \kappa\right)\colon\ \|\phi\|_2^2 = 1\right\}, \end{aligned} \tag{3.7.2}$$

where the second inequality uses the lower bound in (3.7.1). Inserting the scaling $\phi(x) = \delta^{d/2}\psi(\delta x)$, $\delta > 0$, we obtain

$$c\langle\kappa\rangle - J_d^\Pi(c) \geq -\inf\Big\{\frac{1}{2}\delta^2\|\nabla\psi\|_2^2 - c(K-\epsilon)\delta^{d(\gamma-1)}\int_{\mathbb{R}^d} dx\,|\psi(x)|^{2\gamma} \\ \times\int_{\delta^d R_\epsilon\psi^2(x)}^\infty d\kappa\,\kappa^{-1-\gamma}\left(e^{-\kappa}-1+\kappa\right):\ \|\psi\|_2^2 = 1\Big\}. \tag{3.7.3}$$

We choose δ to be the root of $\frac{1}{2}\delta^2 = c(K-\epsilon)\delta^{d(\gamma-1)}$. Since this root is greater or equal than $(cK)^{1/(2-d(\gamma-1))}$, we obtain

$$(2cK)^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^\Pi(c)\right] \geq -\frac{1}{2}\left(\frac{K}{K-\epsilon}\right)^{-2/(2-d(\gamma-1))} \\ \times\inf\left\{\|\nabla\psi\|_2^2 - \int_{\mathbb{R}^d} dx\,|\psi(x)|^{2\gamma}\int_{(cK)^{d/(2-d(\gamma-1))}R_\epsilon\psi^2(x)}^\infty d\kappa\,\kappa^{-1-\gamma}\left(e^{-\kappa}-1+\kappa\right):\ \|\psi\|_2^2 = 1\right\}. \tag{3.7.4}$$

Next we note that

$$\int_0^\infty d\kappa\,\kappa^{-1-\gamma}\left(e^{-\kappa}-1+\kappa\right) = \Gamma(-\gamma) \in (0,\infty). \tag{3.7.5}$$

Let $\beta \in (\gamma, 2]$. Since $e^{-\kappa} - 1 + \kappa \leq \kappa^\beta$, $\kappa \geq 0$, we have

$$\int_0^{(cK)^{d/(2-d(\gamma-1))}R_\epsilon\psi^2(x)} d\kappa\,\kappa^{-1-\gamma}\left(e^{-\kappa}-1+\kappa\right) \leq \frac{1}{\beta-\gamma}\left((cK)^{d/(2-d(\gamma-1))}R_\epsilon\psi^2(x)\right)^{\beta-\gamma}. \tag{3.7.6}$$

By (3.7.4)– (3.7.6) we obtain that

$$(2cK)^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^\Pi(c)\right] \geq -\frac{1}{2}\left(\frac{K}{K-\epsilon}\right)^{-2/(2-d(\gamma-1))} \\ \times\inf\left\{\|\nabla\psi\|_2^2 - \Gamma(-\gamma)\int_{\mathbb{R}^d} dx\,|\psi(x)|^{2\gamma} + E_{\beta,\gamma}(\epsilon,c;\psi^2):\ \|\psi\|_2^2 = 1\right\} \tag{3.7.7}$$

with an error term

$$E_{\beta,\gamma}(\epsilon,c;\psi^2) = \frac{1}{\beta-\gamma}R_\epsilon^{\beta-\gamma}(cK)^{(\beta-\gamma)/(2-d(\gamma-1))}\int_{\mathbb{R}^d} dx\,|\psi(x)|^{2\beta}. \tag{3.7.8}$$

Furthermore for $0 < \alpha < 1$ and conjugate exponents $p, q > 1$, we estimate

$$\int_{\mathbb{R}^d}|\psi|^{2\beta} \leq \left(\int_{\mathbb{R}^d}|\psi|^{2\alpha\beta p}\right)^{1/p}\left(\int_{\mathbb{R}^d}|\psi|^{2(1-\alpha)\beta q}\right)^{1/q}. \tag{3.7.9}$$

Choosing α, β, p, q such that

$$2\alpha\beta p = 2d/(d-2), \qquad p = d/(d-2), \qquad 2(1-\alpha)\beta q = 2, \tag{3.7.10}$$

i.e.,

$$\alpha = d/(d+2), \qquad \beta = (d+2)/d, \qquad p = d/(d-2), \qquad q = d/2, \tag{3.7.11}$$

we obtain from (3.7.9), using $\|\psi\|_2^2 = 1$ and the Sobolev inequality (3.2.15), that

$$\begin{aligned} (2cK)^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^\Pi(c)\right] &\geq -\frac{1}{2}\left(\frac{K}{K-\epsilon}\right)^{-2/(2-d(\gamma-1))} \\ &\times \inf\left\{(1+E_\gamma(\epsilon,c))\,\|\nabla\psi\|_2^2 - \Gamma(-\gamma)\int_{\mathbb{R}^d} dx\,|\psi(x)|^{2\gamma}\colon\ \|\psi\|_2^2 = 1\right\} \end{aligned} \tag{3.7.12}$$

with an error term

$$E_\gamma(\epsilon,c) = R_\epsilon^{(2-d(\gamma-1))/d}\frac{d}{S_d(2-d(\gamma-1))}\,cK. \tag{3.7.13}$$

Finally, we insert the scaling $\phi(x) = \eta^{d/2}\psi(\eta x)$, $\eta > 0$, and choose η the root of $\eta^2(1+E_\gamma(\epsilon,c)) = \Gamma(-\gamma)\eta^{d(\gamma-1)}$, to arrive at

$$\{2cK\Gamma(-\gamma)\}^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^\Pi(c)\right] \geq \frac{1}{2}\left(\frac{K}{K-\epsilon}\,\frac{1}{1+E_\gamma(\epsilon,c)}\right)^{-2/(2-d(\gamma-1))} M_d(\gamma), \tag{3.7.14}$$

where we have used the definition of $M_d(\gamma)$ in (1.4.9). Now let $c \downarrow 0$ and use that $\lim_{c\downarrow 0} E_\gamma(\epsilon,c) = 0$ for all $\epsilon > 0$. Then let $\epsilon \downarrow 0$, to get

$$\liminf_{c\downarrow 0}\{2cK\Gamma(-\gamma)\}^{-2/(2-d(\gamma-1))}\left[c\langle\kappa\rangle - J_d^\Pi(c)\right] \geq \frac{1}{2}M_d(\gamma), \tag{3.7.15}$$

which is the desired lower bound.

The proof of the upper bound runs as follows. Let ϵ and R_ϵ be as before. We estimate, similarly as in (3.7.2),

$$\begin{aligned} &c\langle\kappa\rangle - J_d^\Pi(c) \\ &\leq -\inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 - c\int_{\mathbb{R}^d} dx\int_{R_\epsilon}^\infty d\kappa\;\theta(\kappa)\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right) - E_\theta(\epsilon,c;\phi^2)\colon\ \|\phi\|_2^2 = 1\right\} \end{aligned} \tag{3.7.16}$$

with an error term

$$E_\theta(\epsilon,c;\phi^2) = c\int_{\mathbb{R}^d} dx\int_0^{R_\epsilon} d\kappa\;\theta(\kappa)\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right). \tag{3.7.17}$$

Since $e^{-x} - 1 + x \leq x^{(d+2)/d}$, $x \geq 0$, we may use the Sobolev inequality (3.2.15) to estimate

$$E_\theta(\epsilon,c;\phi^2) \leq c\int_{\mathbb{R}^d} dx\int_0^{R_\epsilon} d\kappa\;\theta(\kappa)\,(\kappa\phi^2(x))^{(d+2)/d} \leq cm_\theta(\epsilon)\,S_d^{-1}\|\nabla\phi\|_2^2, \tag{3.7.18}$$

where we abbreviate $m_\theta(\epsilon) = \int_0^{R_\epsilon} d\kappa\,\theta(\kappa)\,\kappa^{(d+2)/d}$. Combining (3.7.16) and (3.7.18), we obtain, for c small enough,

$$\begin{aligned} &c\langle\kappa\rangle - J_d^\Pi(c) \\ &\leq -\inf\Big\{\left(\frac{1}{2} - cm_\theta(\epsilon)S_d^{-1}\right)\|\nabla\phi\|_2^2 \\ &\qquad - c(K+\epsilon)\int_{\mathbb{R}^d} dx\int_{R_\epsilon}^\infty d\kappa\,\kappa^{-1-\gamma}\left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right)\colon\ \|\phi\|_2^2 = 1\Big\} \\ &\leq -\inf\Big\{\left(\frac{1}{2} - cm_\theta(\epsilon)S_d^{-1}\right)\|\nabla\phi\|_2^2 \\ &\qquad - c(K+\epsilon)\Gamma(-\gamma)\int_{\mathbb{R}^d} dx\,|\phi(x)|^{2\gamma} - E_\gamma(\epsilon,c;\phi^2)\colon\ \|\phi\|_2^2 = 1\Big\}, \end{aligned} \tag{3.7.19}$$

where in the second inequality we use the upper bound in (3.7.1) and the identity in (3.7.5), and introduce an error term

$$E_\gamma(\epsilon, c; \phi^2) = c(K+\epsilon) \int_{\mathbb{R}^d} dx \int_0^{R_\epsilon} d\kappa\, \kappa^{-1-\gamma} \left(e^{-\kappa\phi^2(x)} - 1 + \kappa\phi^2(x)\right). \tag{3.7.20}$$

The integral in (3.7.20) can be estimated from above along the lines of the argument connecting (3.7.7), (3.7.8) with (3.7.12), (3.7.13). This leads to

$$\begin{aligned} &c\langle\kappa\rangle - J_d^\Pi(c) \\ &\leq -\inf\left\{ \left(\frac{1}{2} - E_\gamma(\epsilon,c)\right) \|\nabla\phi\|_2^2 - c(K+\epsilon)\Gamma(-\gamma) \int_{\mathbb{R}^d} dx\, |\phi(x)|^{2\gamma} \colon\ \|\phi\|_2^2 = 1\right\} \end{aligned} \tag{3.7.21}$$

with an error term

$$E_\gamma(\epsilon, c) = cm_\theta(\epsilon) + c(K+\epsilon) R_\epsilon^{(2-d(\gamma-1))/d} \frac{d}{S_d(2-d(\gamma-1))}. \tag{3.7.22}$$

Via the scaling $\phi(x) = \delta^{d/2}\psi(\delta x)$, $\delta > 0$, with δ the root of $\delta^2(\frac{1}{2} - cE_\gamma(\epsilon,c)) = c(K+\epsilon)\Gamma(-\gamma)\delta^{d(\gamma-1)}$, we arrive at

$$\{2Kc\Gamma(-\gamma)\}^{-2/(2-d(\gamma-1))} \left[c\langle\kappa\rangle - J_d^\Pi(c)\right] \leq \frac{1}{2}\left(\frac{K+\epsilon}{K} \frac{1}{1-2E_\gamma(\epsilon,c)}\right)^{2/(2-d(\gamma-1))} M_d(\gamma). \tag{3.7.23}$$

Now let $c \downarrow 0$ and use that $\lim_{c\downarrow 0} E_\gamma(\epsilon, c) = 0$ for all $\epsilon > 0$. Then let $\epsilon \downarrow 0$, to get

$$\limsup_{c\downarrow 0} \{2Kc\Gamma(-\gamma)\}^{-2/(2-d(\gamma-1))} \left[c\langle\kappa\rangle - J_d^\Pi(c)\right] \leq \frac{1}{2} M_d(\gamma), \tag{3.7.24}$$

which is the desired upper bound.

It remains to prove that $M_d(\gamma) \in (0,\infty)$ for all $\gamma \in (1, (d+2)/d)$. By scaling we have, for any $\epsilon > 0$,

$$M_d(\gamma) = -\epsilon^2 \inf\left\{\|\nabla\psi\|_2^2 - \epsilon^{-(2-d(\gamma-1))} \int |\psi|^{2\gamma} \colon\ \|\psi\|_2^2 = 1\right\}. \tag{3.7.25}$$

We get a strictly positive lower bound by choosing for ψ the function

$$\psi(x) = \pi^{-d/4} e^{-|x|^2/2} \tag{3.7.26}$$

and by subsequently choosing ϵ sufficiently small.

To prove that $M_d(\gamma)$ is finite for $\gamma \in (1, \frac{d+2}{d})$, we use the Sobolev inequality (3.2.15) to (1.4.9). This gives

$$M_d(\gamma) \leq -\inf\left\{S_d\|\psi\|_{2d/(d-2)}^2 - \int |\psi|^{2\gamma} \colon\ \|\psi\|_2^2 = 1\right\}. \tag{3.7.27}$$

Since $\|\psi\|_2^2 = 1$ and $\gamma \in (1, d/(d-2))$, Hölder's inequality gives

$$\int |\psi|^{2\gamma} \leq \left(\int |\psi|^{2d/(d-2)}\right)^{(d-2)(\gamma-1)/2}. \tag{3.7.28}$$

Inserting this into (3.7.27), we get

$$\begin{aligned} M_d(\gamma) &\le \sup\left\{\|\psi\|_{2d/(d-2)}^{d(\gamma-1)} - S_d\|\psi\|_{2d/(d-2)}^2\colon\ \|\psi\|_2^2 = 1\right\} \\ &\le \sup_{\rho\in(0,\infty)} \left\{\rho^{d(\gamma-1)} - S_d\rho^2\right\}. \end{aligned} \tag{3.7.29}$$

The supremum in the right-hand side is finite because $d(\gamma-1) < 2$.

This completes the proof of Theorem 1.4.2(ii). The proof of Theorem 1.4.2(iii) is very similar. The argument needs one order less in the expansion since there is no term $c\langle\kappa\rangle$ to subtract.

3.8 Proof of Theorem 1.4.3(ii)

In $d = 2$ the analogue of (3.2.3) reads (recall that κ is replaced by 2π)

$$c^* = \inf\left\{\frac{\frac{1}{2}\|\nabla\phi\|_2^2}{G_2(\phi^2)}\colon\ \|\phi\|_2^2 = 1\right\} \tag{3.8.1}$$

with

$$G_2(\phi^2) = \int_{\mathbb{R}^2} dx\ \left(e^{-2\pi\phi^2(x)} - 1 + 2\pi\phi^2(x)\right). \tag{3.8.2}$$

Lemma 3.8.1 (3.8.1) *has no minimiser. If* (ϕ_n) *is a minimising sequence that is* RSNI*, then* $\lim_{n\to\infty}\int_{\{\phi_n>\delta\}} dx = 0$ *for any* $\delta > 0$.

Proof. Suppose that the variational problem for the right hand side of(3.8.1) has a minimiser, say ψ^*. Then

$$c^* = \frac{\frac{1}{2}\|\nabla\psi^*\|_2^2}{G_2(\psi^{*2})}. \tag{3.8.3}$$

For $\epsilon > 0$, put

$$\psi^*_\epsilon(x) = \epsilon\psi^*(\epsilon x). \tag{3.8.4}$$

Since $\|\psi^*_\epsilon\|_2^2 = 1$, we have

$$c^* \le \frac{\frac{1}{2}\|\nabla\psi^*_\epsilon\|_2^2}{G_2(\psi^{*2}_\epsilon)} = \frac{\frac{1}{2}\|\nabla\psi^*\|_2^2}{\epsilon^{-4}G_2(\epsilon^2\psi^{*2})}. \tag{3.8.5}$$

Next, we claim that

$$y \mapsto \frac{1}{y^2}\left(e^{-\kappa y} - 1 + \kappa y\right), \qquad y > 0, \tag{3.8.6}$$

is strictly decreasing on $(0,\infty)$ for any $\kappa > 0$. Indeed, its derivative at y equals

$$\frac{2}{y^3}\left[\left(1 - \frac{\kappa y}{2}\right) - \left(1 + \frac{\kappa y}{2}\right)e^{-\kappa y}\right]. \tag{3.8.7}$$

Abbreviate $z = \kappa y/2$ and note that $z \mapsto (1+z)e^{-2z} + z$, $z \ge 0$, is strictly increasing on $[0,\infty)$, and equal to 1 at $z = 0$, to get the claim. Finally, using that (3.8.6) is strictly decreasing, we get from (3.8.2) that $\epsilon \mapsto \epsilon^{-4}G_2(\epsilon^2\psi^{*2})$ is strictly decreasing, which clearly contradicts (3.8.3) and (3.8.5) when $\epsilon < 1$.

To prove the last claim, let (ϕ_n) be a minimising sequence for (3.8.1). Then, for any $\delta > 0$,

$$\lim_{n\to\infty} \int_{\{\phi_n > \delta\}} dx = 0. \tag{3.8.8}$$

Indeed, if (3.8.8) fails, then there exists an $\eta > 0$ and a subsequence (ϕ_{n_j}) such that

$$\int_{\{\phi_{n_j} > \delta\}} dx \geq \eta. \tag{3.8.9}$$

But now the above argument shows that the sequence $(\phi^\epsilon_{n_j})$ with $\phi^\epsilon_{n_j}(x) = \epsilon\phi_{n_j}(\epsilon x)$ yields a strictly lower infimum when $\epsilon < 1$, which is a contradiction. ∎

Lemma 3.8.2 *Let $d = 2$. Then*

$$c^* = \frac{1}{4\pi^2} \inf\left\{ \frac{\|\nabla\phi\|_2^2}{\|\phi\|_4^4} \colon \|\phi\|_2^2 = 1 \right\} \tag{3.8.10}$$

and $c^* \in \left[\frac{27}{64\pi}, \frac{1}{2\pi}\right]$.

Proof. Since $e^{-x} \leq 1 - x + \frac{1}{2}x^2$, $x \geq 0$, we get from (3.8.2) that

$$G_2(\phi^2) \leq 2\pi^2 \|\phi\|_4^4. \tag{3.8.11}$$

Substituting (3.8.11) into (3.8.1), we obtain the desired lower bound

$$c^* \geq \frac{1}{4\pi^2} \inf\left\{ \frac{\|\nabla\phi\|_2^2}{\|\phi\|_4^4} \colon \|\phi\|_2^2 = 1 \right\}. \tag{3.8.12}$$

To prove the converse of (3.8.12), let (ϕ_n) be a minimising sequence for (3.8.10) that is RSNI. Then (ϕ^ϵ_n) with $\phi^\epsilon_n(x) = \epsilon\phi_n(\epsilon x)$ is a minimising sequence too. Replacing ϵ by $\epsilon/\phi_n(0)$, we may assume that $\phi_n(0) = 1$. It suffices to show that

$$\limsup_{\epsilon\downarrow 0} \limsup_{n\to\infty} \left(\frac{\|\nabla\phi^\epsilon_n\|_2^2}{G_2({\phi^\epsilon_n}^2)} - \frac{\|\nabla\phi^\epsilon_n\|_2^2}{2\pi^2\|\phi^\epsilon_n\|_4^4} \right) \leq 0. \tag{3.8.13}$$

Since (ϕ^ϵ_n) is a minimising sequence, there exists an N such that for $n \geq N$,

$$\phi^\epsilon_n(0) = \epsilon, \quad \|\nabla\phi^\epsilon_n\|_2^2/\|\phi^\epsilon_n\|_4^4 < \infty, \quad \|\phi^\epsilon_n\|_2^2 = 1. \tag{3.8.14}$$

Since $e^{-x} - 1 + x - \frac{1}{2}x^2 \geq -\frac{1}{6}x^3$, $x \geq 0$, it follows from (3.8.2) that

$$G_2({\phi^\epsilon_n}^2) \geq \int \left[\frac{1}{2}(2\pi{\phi^\epsilon_n}^2)^2 - \frac{1}{6}(2\pi{\phi^\epsilon_n}^2)^3 \right] \geq 2\pi^2 \left[1 - \frac{2\pi\epsilon^2}{3} \right] \int {\phi^\epsilon_n}^4, \tag{3.8.15}$$

where we use that $\phi_n \leq \phi_n(0) = \epsilon$. Hence, for $n \geq N$,

$$\|\nabla\phi^\epsilon_n\|_2^2 \left(\frac{1}{G_2({\phi^\epsilon_n}^2)} - \frac{1}{2\pi^2\|\phi^\epsilon_n\|_4^4} \right) \leq \frac{1}{2\pi^2} \left(\left[1 - \frac{2\pi\epsilon^2}{3} \right]^{-1} - 1 \right) \frac{\|\nabla\phi^\epsilon_n\|_2^2}{\|\phi^\epsilon_n\|_4^4}. \tag{3.8.16}$$

As $n \to \infty$, the quotient in the right-hand side converges to $2c^*$. Now let $\epsilon \downarrow 0$, to get the claim in (3.8.13).

Finally, the numerical bounds on c^* are obtained as follows. First note that in $d = 2$ we have the Sobolev inequality

$$\|\nabla\phi\|_2^2 \geq S_{2,4}^{-2}\|\phi\|_4^2 - \|\phi\|_2^2 \tag{3.8.17}$$

(see Lieb and Loss [6] page 190). With the substitution $\phi_p(x) = \phi(x/p)$, $p > 0$, this inequality transforms into

$$\|\nabla\phi\|_2^2 \geq S_{2,4}^{-2}\, p\, \|\phi\|_4^2 - p^2\, \|\phi\|_2^2. \tag{3.8.18}$$

After optimisation over p this yields the Sobolev inequality

$$\|\nabla\phi\|_2^2 \geq \tfrac{1}{4} S_{2,4}^{-4}\|\phi\|_4^4\, \|\phi\|_2^{-2}. \tag{3.8.19}$$

Substituting (3.8.19) into (3.8.12), we find the lower bound

$$c^* \geq \frac{1}{16\pi^2} S_{2,4}^{-4}. \tag{3.8.20}$$

This implies that $c^* \geq 27/64\pi$, because $S_{2,4}^{-4} = 27\pi/4$. To obtain the upper bound on c^*, we pick ψ as in (3.7.26) with $d = 2$. Since $\|\nabla\psi\|_2^2 = 1$, $\|\psi\|_2^2 = 1$ and $\|\psi\|_4^4 = \frac{1}{2\pi}$, subsitution into (3.8.10) yields that $c^* \leq 1/2\pi$. ∎

Lemma 3.8.3 *Let $d = 2$. Then $\lim_{c\downarrow c^*}[J_2(c) - J_2(c^*)]/(c - c^*) = 2\pi$.*

Proof. By the concavity of $c \mapsto J_2(c)$ stated in Theorem 1.4.3(i), it suffices to prove that

$$\liminf_{c\downarrow c^*} \frac{J_2(c) - J_2(c^*)}{c - c^*} \geq 2\pi. \tag{3.8.21}$$

Since J_2 does not depend on Π, it is given by the expression we obtained in vdBBdH Theorem 2 and Corollary 2 for the case where $\Pi = \delta_{B_a(0)}$ with $a > 0$ arbitrary, namely,

$$J_2(c) = \inf_{0<b\leq 2\pi} [bc + I_2(b)] \tag{3.8.22}$$

with

$$I_2(b) = \left\{\frac{1}{2}\|\nabla\phi\|_2^2 \colon\ \phi \in H^1(\mathbb{R}^2),\ \|\phi\|_2^2 = 1,\ \int (1 - e^{-2\pi\phi^2}) = b\right\} \tag{3.8.23}$$

(see also vdBBdH Equations (5.7) and (5.13)). Now, from vdBBdH Theorems 3(i) and 4(ii) we know that

$$b \mapsto \frac{I_2(b)}{2\pi - b} \tag{3.8.24}$$

is strictly decreasing on $(0, 2\pi)$, with

$$\lim_{b\uparrow 2\pi} \frac{I_2(b)}{2\pi - b} = \frac{1}{4\pi^2} \inf \left\{\|\nabla\phi\|_2^2 \colon\ \phi \in H^1(\mathbb{R}^2),\ \|\phi\|_2^2 = 1,\ \|\phi\|_4^4 = 1\right\} = c^* \tag{3.8.25}$$

(compare with (3.8.10)). Put

$$\Delta(b) = \frac{I_2(b)}{2\pi - b} - c^*. \tag{3.8.26}$$

Using (3.8.22), we may then write

$$\frac{J_2(c) - J_2(c^*)}{c - c^*} = \inf_{0<b\leq 2\pi} \left[b + \frac{(2\pi - b)\Delta(b)}{c - c^*}\right]. \tag{3.8.27}$$

Since $\Delta(b) > 0$ for all $0 < b < 2\pi$, the minimiser in the right-hand side tends to 2π as $c \downarrow c^*$, which yields (3.8.21). ∎

3.9 Proof of Theorem 1.4.3(iv)

The proof is the same as that of Lemma 3.5.2.

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