PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 3, Pages 643–646
S 0002-9939(99)05655-5
Article electronically published on October 25, 1999

EIGENVALUE COMPLETIONS BY AFFINE VARIETIES

JOACHIM ROSENTHAL AND XIAOCHANG WANG

(Communicated by John A. Burns)

ABSTRACT. In this paper we provide new necessary and sufficient conditions for a general class of eigenvalue completion problems.

1. Preliminaries

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. Let $Mat_{n\times n}$ be the space of all $n \times n$ matrices defined over the field $\mathbb{F}$. We will identify $Mat_{n\times n}$ with the vector space $\mathbb{F}^{n^2}$. Let $\mathcal{X} \subset Mat_{n\times n}$ be an affine variety. If $M \in Mat_{n\times n}$ is a particular matrix we will denote by $\sigma_i(M)$ the i-th elementary symmetric function in the eigenvalues of M, i.e. $\sigma_i(M)$ denotes up to sign the i-th coefficient of the characteristic polynomial of M.

In this note we will be interested in conditions on the variety $\mathcal{X}$ which guarantee that the morphism

$$\chi : \mathcal{X} \longrightarrow \mathbb{F}^n, \quad X \longmapsto (\sigma_1(A+X), \ldots, \sigma_n(A+X)) \tag{1.1}$$

is dominant for a particular matrix A. In other words we are interested under what conditions the image misses at most a proper algebraic subset, i.e. the image forms a generic subset of $\mathbb{F}^n$. This problem was treated in [HRW97] when $\mathcal{X}$ is a linear subspace of $Mat_{n\times n}$ and the base field $\mathbb{F}$ consists of the complex numbers. In this paper we generalize those results to the situation when $\mathcal{X}$ represents a general affine (irreducible) variety defined over $\mathbb{F}$.

Our study is motivated in part by an extensive literature on matrix completion problems and by several applications arising in the control literature. We refer to the research monograph [GKS95], which provides a good overview on the large linear algebra literature on matrix completions and to the survey articles [Byr89, RW97] for the connections to the control literature and further references.

2. Main result

Theorem 2.1. *The characteristic map χ introduced in* (1.1) *is dominant for a generic set of matrices $A \in Mat_{n\times n} \cong \mathbb{F}^{n^2}$ if and only if $\dim \mathcal{X} \geq n$ and the trace function $\mathrm{tr} \in \mathcal{O}(\mathcal{X})$ is not a constant.*

Received by the editors March 4, 1997 and, in revised form, April 2, 1998.
2000 *Mathematics Subject Classification.* Primary 15A18; Secondary 93B60.
Key words and phrases. Eigenvalue completions, pole placement problems, dominant morphism theorem, inverse eigenvalue problems.
The first author was supported in part by NSF grant DMS-9400965.
The second author was supported in part by NSF grant DMS-9500594.

The stated conditions are obviously necessary. Our proof is mainly based on two propositions. The first one is a strong version of the Dominant Morphism Theorem. Our formulation is immediately deduced from [Bor91, Chapter AG, §17, Theorem 17.3].

Proposition 2.2 (Dominant Morphism Theorem)**.** *Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a morphism of affine varieties. Then ϕ is dominant if and only if there is a smooth point $P \in \mathcal{X}$ having the property that $\phi(P)$ is smooth and the Jacobian $d\phi_P : T_P(\mathcal{X}) \to T_{\phi(P)}(\mathcal{Y})$ is surjective.*

The second proposition which we will need in the proof of Theorem 2.1 is:

Proposition 2.3 ([HRW97])**.** *Let $\mathcal{L} \subset Mat_{n\times n}$ be a linear subspace of dimension $\geq n$, $\mathcal{L} \not\subset sl_n$ (i.e. $\mathcal{L}$ contains an element with nonzero trace). Let*

$$\pi(L) = (l_{11}, l_{22}, \ldots, l_{nn})$$

be the projection onto the diagonal entries. Then there exists an $S \in Gl_n$ such that

$$\pi(S\mathcal{L}S^{-1}) = \mathbb{F}^n.$$

This proposition was formulated in [HRW97, Lemma 2.8] when the base field $\mathbb{F}$ consists of the complex numbers. The proof presented in [HRW97] only requires linearizations of rational functions and it is therefore valid mutatis mutandis for an arbitrary base field $\mathbb{F}$.

Proof of Theorem 2.1. As mentioned earlier it is enough to show the sufficiency of the stated conditions. It has been pointed out in [HRW97] that the characteristic map χ is dominant, respectively surjective, if and only if the *trace map*

$$\psi :\ \mathcal{X} \longrightarrow \mathbb{F}^n, \quad X \longmapsto (\mathrm{tr}(A+X), \ldots, \mathrm{tr}(A+X)^n) \tag{2.1}$$

is dominant, respectively surjective. This follows from the so-called Newton identities which express the elementary symmetric functions $\sigma_i(M)$ in terms of the power sum symmetric functions $\{\mathrm{tr}(M^j) \mid 1 \leq j \leq n\}$.

It is the strategy of our proof to show the existence of a smooth point $P \in \mathcal{X}$ which has the property that the Jacobian $d\psi_P$ is surjective for a generic set of matrices $A \in Mat_{n\times n}$. Since the range of the trace map ψ is a smooth variety, the proof would be complete.

Let $Q \in \mathcal{X}$ be a smooth point and consider the polynomial function

$$f(M) := \mathrm{tr}(M) - \mathrm{tr}(Q) \in \mathcal{O}(Mat_{n\times n}) = \mathbb{F}[x_{11}, \ldots, x_{nn}].$$

Let $\mathcal{H}$ be the linear hypersurface

$$\mathcal{H} := \{U \in Mat_{n\times n} \mid f(U) = 0\}.$$

Since $Q \in \mathcal{H} \cap \mathcal{X}$, it follows by the affine dimension theorem that the dimension of $\mathcal{H} \cap \mathcal{X}$ is at least $\dim \mathcal{X} - 1$. Since by assumption $\mathcal{X}$ is irreducible and $\mathrm{tr} \in \mathcal{O}(\mathcal{X})$ is not a constant, it follows that

$$\dim(\mathcal{H} \cap \mathcal{X}) = \dim \mathcal{X} - 1.$$

Let $\mathcal{S} \subset \mathcal{X}$ be the singular locus and let $\mathcal{I} \subseteq \mathcal{H} \cap \mathcal{X}$ be the irreducible component of $\mathcal{H} \cap \mathcal{X}$ which contains the smooth point Q. It follows that $\mathcal{S} \cap \mathcal{I}$ is a proper algebraic subset of $\mathcal{I}$. Because of this there exists a point $P \in \mathcal{I} \subset \mathcal{X}$ which is both smooth inside $\mathcal{I}$ as well as inside $\mathcal{X}$.

By construction the tangent space $T_P(\mathcal{I})$ is properly contained inside the tangent space $T_P(\mathcal{X})$ and one has the relation

$$T_P(\mathcal{I}) = T_P(\mathcal{X}) \cap sl_n.$$

By Proposition 2.3 there exists an $S \in Gl_n$ such that

$$\pi(S(T_P(\mathcal{X}))S^{-1}) = \mathbb{F}^n.$$

Consider the trace map ψ introduced in (2.1). A direct computation shows that the Jacobian at the point P is given through:

$$d\psi_P : \ T_P(\mathcal{X}) \longrightarrow \mathbb{F}^n, \quad L \longmapsto \left(\mathrm{tr}(L), 2\mathrm{tr}\left((A+P)L\right), \dots, n \cdot \mathrm{tr}\left((A+P)^{n-1}L\right)\right).$$

Since the characteristic of $\mathbb{F}$ is zero, $\mathbb{F}$ contains as a prime field the rational numbers $\mathbb{Q}$. The matrices

$$D := \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{pmatrix}, \quad V := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & n & \cdots & n^{n-1} \end{pmatrix} \tag{2.2}$$

are therefore invertible. Define $A := S^{-1}DS - P$. With our choice of the matrix A the Jacobian is given through:

$$\begin{aligned} d\psi_P(L) &= \left(\mathrm{tr}(SLS^{-1}), 2\mathrm{tr}(SLS^{-1}D), \dots, n\mathrm{tr}(SLS^{-1}D^{n-1})\right) \\ &= \pi(SLS^{-1})VD. \end{aligned}$$

Since both the matrices V and D describe invertible transformations on $\mathbb{F}^n$, it follows that $d\psi_P$ is surjective for the particular choice of the matrix A. By the Dominant Morphism Theorem 2.2, ψ and therefore χ is dominant.

Since the set of matrices A whose associated Jacobian $d\psi_P$ forms a Zariski open set, and since we just showed that it is nonempty, it follows that for a generic set of matrices the map χ is dominant. □

In the remainder of the paper we assume that $\mathcal{X} \subset \mathbb{F}^{n^2}$ is a fixed affine variety of dimension $\dim \mathcal{X} = m \geq n$, with coordinate ring $\mathcal{O}(\mathcal{X})$ and vanishing ideal $I(\mathcal{X})$. We conclude the paper with an algebraic description of all matrices A whose characteristic map is dominant.

Let $(f_1(X), \dots, f_k(X))$ be generators of $I(\mathcal{X})$ and define

$$T(X) = \begin{pmatrix} \frac{\partial f_1(X)}{\partial x_{11}} & \cdots & \frac{\partial f_k(X)}{\partial x_{11}} \\ \vdots & & \vdots \\ \frac{\partial f_1(X)}{\partial x_{1n}} & \cdots & \frac{\partial f_k(X)}{\partial x_{1n}} \\ \vdots & & \vdots \\ \frac{\partial f_1(X)}{\partial x_{nn}} & \cdots & \frac{\partial f_k(X)}{\partial x_{nn}} \end{pmatrix}.$$

Let $\iota_{ij}(M)$ denote the ijth entry of the matrix M and let

$$J(X) := \begin{pmatrix} \iota_{11}(I) & \iota_{11}(A^t + X^t) & \dots & \iota_{11}((A^t + X^t)^{n-1}) \\ \vdots & \vdots & & \vdots \\ \iota_{1n}(I) & \iota_{1n}(A^t + X^t) & \dots & \iota_{1n}((A^t + X^t)^{n-1}) \\ \vdots & \vdots & & \vdots \\ \iota_{nn}(I) & \iota_{nn}(A^t + X^t) & \dots & \iota_{nn}((A^t + X^t)^{n-1}) \end{pmatrix}.$$

Theorem 2.4. *Let $\Delta \subset \mathbb{F}[x_{11}, \ldots, x_{nn}]$ be the ideal generated by the $(m+n) \times (m+n)$ minors of the matrix $[J(X), T(X)]$. Then χ is not dominant for a particular matrix A if and only if $\Delta \subset I(\mathcal{X})$.*

Proof. With respect to the standard basis of $Mat_{n\times n}$ the matrix $T(X)$ defines a linear transformation $T(X): \mathbb{F}^{n^2} \to \mathbb{F}^k, x \mapsto xT(X)$ and $\ker(T(X)) = T_X(\mathcal{X})$. The rank of $T(X)$ is by assumption at most m, the dimension of $\mathcal{X}$. Similarly the matrix $J(X)$ defines a linear transformation $J(X): \mathbb{F}^{n^2} \to \mathbb{F}^n, x \mapsto xJ(X)$ and $J(X)$ restricted to $T_X(\mathcal{X})$ is exactly $d\psi_X$. The concatenated matrix $[J(X), T(X)]$ induces a linear map $\tau: \mathbb{F}^{n^2} \to \mathbb{F}^n \oplus \mathbb{F}^k$.

If $\Delta \subset I(\mathcal{X})$, then $\mathcal{X} \subset V(\Delta)$, the algebraic set defined by Δ. It follows that the rank of τ is strictly less than $m+n$ for all matrices $X \in \mathcal{X}$. It is therefore not possible to find a smooth point P whose associated map $d\psi_P$ has full rank n. By the dominant morphism theorem ψ and therefore χ is not dominant.

On the other hand if $\Delta \not\subset I(\mathcal{X})$, then there is a smooth point P such that $[J(P), T(P)]$ has rank $m+n$. Since $T(P)$ has rank m it follows that for every $y \in \mathbb{F}^n$ the point $(y, 0) \in \mathbb{F}^n \oplus \mathbb{F}^k$ is in the image of τ. It follows that $d\psi_P$ is surjective and once more by the dominant morphism theorem ψ and χ are dominant. $\square$

The following statement is a reformulation:

Corollary 2.5. *χ is dominant for a particular matrix A if and only if the matrix $[J(X), T(X)]$ has rank $m+n$ over the ring $\mathcal{O}(\mathcal{X})$.*

Remark 2.6. Theorem 2.1 did assume that the characteristic of $\mathbb{F}$ is zero. If the characteristic of $\mathbb{F}$ is p and $p > n$, then our proof of Theorem 2.1 is still valid. If $0 < p \leq n$, then the Newton identities expressing the elementary symmetric functions $\sigma_i(M)$ in terms of the power sum symmetric functions $\{\mathrm{tr}(M^j) \mid 1 \leq j \leq n\}$ do not exist and our proof method does not go through. It therefore remains an open question if Theorem 2.1 is also true in characteristic p, where $0 < p \leq n$.

References

[Bor91] A. Borel, *Linear algebraic groups*, second enlarged edition, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR **92d:**20001

[Byr89] C. I. Byrnes, *Pole assignment by output feedback*, Three Decades of Mathematical System Theory (H. Nijmeijer and J. M. Schumacher, eds.), Lect Notes in Control and Information Sciences # 135, Springer Verlag, 1989, pp. 31–78. MR **90k:**93001

[GKS95] I. Gohberg, M. A. Kaashoek, and F. van Schagen, *Partially specified matrices and operators: Classification, completion, applications*, Birkhäuser, Boston-Basel-Berlin, 1995. MR **97i:**47002

[HRW97] W. Helton, J. Rosenthal, and X. Wang, *Matrix extensions and eigenvalue completions, the generic case*, Trans. Amer. Math. Soc. **349** (1997 no. 8, 3401–3408.) MR **97m:**15010

[RW97] J. Rosenthal and X. Wang, *Inverse eigenvalue problems for multivariabl linear systems*, Systems and Control in the Twenty-First Century (Boston-Basel-Berlin) (C. I. Byrnes, B. N. Datta, D. Gilliam, and C. F. Martin, eds.), Birkäuser, Boston-Basel-Berlin, 1997, pp. 289–311. MR **97k:**93027

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556-5683
E-mail address: `Rosenthal.1@nd.edu`

Department of Mathematics, Texas Tech University, Lubbock, Texas 79409-2013
E-mail address: `mdxia@ttacs1.ttu.edu`