SIMPLE PROOF OF TWO-WELL RIGIDITY

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Abstract. We give a short proof of the rigidity estimate of Müller and Chaudhuri [3] for two strongly incompatible wells. Making strong use of the arguments of Ball and James our approach shows that incompatibility for gradient Young measures can be used to reduce rigidity estimates for several wells to one-well rigidity.

1. Introduction

A crucial ingredient in rigorous derivations of plate theories from three-dimensional elasticity [4, 5] is a quantitative rigidity estimate in terms of the bulk energy of deformations close to zero-energy configurations. In nonlinear elasticity one usually considers sets of the form $K = \bigcup_{i=1}^n SO(3)A_i$ as the set of deformations which carry zero bulk energy. The different copies of $SO(n)$ are called energy wells. In the rigid situation, when the only deformations with zero energy (i.e. maps $u : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $\nabla u \in K$) are affine maps, it is of interest to find estimates on the precise rate of convergence of approximating sequences. The starting point of such an analysis is the rigidity estimate of Friesecke, James and Müller [5], which says that for any Lipschitz domain $\Omega \subset \mathbb{R}^n$ and for any $p \in [1, \infty]$, there exists a constant $C(p, \Omega)$ so that

$$\inf_{R \in SO(n)} \| \nabla u - R \|_{L^p(\Omega)} \leq C(p, \Omega) \| \text{dist}(\nabla u, K) \|_{L^p(\Omega)} \quad \forall u \in W^{1,p}(\Omega, \mathbb{R}^n),$$

(1)

where $K = SO(n)$ (Friesecke, James and Müller proved in [5] only the case $p = 2$ of (1), but the corresponding inequalities for $p \in ]1, \infty[$ can be obtained by minor modifications of the arguments.

Building on the methods developed in [5] Chaudhuri and Müller in [3] obtained the corresponding rigidity estimate for the case of two strongly incompatible wells $K = SO(n)A_1 \cup SO(n)A_2$. An important ingredient in the proof of Chaudhuri and Müller is the result of Matos in [6] that under certain conditions on the matrices $A_1$ and $A_2$ the exact solutions of the inclusion problem $\nabla u \in K$ are solutions of a certain strongly elliptic system. We also note that Matos used this observation in [6] to deduce incompatibility for gradient Young measures in the sense of Definition 1.1 below.

Our aim in this note is to give a simple proof of how under the condition of incompatibility for gradient Young measures the rigidity estimate of the two-well problem reduces to the rigidity estimate [5] for the one-well problem. Our argument is very much based on the unpublished but well-known argument of Ball and James [1] for obtaining a transition-layer
estimate for approximate solutions to differential inclusions with incompatible wells. Indeed, our estimate in Theorem 1.2 is very similar in spirit to the transition-layer estimate.

**Definition 1.1.** Let \( K_1, K_2 \subset \mathbb{R}^{m \times n} \) be disjoint compact sets. We say that \( K_1, K_2 \) are incompatible for gradient Young measures if whenever \( \nu_x \) is a gradient Young measure on some connected domain \( \Omega \) such that \( \text{supp} \nu_x \subset K_1 \cup K_2 \) for almost every \( x \in \Omega \), then

\[
\text{supp} \nu_x \subset K_1 \text{ for a.e. } x \in \Omega \quad \text{or} \quad \text{supp} \nu_x \subset K_2 \text{ for a.e. } x \in \Omega.
\]

**Theorem 1.2.** Suppose \( K_1, K_2 \subset \mathbb{R}^{m \times n} \) are disjoint compact sets which are incompatible for gradient Young measures and let \( K = K_1 \cup K_2 \). Let \( p \in [1, \infty[ \) and \( \Omega \subset \mathbb{R}^n \) be a connected Lipschitz domain. Then there exists a constant \( C = C(p, \Omega) \) such that

\[
\min \left( \int_{\Omega} d_{K_1}^p(\nabla u)dx, \int_{\Omega} d_{K_2}^p(\nabla u)dx \right) \leq C(p, \Omega) \int_{\Omega} d_K^p(\nabla u)dx \quad \text{for all } u \in W^{1,p}(\Omega, \mathbb{R}^m).
\]

(2)

**Remark 1.** Note that (2) holds even in the critical case \( p = 1 \), in contrast with estimate (1), (see [2]).

**Proof.** By a truncation argument it suffices to prove the inequality when \( \|\nabla u\|_{L^\infty(\Omega)} \leq M \) for some constant \( M \) depending on \( \Omega \), \( K \) and \( p \). Indeed, since \( K \) is compact, we can choose positive constants \( R \) and \( C \) such that \( |A| \leq C d_K(A) \) for every \( A \subset \mathbb{R}^{m \times n} \) with \( |A| \geq R \). By Proposition A.1 of [5] there exists a constant \( C = C(p, \Omega) \) such that, for every \( v \in W^{1,p}(\Omega) \) there exists \( u \in W^{1,\infty}(\Omega) \) satisfying the following properties:

\[
(i) \quad \|\nabla u\|_{L^\infty} \leq CR;
(ii) \quad |\{ x \in \Omega : u(x) \neq v(x) \}| \leq CR^{-p} \int_{\{x \in \Omega : |\nabla v(x)| > R\}} |\nabla v|^p dx;
(iii) \quad \|\nabla v - \nabla u\|_{L^p} \leq C \int_{\{x \in \Omega : |\nabla v(x)| > R\}} |\nabla v|^p dx.
\]

Recall that \( d_K(A) \leq d_{K_1}(A) \leq C(1 + |A|) \). Hence, from (i), (ii), (iii) and the choice of \( R \), it follows easily that

\[
\int d_{K_1}^p(\nabla u)dx \leq C \int d_K^p(\nabla v)dx
\]

and

\[
\int d_{K_2}^p(\nabla v)dx \leq C \left( \int d_{K_1}^p(\nabla u)dx + \int d_K^p(\nabla v)dx \right).
\]

These inequalities show that it suffices to prove (2) for functions \( u \) which enjoy (i).

Let \( B_1 \) denote the unit ball of \( \mathbb{R}^n \). Then, without loss of generality we can assume that \( \Omega \) satisfies the following connectedness property:

(C) For any sequence of points \( x_j \in \Omega \) and any sequence of positive numbers \( r_j \leq \text{diam} (\Omega) \), there exists a subsequence of the sets \( U_j = \frac{1}{r_j} (\Omega - x_j) \cap B_1 \) converging in measure to a connected open set \( U \).
Indeed, (C) is satisfied, for instance, when \( \Omega \) is a standard Lipschitz domain, i.e. a domain of the form \( \{ (x, x') : x \in [0, 1]^{n-1}, f(x) < x' < 1 \} \) for some Lipschitz function \( f \). Therefore, a general bounded Lipschitz open set \( \Omega \) can be covered by finitely many Lipschitz subsets enjoying property (C).

Finally, note that the combination of Lipschitz regularity and (C) gives that

(C1) There exists \( \gamma > 0 \), independent of the choice of \( x_j \)’s and \( r_j \)’s, such that, if \( U \) is as in (C), then \( |U| \geq \gamma \);

(C2) If \( \delta > 0 \) and \( j \) is large enough, there exists a connected \( \tilde{U} \subset U \cap U_j \) with \( |U \setminus \tilde{U}| < \delta \).

We argue by contradiction and assume that the assertion of the theorem is not true. Then there exists a Lipschitz open set \( \Omega \) satisfying (C) and a sequence of maps \( u_j : \Omega \rightarrow \mathbb{R}^m \) with \( \| \nabla u_j \|_{L^\infty(\Omega)} \leq M \) and

\[
\min \left( \int_{\Omega} d_{K_1}^p(\nabla u_j) dx, \int_{\Omega} d_{K_2}^p(\nabla u_j) dx \right) \geq j \int_{\Omega} d_K^p(\nabla u_j) dx. \tag{3}
\]

By considering a suitable subsequence we may assume that \( \nabla u_j \) generates a gradient Young measure \( \nu_x \). From (3) and the uniform Lipschitz bound we deduce that \( \text{supp} \nu_x \subset K \) a.e., hence by incompatibility \( \text{supp} \nu_x \subset K_2 \) a.e., say. Therefore there exists \( c > 0 \) such that

\[
\int_{\Omega} d_{K_1}^p(\nabla u_j) dx \geq c \text{ for all } j \quad \text{and} \quad \int_{\Omega} d_{K_2}^p(\nabla u_j) dx \rightarrow 0 \text{ as } j \rightarrow \infty. \tag{4}
\]

Next, we define \( S_j := \{ x \in \Omega : d_{K_1}(\nabla u_j) \leq d_{K_2}(\nabla u_j) \} \) and \( f_j, g_j : \mathbb{R}^n \rightarrow \mathbb{R} \) by

\[
f_j := \chi_{S_j} d_{K_2}^p(\nabla u_j) \quad \text{and} \quad g_j := \chi_{\Omega \setminus S_j} d_{K_1}^p(\nabla u_j).
\]

If \( |S_j| = 0 \) for some \( j > 1 \), then \( d_K(\nabla u_j(x)) = d_{K_2}(\nabla u_j(x)) \leq d_{K_1}(\nabla u_j(x)) \) a.e. in \( \Omega \), in contradiction with (3). Therefore we may assume that \( |S_j| > 0 \). Using the definition of \( S_j \) we have

\[
\int f_j = \int_{\Omega} d_{K_2}^p(\nabla u_j) dx - \int_{\Omega \setminus S_j} d_{K_2}^p(\nabla u_j) dx = \int_{\Omega} d_{K_2}^p(\nabla u_j) dx - \int_{\Omega \setminus S_j} d_{K}^p(\nabla u_j) dx,
\]

hence (3) implies that \( \int f_j \geq (j - 1) \int_{\Omega} d_{K}^p(\nabla u_j) dx \). On the other hand (4) implies \( \int f_j \rightarrow 0 \), consequently

\[
\int f_j + \int g_j = \int_{S_j} d_{K_2}^p(\nabla u_j) dx + \int_{\Omega \setminus S_j} d_{K_2}^p(\nabla u_j) dx \geq \int_{S_j} d_{K}^p(\nabla u_j) dx + \int_{\Omega \setminus S_j} d_{K}^p(\nabla u_j) dx \geq c.
\]

Therefore by taking a subsequence if necessary we may assume that

\[
\int (f_j - g_j) \leq -c/2. \tag{5}
\]

Let us fix \( j \) for the moment. For a.e. \( x \in S_j \),

\[
\frac{1}{|B_r(x)|} \int_{B_r(x)} f_j \rightarrow f_j(x) \quad \text{and} \quad \frac{1}{|B_r(x)|} \int_{B_r(x)} g_j \rightarrow 0 \quad \text{as } r \rightarrow 0,
\]

where \( f_j(x) = \chi_{S_j} f_j \).
by Lebesgue’s differentiation theorem. Hence
\[
\frac{1}{|B_r(x)|} \int_{B_r(x)} (f_j - g_j) \to d_{K_2}^p (\nabla u_j(x)) > 0 \quad \text{as } r \downarrow 0,
\]
by the definition of \( S_j \) and since \( K_1 \) and \( K_2 \) are disjoint. On the other hand as \( r \to \text{diam } \Omega \) by (5) we have
\[
\frac{1}{|B_r(x)|} \int_{B_r(x)} (f_j - g_j) \to \frac{1}{(\text{diam } \Omega)^n} \int_\Omega (f_j - g_j) \leq -\frac{c}{2(\text{diam } \Omega)^n}.
\]
(6)

Since \( r \to \frac{1}{|B_r(x)|} \int_{B_r(x)} (f_j - g_j) \) is continuous, we deduce the existence of a radius \( r(x) \in (0, \text{diam } \Omega) \) for which \( \int_{B_{r(x)}(x)} f_j = \int_{B_{r(x)}(x)} g_j \). The set of balls \( \{B_{r(x)}(x) : x \in S_j\} \) forms a cover for \( S_j \), so by the Besicovitch covering theorem there exists a number \( N \in \mathbb{N} \) depending only on the dimension \( n \), and \( N \) countable subfamilies of disjoint balls
\[
B_k \subset \{B_{r(x)}(x) : x \in S_j\} \quad \text{for } k = 1, \ldots, N
\]
such that \( \bigcup_{k=1}^N B_k \) forms a cover for \( S_j \). Then
\[
\sum_{k=1}^N \sum_{B \in B_k} \int_B f_j \geq \int f_j,
\]
so there exists \( k \) such that
\[
\sum_{B \in B_k} \int_B f_j \geq \frac{1}{N} \int f_j \geq \frac{1}{N} (j - 1) \int_{\Omega} d_{K}^p (\nabla u_j) dx \geq \frac{1}{N} (j - 1) \sum_{B \in B_k} \int_{B \cap \Omega} d_{K}^p (\nabla u_j) dx.
\]
(7)

Therefore there exists a ball \( B = B_{r_j}(x_j) \in B_k \) such that
\[
\int_{B_{r_j}(x_j)} f_j = \int_{B_{r_j}(x_j)} g_j \geq \frac{1}{N} (j - 1) \int_{B_{r_j}(x_j) \cap \Omega} d_{K}^p (\nabla u_j) dx.
\]
(8)

Let \( U_j = \frac{1}{r_j} (\Omega - x_j) \cap B_1 \), \( \Sigma_j = \frac{1}{r_j} (S_j - x_j) \cap B_1 \) and \( v_j : U_j \to \mathbb{R}^m \) be defined as
\[
v_j(x) = \frac{u_j(r_j(x - x_j)) - (u_j)_{x_j,r_j}}{r_j},
\]
where \( (u_j)_{x,r} \) denotes the average of \( u_j \) in \( B_{r}(x_j) \cap \Omega \). Then \( \| \nabla v_j \|_{L^\infty(U_j)} \leq M \) and
\[
\int_{\Sigma_j \cap U_j} d_{K_2}^p (\nabla v_j) = \int_{U_j \setminus \Sigma_j} d_{K_1}^p (\nabla v_j) \geq \frac{j - 1}{N} \int_{U_j} d_{K}^p (\nabla v_j)
\]
(9)

From the properties (C) and (C1), we can assume that a suitable subsequence of \( U_j \) converges to a connected open set \( U \) with \( |U| \geq \gamma \). Let \( \delta > 0 \), to be fixed later. By (C2) there exists a connected open set \( \tilde{U} \subset U \) with \( |U \setminus \tilde{U}| < \delta \), and \( \tilde{U} \subset U_j \) for sufficiently large \( j \). After taking a further subsequence we may assume that the sequence \( \{\nabla v_j\} \) generates a gradient
Young measure $\nu_x$ for $x \in \bar{U}$. In particular from (9) we deduce that $\text{supp} \, \nu_x \subset K$ for almost every $x \in \bar{U}$, hence by incompatibility

$$\int_{\bar{U}} d_{K_1}^p(\nabla v_j) \, dx \to 0 \quad \text{as} \quad j \to 0 \quad \text{or} \quad \int_{\bar{U}} d_{K_2}^p(\nabla v_j) \, dx \to 0 \quad \text{as} \quad j \to 0.$$  

(10)

By the Lipschitz bound and (9) we also have

$$\left| \int_{\Sigma_j \cap \bar{U}} d_{K_2}^p(\nabla v_j) - \int_{\bar{U} \setminus \Sigma_j} d_{K_1}^p(\nabla v_j) \right| \leq M|U \setminus \bar{U}| \leq M\delta.$$

In either case from (10) we deduce that for large enough $j \in \mathbb{N}$

$$\int_{\Sigma_j \cap \bar{U}} d_{K_2}^p(\nabla v_j) \leq 2M\delta \quad \text{and} \quad \int_{\bar{U} \setminus \Sigma_j} d_{K_1}^p(\nabla v_j) \leq 2M\delta,$$

and also $\int_{\bar{U}} d_{K}^p(\nabla v_j) \leq M\delta$. But then, from the definition of $\Sigma_j$ we get $\int_{\bar{U}} d_{K_2}^p(\nabla v_j) \leq 3M\delta$ and $\int_{\bar{U}} d_{K_1}^p(\nabla v_j) \leq 3M\delta$ for sufficiently large $j$, which contradicts disjointness of $K_1$ and $K_2$ if $\delta$ is chosen sufficiently small.

\[\square\]

References


