A FINITENESS RESULT FOR ASSOCIATED PRIMES OF CERTAIN EXT-MODULES

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Abstract ¹. Using some properties of unconditioned M -sequences in dimension $> s$, we give a finiteness result for the set $\bigcup_{n \in \mathbb{N}} \text{Ass}_R(\text{Ext}_R^i(R/I^n, M))$.

1 Introduction

Throughout this paper, let R be a Noetherian commutative ring, let M be a finitely generated R -module, and A an Artinian R -module.

For an ideal I of R , it was shown in [B] that the two sequences of associated primes

$$\text{Ass}_R(M/I^n M) \text{ and } \text{Ass}_R(I^n M/I^{n+1} M), n = 1, 2, \dots$$

eventually become constant for large n . Sharp [Sh] proved the dual result for Artinian modules: $\text{Att}_R(0 :_A I^n)$ and $\text{Att}_R((0 :_A I^n)/(0 :_A I^{n-1}))$ do not depend on n for n large. Starting from the observation that $M/I^n M \cong \text{Tor}_0^R(R/I^n, M)$ and $0 :_A I^n \cong \text{Ext}_R^0(R/I^n, A)$ for any n , Melkersson and Schenzel [MS] extended the above results as follows: For any given integer $i \geq 0$, the sequences

$$\text{Ass}_R(\text{Tor}_i^R(R/I^n, M)) \text{ and } \text{Att}_R(\text{Ext}_R^i(R/I^n, A)), n = 1, 2, \dots$$

become independent of n for large n . Melkersson and Schenzel [MS] also asked whether the set $\text{Ass}_R(\text{Ext}_R^i(R/I^n, M))$ is independent of n for large n .

In fact, $\bigcup_{n \in \mathbb{N}} \text{Ass}_R(\text{Ext}_R^i(R/I^n, M))$ is not a finite set in general, and therefore the set $\text{Ass}_R(\text{Ext}_R^i(R/I^n, M))$ depends on n for n large. Indeed, Katzman [Ka, Corollary 1.3] gave an example of a Noetherian local ring $(R, \mathfrak{m})$ with two elements $x, y \in \mathfrak{m}$ such that $\text{Ass}_R(H_{(x,y)R}^2(R))$ is an infinite set. Therefore the set $\bigcup_{n \in \mathbb{N}} \text{Ass}_R(\text{Ext}_R^2(R/(x, y)^n, R))$ is infinite.

For convenience, for a subset T of $\text{Spec } R$ and an integer $i \geq 0$, we set

$$(T)_i := \{\mathfrak{p} \in T : \dim R/\mathfrak{p} = i\};$$

$$(T)_{\geq i} := \{\mathfrak{p} \in T : \dim R/\mathfrak{p} \geq i\}.$$

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For an integer $i \geq 0$, an ideal I of R , and a system $\underline{a} = (a_1, \dots, a_k)$ of elements in R , we set

$$T^i(I, M) := \bigcup_{n \in \mathbb{N}} \text{Ass}_R(\text{Ext}_R^i(R/I^n, M));$$

$$T^i(\underline{a}, M) := \bigcup_{n_1, \dots, n_k \in \mathbb{N}} \text{Ass}_R(\text{Ext}_R^i(R/(a_1^{n_1}, \dots, a_k^{n_k}), M)).$$

In this paper, we prove the following finiteness result for the sets $T^i(I, M)$ and $T^i(\underline{a}, M)$.

Theorem 1.1. *Let $s \geq 0$ and $r \geq 1$ be integers. Assume that $\dim(\text{Supp}(H_I^i(M))) \leq s$ for all $i < r$. Then for any system of generators $\underline{a} = (a_1, \dots, a_k)$ of I and for all integers $t \leq r$, the sets $\left(T^t(I, M)\right)_{\geq s}$ and $\left(T^t(\underline{a}, M)\right)_{\geq s}$ are contained in the finite set $\bigcup_{i=0}^t \text{Ass}_R \text{Ext}_R^i(R/I, M)$.*

Theorem 1.2. *Let $s \geq 0$ and $r \geq 1$ be integers. Assume that $\dim(\text{Supp}(H_I^i(M))) \leq s$ for all $i < r$. Let $x_1, \dots, x_r \in I$ be a sequence which is at the same time an unconditioned M -sequence in dimension $> s$ and an unconditioned I -filter regular sequence with respect to M (such sequences exist by Proposition 2.5). Then for any system of generators $\underline{a} = (a_1, \dots, a_k)$ of I and for all integers $t \leq r$, the sets $\left(T^t(I, M)\right)_{\geq s}$ and $\left(T^t(\underline{a}, M)\right)_{\geq s}$ are contained in the finite set*

$$\left(\text{Ass}_R(M/(x_1, \dots, x_t)M)\right)_{\geq s+1} \cup \left(\bigcup_{i=0}^t \text{Ass}_R(M/(x_1, \dots, x_i)M)\right)_s.$$

2 Unconditioned M -sequences in dimension $> s$

Definition 2.1. Let $s \geq 0$ be an integer, let $x_1, \dots, x_r \in R$ be a sequence. We say that $x_1, \dots, x_r$ is an M -sequence in dimension $> s$ if $x_1, \dots, x_r$ is a poor $M_{\mathfrak{p}}$ -sequence for all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim(R/\mathfrak{p}) > s$.

Observe that $x_1, \dots, x_r$ is an M -sequence in dimension $> s$ if and only if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \left(\text{Ass}_R(M/(x_1, \dots, x_{i-1})M)\right)_{\geq s+1}$ and all $i = 1, \dots, r$.

Assume that R is local. Then $x_1, \dots, x_r$ is an M -sequence in dimension > 0 if and only if it is a filter regular sequence with respect to M in sense of [Cst]. Moreover, $x_1, \dots, x_r$ is an M -sequence in dimension > 1 if and only if it is a generalized regular sequence with respect to M in sense of [Nh].

Reminder 2.2. (a) Let I be an ideal. A sequence $x_1, \dots, x_r \in I$ is called an I -filter regular sequence with respect to M if $x_1, \dots, x_r$ is an $M_{\mathfrak{p}}$ -sequence for all $\mathfrak{p} \in \text{Spec}(R) \setminus \text{Var}(I)$. It is equivalent to say that $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}_R(M/(x_1, \dots, x_{i-1})M) \setminus \text{Var}(I)$ and all $i = 1, \dots, r$.

(b) (cf. [NS, 3.4], [Kh, 2.1]). If $x_1, \dots, x_r$ is an I -filter regular sequence with respect to M then

$$H_I^j(M) = \begin{cases} H_{(x_1, \dots, x_r)R}^j(M), & \text{if } j < r \\ H_I^{j-r}(H_{(x_1, \dots, x_r)R}^r(M)), & \text{if } j \geq r. \end{cases}$$

Definition 2.3. A sequence $x_1, \dots, x_r \in R$ is called an *unconditioned M -sequence in dimension $> s$* if $x_{\sigma(1)}, \dots, x_{\sigma(r)}$ is an M -sequence in dimension $> s$ for all permutations $\sigma \in \mathbb{S}_r$. Similarly, a sequence $x_1, \dots, x_r \in I$ is called an *unconditioned I -filter regular sequence with respect to M* if $x_{\sigma(1)}, \dots, x_{\sigma(r)}$ is an I -filter regular sequence with respect to M for all permutations $\sigma \in \mathbb{S}_r$.

Lemma 2.4. Let $s \geq 0$ be an integer, let I be an ideal of R .

(a) Let $r > 0$ be an integer. Then $\dim(\text{Supp}(H_I^i(M))) \leq s$ for all $i < r$ if and only if there exists an M -sequence in dimension $> s$ of length r in I .

(b) If $\dim(M/IM) > s$ then each M -sequence in dimension $> s$ in I may be extended to a maximal M -sequence in dimension $> s$ in I . Moreover, all maximal M -sequences in dimension $> s$ in I have the same length, and this common length is equal to the least integer i such that $\dim(\text{Supp}(H_I^i(M))) > s$.

(c) If $\dim(M/IM) \leq s$ then there exists an M -sequence in dimension $> s$ in I of length n for any integer $n > 0$.

Proof. (a). Assume that $\dim(\text{Supp}(H_I^i(M))) \leq s$ for all $i < r$. We prove by induction on r that there is a sequence $x_1, \dots, x_r \in I$ which is an M -sequence in dimension $> s$. Let $r \geq 1$. Then $\dim(\text{Supp}(H_I^0(M))) \leq s$. Hence $I \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in (\text{Ass}_R M)_{\geq s+1}$. Therefore there exists an element $x_1 \in I$ which is M -regular in dimension $> s$. This proves the case $r = 1$. Let $r > 1$ and set $x_1 = x$. Then $\dim(0 :_M x) \leq s$. From the exact sequence $0 \longrightarrow 0 :_M x \longrightarrow M \longrightarrow M/(0 :_M x) \longrightarrow 0$ we get an exact sequence

$$H_I^i(M) \longrightarrow H_I^i(M/(0 :_M x)) \longrightarrow H_I^{i+1}(0 :_M x)$$

for all $i \geq 0$. As $\dim(0 :_M x) \leq s$, we have $\dim(\text{Supp}(H_I^i(0 :_M x))) \leq s$ for all $i \geq 0$. Therefore, by our hypothesis, $\dim(\text{Supp}(H_I^i(M/(0 :_M x)))) \leq s$ for all $i < r$. From the exact sequence

$$0 \longrightarrow M/(0 :_M x) \longrightarrow M \longrightarrow M/xM \longrightarrow 0$$

we get an exact sequence $H_I^i(M) \longrightarrow H_I^i(M/xM) \longrightarrow H_I^{i+1}(M/(0 :_M x))$ for all $i \geq 0$. So, $\dim(\text{Supp}(H_I^i(M/xM))) \leq s$ for all $i < r-1$. By induction, there exists a sequence $x_2, \dots, x_r$ in I which is an M/xM -sequence in dimension $> s$. So, $x_1, \dots, x_r$ is an M -sequence in dimension $> s$ in I .

Conversely, assume $x_1, \dots, x_r$ is an M -sequence in dimension $> s$ in I . Let $\mathfrak{p} \in \text{Spec } R$ such that $\dim(R/\mathfrak{p}) > s$. Then $\frac{x_1}{1}, \dots, \frac{x_r}{1}$ is a poor $M_{\mathfrak{p}}$ -sequence in $I_{\mathfrak{p}}$. So, $H_{I_{\mathfrak{p}}}^i(M_{\mathfrak{p}}) = 0$, i.e. $\mathfrak{p} \notin \text{Supp}(H_I^i(M))$ for all $i < r$. Therefore $\dim(\text{Supp } H_I^i(M)) \leq s$ for all $i < r$.

(b). Since $\dim(M/IM) > s$, there is a maximal ideal $\mathfrak{m}$ such that $\dim(M_{\mathfrak{m}}/IM_{\mathfrak{m}}) > s$. Note that each M -sequence in dimension $> s$ in I is an $M_{\mathfrak{m}}$ -sequence in dimension $> s$ in $IR_{\mathfrak{m}}$.

As $\dim(M_{\mathfrak{m}}/IM_{\mathfrak{m}}) > s$, each $M_{\mathfrak{m}}$ -sequence in dimension $> s$ in $IR_{\mathfrak{m}}$ is a part of a system of parameters of $M_{\mathfrak{m}}$. Therefore the length of an M -sequence in dimension $> s$ in I is at most $\dim M_{\mathfrak{m}} - s - 1$. So, there is a common bound on the lengths of all M -sequences in dimension $> s$ which consist of elements in I . Therefore, each M -sequence in dimension $> s$ in I may be extended to a maximal M -sequence in dimension $> s$ in I .

Let $x_1, \dots, x_r$ and $y_1, \dots, y_t$ be maximal M -sequences in dimension $> s$ in I . Assume that $r \neq t$, say $r < t$. By (a), $\dim(\text{Supp } H_I^i(M)) \leq s$ for all $i \leq r$. Similar as in the proof of (a), it follows by induction on k that $\dim(\text{Supp}(H_I^i(M/(x_1, \dots, x_k)M))) \leq s$ for all $i \leq r - k$ and all $k \leq r$. Thus $\dim(H_I^0(M/(x_1, \dots, x_r)M)) \leq s$. Therefore there is an element in I which is $M/(x_1, \dots, x_r)M$ -regular in dimension $> s$. This is a contradiction to the maximality of the sequence $(x_1, \dots, x_r)$. So, all maximal M -sequences in dimension $> s$ in I have the same length and by (a) this length has the stated value.

(c) is clear. □

Proposition 2.5. *Let $s \geq 0$ and $r \geq 1$ be integers, and let $I \subseteq R$ be an ideal. If $\dim(\text{Supp}(H_I^i(M))) \leq s$ for all $i < r$ then there is a sequence $x_1, \dots, x_r$ in I which is at the same time an unconditioned M -sequence in dimension $> s$ and an unconditioned I -filter regular sequence with respect to M .*

Proof. We proceed by induction on r . Let $r = 1$. Set

$$C_1 := (\text{Ass}_R M)_{\geq s+1} \cup (\text{Ass}_R M \setminus \text{Var}(I)).$$

Since $\dim(H_I^0(M)) \leq s$, it follows that $I \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in (\text{Ass}_R M)_{\geq s+1}$. Therefore, by Prime Avoidance, there exists an element $x_1 \in I$ such that $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in C_1$. It is clear that x_1 is an unconditioned M -sequence in dimension $> s$ and an unconditioned I -filter regular sequence w.r.t. M .

Let $r > 1$ and assume that the result is true for $r - 1$. Then there exists a sequence $x_1, \dots, x_{r-1}$ in I which is an unconditioned M -sequence in dimension $> s$ and an unconditioned I -filter regular sequence w.r.t. M . By Lemma 2.4 and by the assumption, for any subset J of $\{1, \dots, r - 1\}$, the sequence $(x_j)_{j \in J}$ can be extended to an M -sequence in dimension $> s$ in I of length r . Therefore for any subset J of $\{1, \dots, r - 1\}$, there exists an $(M/\sum_{j \in J} x_j M)$ -regular element in dimension $> s$ in I . It follows that $I \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in (\text{Ass}_R (M/\sum_{j \in J} x_j M))_{\geq s+1}$ and all subsets J of $\{1, \dots, r - 1\}$. By Prime Avoidance, we can choose an element $x_r \in I$ such that $x_r \notin \mathfrak{p}$ for all $\mathfrak{p} \in C_r$, where

$$C_r := \left(\bigcup_{J \subseteq \{1, \dots, r-1\}} \text{Ass}_R(M/\sum_{j \in J} x_j M) \right)_{\geq s+1} \cup \left(\bigcup_{J \subseteq \{1, \dots, r-1\}} \text{Ass}_R(M/\sum_{j \in J} x_j M) \setminus \text{Var}(I) \right).$$

We first show that $x_1, \dots, x_r$ is an unconditioned M -sequence in dimension $> s$. Let $\sigma \in \mathbb{S}_r$ be a permutation of $1, \dots, r$. Assume that $x_{\sigma(1)}, \dots, x_{\sigma(r)}$ is not an M -sequence in dimension $> s$. Let $n \in \{1, \dots, r\}$ be the least integer such that $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ is not an M -sequence in dimension $> s$. Then $r = \sigma(i)$ for some $i < n$ by our choice

of x_r , and there is some $\mathfrak{p} \in \left(\text{Ass}_R(M/(x_{\sigma(1)}, \dots, x_{\sigma(n-1)})M) \right)_{\geq s+1}$ such that $x_{\sigma(n)} \in \mathfrak{p}$. So $x_{\sigma(1)}, \dots, x_{\sigma(n)} \in \mathfrak{p}$ and $\frac{x_{\sigma(1)}}{1}, \dots, \frac{x_{\sigma(n)}}{1}$ is not a regular sequence w.r.t. $M_{\mathfrak{p}}$. Therefore $\frac{x_{\sigma(1)}}{1}, \dots, \frac{x_{\sigma(i-1)}}{1}, \frac{x_{\sigma(i+1)}}{1}, \dots, \frac{x_{\sigma(n)}}{1}, \frac{x_{\sigma(i)}}{1}$ is not a regular sequence w.r.t. $M_{\mathfrak{p}}$. Set $J := \{j \in \mathbb{N} : j \leq n, j \neq i\}$. As $\dim(R/\mathfrak{p}) > s$, we know that $\left(\frac{x_{\sigma(j)}}{1}\right)_{j \in J}$ is a regular sequence w.r.t. $M_{\mathfrak{p}}$. Therefore $\frac{x_r}{1} = \frac{x_{\sigma(i)}}{1}$ is not a regular element w.r.t. $M_{\mathfrak{p}}/\sum_{j \in J} x_{\sigma(j)}M_{\mathfrak{p}}$. So, there exists $\mathfrak{q} \in \text{Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\frac{x_r}{1} \in \mathfrak{q} R_{\mathfrak{p}} \in \text{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/\sum_{j \in J} x_{\sigma(j)}M_{\mathfrak{p}})$. It follows that $x_r \in \mathfrak{q} \in \left(\text{Ass}_R(M/\sum_{j \in J} x_{\sigma(j)}M) \right)_{\geq s+1} \subseteq C_r$. This gives a contradiction.

Finally, we need to prove that $x_1, \dots, x_r$ is an unconditioned I -filter regular sequence w.r.t. M . Assume that $x_{\sigma(1)}, \dots, x_{\sigma(r)}$ is not an I -filter regular sequence w.r.t. M for some $\sigma \in \mathbb{S}_r$. Let n be the least integer such that $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ is not an I -filter regular sequence w.r.t. M . By our choice of x_r , we have $r = \sigma(i)$ for some $i < n$ and there exists some $\mathfrak{p} \in \text{Ass}_R(M/(x_{\sigma(1)}, \dots, x_{\sigma(n-1)})M) \setminus \text{Var}(I)$ such that $x_{\sigma(n)} \in \mathfrak{p}$. So $x_{\sigma(1)}, \dots, x_{\sigma(n)} \in \mathfrak{p}$ and $\frac{x_{\sigma(1)}}{1}, \dots, \frac{x_{\sigma(n)}}{1}$ is not a regular sequence w.r.t. $M_{\mathfrak{p}}$. Therefore $\frac{x_{\sigma(1)}}{1}, \dots, \frac{x_{\sigma(i-1)}}{1}, \frac{x_{\sigma(i+1)}}{1}, \dots, \frac{x_{\sigma(n)}}{1}, \frac{x_{\sigma(i)}}{1}$ is not a regular sequence w.r.t. $M_{\mathfrak{p}}$. Set $J := \{j \in \mathbb{N} : j \leq n, j \neq i\}$. As $\mathfrak{p} \notin \text{Var}(I)$, we know that $\left(\frac{x_{\sigma(j)}}{1}\right)_{j \in J}$ is a regular sequence w.r.t. $M_{\mathfrak{p}}$. Therefore $\frac{x_r}{1} = \frac{x_{\sigma(i)}}{1}$ is not a regular element w.r.t. $M_{\mathfrak{p}}/\sum_{j \in J} x_{\sigma(j)}M_{\mathfrak{p}}$. So, there exists some $\mathfrak{q} \in \text{Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\frac{x_r}{1} \in \mathfrak{q} R_{\mathfrak{p}} \in \text{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/\sum_{j \in J} x_{\sigma(j)}M_{\mathfrak{p}})$. It follows that $x_r \in \mathfrak{q} \in \left(\text{Ass}_R(M/\sum_{j \in J} x_{\sigma(j)}M) \right)_{\geq s+1} \setminus \text{Var}(I) \subseteq C_r$, a contradiction. $\square$

Proposition 2.6. *Let s be a non-negative integer. Let $x_1, \dots, x_t$ be an unconditioned M -sequence in dimension $> s$. Then*

$$\left(\bigcup_{n_1, \dots, n_t \in \mathbb{N}} \text{Ass}_R(M/(x_1^{n_1}, \dots, x_t^{n_t})M) \right)_{\geq s} = \left(\text{Ass}_R(M/(x_1, \dots, x_t)M) \right)_{\geq s}.$$

In particular, the set $\left(\bigcup_{n_1, \dots, n_t \in \mathbb{N}} \text{Ass}_R(M/(x_1^{n_1}, \dots, x_t^{n_t})M) \right)_{\geq s}$ is finite.

Proof. We prove the result by induction on t . Let $t = 1$ and we write $x_1 = x$. We will show by induction on $n_1 = n$ that $\left(\text{Ass}_R(M/x^n M) \right)_{\geq s} \subseteq \text{Ass}_R(M/xM)$. The case $n = 1$ is clear.

Let $n > 1$ and assume that the result is true for $n - 1$. Let $\mathfrak{p} \in \left(\text{Ass}_R(M/x^n M) \right)_{\geq s}$. Then $\mathfrak{p} R_{\mathfrak{p}} \in \text{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/x^n M_{\mathfrak{p}})$. If $\dim(R/\mathfrak{p}) > s$ then x is a regular element w.r.t. $M_{\mathfrak{p}}$ and therefore $\mathfrak{p} R_{\mathfrak{p}} \in \text{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/x M_{\mathfrak{p}})$. It follows that $\mathfrak{p} \in \text{Ass}_R(M/x M)$. Assume that $\dim(R/\mathfrak{p}) = s$. From the exact sequence

$$0 \longrightarrow x^{n-1} M/x^n M \longrightarrow M/x^n M \longrightarrow M/x^{n-1} M \longrightarrow 0,$$

we have $\mathfrak{p} \in \text{Ass}_R(M/x^{n-1} M) \cup \text{Ass}_R(x^{n-1} M/x^n M)$. If $\mathfrak{p} \in \text{Ass}_R(M/x^{n-1} M)$ then by induction $\mathfrak{p} \in \text{Ass}_R(M/x M)$. So, assume that $\mathfrak{p} \in \text{Ass}_R(x^{n-1} M/x^n M)$. Consider the exact sequence

$$0 \longrightarrow (x^n M :_M x^{n-1})/x M \longrightarrow M/x M \longrightarrow x^{n-1} M/x^n M \longrightarrow 0.$$

Set $K = (x^n M :_M x^{n-1})/x M$. Assume that $\mathfrak{p} \notin \text{Supp}(K)$. Then the above exact sequence shows that $(M/x M)_{\mathfrak{p}} \cong (x^{n-1} M/x^n M)_{\mathfrak{p}}$. Since $\mathfrak{p} \in \text{Ass}_R(x^{n-1} M/x^n M)$, it follows that $\mathfrak{p} \in \text{Ass}_R(M/x M)$. So, we assume that $\mathfrak{p} \in \text{Supp}(K)$. Note that for any $\mathfrak{q} \in \text{Supp}(M)$ satisfying $\dim(R/\mathfrak{q}) > s$, we have $K_{\mathfrak{q}} = 0$ as x is a poor regular element w.r.t. $M_{\mathfrak{q}}$. So $\dim(K) \leq s$. Hence $\mathfrak{p}$ is a minimal element in $\text{Supp}(K)$, and hence $\mathfrak{p} \in \text{Ass}_R(K) \subseteq \text{Ass}_R(M/x M)$. So, the result is true for $t = 1$.

Let $t > 1$ and assume that the result is true for $t - 1$. Let $n_1, \dots, n_t$ be arbitrary positive integers. Since x_t is an $M/(x_1^{n_1}, \dots, x_{t-1}^{n_{t-1}})M$ -regular element in dimension $> s$, we get by the case $t = 1$ that

$$\left(\text{Ass}_R(M/(x_1^{n_1}, \dots, x_t^{n_t})M) \right)_{\geq s} \subseteq \text{Ass}_R(M/(x_1^{n_1}, \dots, x_{t-1}^{n_{t-1}}, x_t)M).$$

Since $x_1, \dots, x_{t-1}$ is an unconditioned $M/x_t M$ -sequence in dimension $> s$, we have by induction that

$$\begin{aligned} \left(\text{Ass}_R(M/(x_1^{n_1}, \dots, x_{t-1}^{n_{t-1}}, x_t)M) \right)_{\geq s} &= \left(\text{Ass}_R(M/(x_t, x_1^{n_1}, \dots, x_{t-1}^{n_{t-1}})M) \right)_{\geq s} \\ &\subseteq \text{Ass}_R(M/(x_1, \dots, x_t)M). \end{aligned}$$

This proves our claim. $\square$

In general, a permutation of an M -sequence in dimension $> s$ is not an M -sequence in dimension $> s$. In particular, a permutation of a generalized regular sequence is not necessarily a generalized regular sequence. For example, let $R = k[[x_1, x_2, x_3, x_4, x_5]]$ be the ring of power series in 5 variables over a field k . Let $M = R/(x_1) \cap (x_2, x_3, x_4)$. Then $\dim M = 4$ and x_5, x_2 is a generalized regular sequence w.r.t M , but x_2, x_5 is not a generalized regular sequence w.r.t M . This fact shows that Lemma 2.2(i),(ii) of [Nh] is not correct. It follows that the proof of [Nh, 3.1] is not correct, too. The next corollary is an improvement and also a correction of [Nh, 3.1].

Corollary 2.7. *Assume that $x_1, \dots, x_t$ is an unconditioned generalized regular sequence w.r.t. M . Then*

$$\bigcup_{n_1, \dots, n_t \in \mathbb{N}} \text{Ass}_R(M/(x_1^{n_1}, \dots, x_t^{n_t})M) \setminus \text{Max } R \subseteq \text{Ass}_R(M/(x_1, \dots, x_t)M).$$

In particular, if in addition R is local, then $\bigcup_{n_1, \dots, n_t \in \mathbb{N}} \text{Ass}_R(M/(x_1^{n_1}, \dots, x_t^{n_t})M)$ is a finite set.

Proof. The proof is immediate by setting $s = 1$ in Proposition 2.6. $\square$

3 Finiteness results

Remark 3.1. Let $I \subseteq R$ be an ideal with $\text{depth}(I, M) = t$. Then it is well known that

$$\text{Ass}_R(\text{Ext}_R^t(R/I, M)) = \text{Ass}_R(H_I^t(M)).$$

Lemma 3.2. Let t be a positive integer. Set $P_t = \bigcup_{i=0}^{t-1} \text{Supp}(\text{Ext}_R^i(R/I, M))$. Then

$$\begin{aligned} \text{Ass}_R(\text{Ext}_R^t(R/I^n, M)) \cup P_t &= \text{Ass}_R(\text{Ext}_R^t(R/(a_1^{n_1}, \dots, a_k^{n_k}), M)) \cup P_t \\ &= \text{Ass}_R(\text{Ext}_R^t(R/I, M)) \cup P_t = \text{Ass}_R(H_I^t(M)) \cup P_t \end{aligned}$$

for any system of generators $(a_1, \dots, a_k)$ of I and all positive integers $n, n_1, \dots, n_k$.

Proof. Let $\mathfrak{p} \in \text{Supp}(M)$ such that $\mathfrak{p} \notin P_t$. Then for any $i < t$ we have

$$\text{Ext}_{R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \cong (\text{Ext}_R^i(R/I, M))_{\mathfrak{p}} = 0.$$

Therefore $\text{depth}(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \geq t$. Let $n, n_1, \dots, n_k$ be positive integers. It is clear that

$$\text{depth}(I^n R_{\mathfrak{p}}, M_{\mathfrak{p}}) = \text{depth}(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = \text{depth}((a_1^{n_1}, \dots, a_k^{n_k})R_{\mathfrak{p}}, M_{\mathfrak{p}}).$$

We first suppose that $\text{depth}(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) > t$. Then the $R_{\mathfrak{p}}$ -modules $\text{Ext}_{R_{\mathfrak{p}}}^t(R_{\mathfrak{p}}/I^n R_{\mathfrak{p}}, M_{\mathfrak{p}})$, $\text{Ext}_{R_{\mathfrak{p}}}^t(R_{\mathfrak{p}}/(a_1^{n_1}, \dots, a_k^{n_k})R_{\mathfrak{p}}, M_{\mathfrak{p}})$, $\text{Ext}_{R_{\mathfrak{p}}}^t(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}})$, $H_{IR_{\mathfrak{p}}}^t(M_{\mathfrak{p}})$ are zero. So, $\mathfrak{p}$ does not belong to any of the four sets $\text{Ass}_R(\text{Ext}_R^t(R/I^n, M))$, $\text{Ass}_R(\text{Ext}_R^t(R/(a_1^{n_1}, \dots, a_k^{n_k}), M))$, $\text{Ass}_R(\text{Ext}_R^t(R/I, M))$ and $\text{Ass}_R(H_I^t(M))$.

Assume that $\text{depth}(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = t$. Since $\text{rad}(I) = \text{rad}(I^n) = \text{rad}((a_1^{n_1}, \dots, a_k^{n_k})R)$, we have by Remark 3.1 that

$$\begin{aligned} \text{Ass}_{R_{\mathfrak{p}}}(\text{Ext}_{R_{\mathfrak{p}}}^t(R_{\mathfrak{p}}/I^n R_{\mathfrak{p}}, M_{\mathfrak{p}})) &= \text{Ass}_{R_{\mathfrak{p}}}(H_{I^n R_{\mathfrak{p}}}^t(M_{\mathfrak{p}})) = \text{Ass}_{R_{\mathfrak{p}}}(H_{IR_{\mathfrak{p}}}^t(M_{\mathfrak{p}})) \\ &= \text{Ass}_{R_{\mathfrak{p}}}(\text{Ext}_{R_{\mathfrak{p}}}^t(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}})), \end{aligned}$$

$$\text{Ass}_{R_{\mathfrak{p}}}(\text{Ext}_{R_{\mathfrak{p}}}^t(R_{\mathfrak{p}}/(a_1^{n_1}, \dots, a_k^{n_k})R_{\mathfrak{p}}, M_{\mathfrak{p}})) = \text{Ass}_{R_{\mathfrak{p}}}(H_{IR_{\mathfrak{p}}}^t(M_{\mathfrak{p}})).$$

It follows that

$$\begin{aligned} \mathfrak{p} \in \text{Ass}_R(\text{Ext}_R^t(R/I^n, M)) &\Leftrightarrow \mathfrak{p} \in \text{Ass}_R(H_I^t(M)) \Leftrightarrow \mathfrak{p} \in \text{Ass}_R(\text{Ext}_R^t(R/I, M)) \\ &\Leftrightarrow \mathfrak{p} \in \text{Ass}_R(\text{Ext}_R^t(R/(a_1^{n_1}, \dots, a_k^{n_k}), M)). \end{aligned}$$

Now the result follows immediately. $\square$

The next lemma follows easily by induction on t using Remark 3.1 and Lemma 3.2.

Lemma 3.3. *For any integer $t \geq 0$ we have*

$$\bigcup_{i=0}^t \left(\bigcup_{n \in \mathbb{N}} \text{Supp}(\text{Ext}_R^i(R/I^n, M)) \right) = \bigcup_{i=0}^t \text{Supp}(\text{Ext}_R^i(R/I, M)) = \bigcup_{i=0}^t \text{Supp}(H_I^i(M)).$$

It should be mentioned that the equalities $\text{Ass}_R(\text{Ext}_R^t(R/I, M)) \cup P_t = \text{Ass}_R(H_I^t(M)) \cup P_t$ in Lemma 3.2 and $\bigcup_{i=0}^t \text{Supp}(\text{Ext}_R^i(R/I, M)) = \bigcup_{i=0}^t \text{Supp}(H_I^i(M))$ in Lemma 3.3 have been proved by Cuong-Hoang [CH].

Proof of Theorem 1.1. Let $t \leq r$ be a non-negative integer. Set $P_t = \bigcup_{i=0}^{t-1} \text{Supp}(\text{Ext}_R^i(R/I, M))$.

Then $P_t = \bigcup_{i=0}^{t-1} \text{Supp}(H_I^i(M))$ by Lemma 3.3. Therefore, by our assumption, $\dim(R/\mathfrak{p}) \leq s$ for all $\mathfrak{p} \in P_t$. Now, let $\mathfrak{p} \in (T^t(I, M))_{\geq s} \cup (T^t(\underline{a}, M))_{\geq s}$. If $\dim(R/\mathfrak{p}) > s$ then $\mathfrak{p} \notin P_t$. So, we get by Lemma 3.2 that $\mathfrak{p} \in \text{Ass}_R(\text{Ext}_R^t(R/I, M))$.

Let $\dim(R/\mathfrak{p}) = s$. It follows by Lemma 3.2 that $\mathfrak{p} \in \left(\text{Ass}(\text{Ext}_R^t(R/I, M)) \cup P_t \right)_s$. If $\mathfrak{p} \notin \text{Ass}_R(\text{Ext}_R^t(R/I, M))$ then $\mathfrak{p} \in P_t$, and hence $\mathfrak{p}$ is a minimal prime ideal of $\text{Ext}_R^i(R/I, M)$ for some $i < t$. Therefore $\mathfrak{p} \in \text{Ass}_R(\text{Ext}_R^i(R/I, M))$ for some $i < t$. Thus, $T^t(I, M)$ and $T^t(\underline{a}, M)$ are subsets of $\bigcup_{i=0}^t \text{Ass}(\text{Ext}_R^i(R/I, M))$. $\square$

Proof of Theorem 1.2. Let $t \leq r$ be a non-negative integer. Let $\mathfrak{p} \in (T^t(I, M))_{\geq s} \cup (T^t(\underline{a}, M))_{\geq s}$. Let $i < t$. Since $\dim(\text{Supp}(H_I^i(M))) \leq s$, any prime ideal in $\left(\text{Supp}(H_I^i(M)) \right)_{\geq s}$ is minimal in $\text{Supp}(H_I^i(M))$ and hence associated to $H_I^i(M)$. It follows that

$$\left(\text{Supp}(H_I^i(M)) \right)_{\geq s} = \left(\text{Ass}_R(H_I^i(M)) \right)_s \text{ for all } i < t.$$

So, we get by Lemma 3.2 and Lemma 3.3 that $\mathfrak{p} \in \left(\bigcup_{i=0}^{t-1} \text{Ass}_R(H_I^i(M)) \right)_s \cup \left(\text{Ass}_R(H_I^t(M)) \right)_{\geq s}$.

As $x_1, \dots, x_r \in I$ is an unconditioned I -filter regular sequence w.r.t. M , we have by 2.2 that $H_I^i(M) \cong H_I^0(H_{(x_1, \dots, x_i)R}^i(M))$ for all $i = 1, \dots, t$. Therefore, for each $i = 1, \dots, t$, we have

$$\text{Ass}_R H_I^i(M) \subseteq \text{Ass}_R(H_{(x_1, \dots, x_i)R}^i(M)) \subseteq \bigcup_{n \in \mathbb{N}} \text{Ass}_R(M/(x_1^n, \dots, x_i^n)M).$$

As $x_1, \dots, x_r$ is an unconditioned M -sequence in dimension $> s$, we get by Proposition 2.6 that

$$\left(\bigcup_{n \in \mathbb{N}} \text{Ass}_R(M/(x_1^n, \dots, x_i^n)M) \right)_s = \left(\text{Ass}_R(M/(x_1, \dots, x_i)M) \right)_s$$

for all $i = 1, \dots, t$, and

$$\left(\bigcup_{n \in N} \text{Ass}_R(M/(x_1^n, \dots, x_t^n)M) \right)_{\geq s+1} = \left(\text{Ass}_R(M/(x_1, \dots, x_t)M) \right)_{\geq s+1}.$$

If $\dim(R/\mathfrak{p}) > s$ then $t = r$ and $\mathfrak{p} \in \text{Ass}_R H_I^t(M)$, hence $\mathfrak{p} \in \left(\text{Ass}_R(M/(x_1, \dots, x_t)M) \right)_{\geq s+1}$.

If $\dim(R/\mathfrak{p}) = s$ then

$$\mathfrak{p} \in \bigcup_{i=0}^t \left(\text{Ass}_R(H_I^i(M)) \right)_s = \bigcup_{i=0}^t \left(\text{Ass}_R(M/(x_1, \dots, x_i)M) \right)_s.$$

Now the result follows immediately. $\square$

Replacing s by 1 in Theorems 1,2 and on use of Lemma 3.2 we get the following result.

Corollary 3.4. *Let $r > 0$ be an integer such that $\dim(\text{Supp}(H_I^i(M))) \leq 1$ for all $i < r$. Let $\underline{a} = (a_1, \dots, a_k)$ be a system of generators of I . Then*

(a) *For any integer $t \leq r$, the sets $T^t(I, M)$ and $T^t(\underline{a}, M)$ are contained in the set*

$$\bigcup_{i=0}^t \text{Ass}_R(\text{Ext}_R^i(R/I, M)) \cup \left(\text{Max}(R) \cap \bigcup_{i=0}^{t-1} \text{Supp}(H_I^i(M)) \right).$$

(b) *Let $x_1, \dots, x_r \in I$ be an unconditioned generalized regular sequence with respect to M which is an unconditioned I -filter regular sequence with respect to M (such sequences exist by Proposition 2.5). Then for any integer $t \leq r$, the sets $T^t(I, M)$ and $T^t(\underline{a}, M)$ are contained in the set $\left(\text{Ass}_R(M/(x_1, \dots, x_t)M) \right)_{\geq 2} \cup \left(\bigcup_{i=0}^t \text{Ass}_R(M/(x_1, \dots, x_i)M) \right)_1 \cup \text{Max}(R)$.*

Remark 3.5. Let R be local. In Khashyarmansh [Kh], it is shown that if $H_I^i(M)$ is of finite support for all $i < r$ then the set $(T^r(I, M))_{\geq 2} = \{\mathfrak{p} \in T^r(I, M) : \dim R/\mathfrak{p} > 1\}$ is finite. Corollary 3.4 shows that even $T^r(I, M)$ is a finite set.

Setting $s = 0$ in Theorems 1,2 we get the following result.

Corollary 3.6. *Let $r > 0$ be an integer such that $\dim(\text{Supp}(H_I^i(M))) \leq 0$ for all $i < r$. Let $\underline{a} = (a_1, \dots, a_k)$ be a system of generators of I . Then*

(a) *For any integer $t \leq r$, the sets $T^t(I, M)$ and $T^t(\underline{a}, M)$ are contained in the finite set $\bigcup_{i=0}^t \text{Ass}_R(\text{Ext}_R^i(R/I, M))$.*

(b) *Let $x_1, \dots, x_r \in I$ be an unconditioned filter regular sequence with respect to M which is an unconditioned I -filter regular sequence with respect to M (such sequences exist by Proposition 2.5). Then for any integer $t \leq r$, the sets $T^t(I, M)$ and $T^t(\underline{a}, M)$ are contained in the finite set $\left(\text{Ass}_R(M/(x_1, \dots, x_t)M) \right)_{\geq 1} \cup \bigcup_{i=0}^t \text{Ass}_R(M/(x_1, \dots, x_i)M)$.*

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