PUBLIC KEY CRYPTOGRAPHY BASED
ON SEMIGROUP ACTIONS

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Abstract. A generalization of the original Diffie-Hellman key exchange in $(\mathbb{Z}/p\mathbb{Z})^*$ found a new depth when Miller [27] and Koblitz [16] suggested that such a protocol could be used with the group over an elliptic curve. In this paper, we propose a further vast generalization where abelian semigroups act on finite sets. We define a Diffie-Hellman key exchange in this setting and we illustrate how to build interesting semigroup actions using finite (simple) semirings. The practicality of the proposed extensions rely on the orbit sizes of the semigroup actions and at this point it is an open question how to compute the sizes of these orbits in general and also if there exists a square root attack in general.

In Section 5 a concrete practical semigroup action built from simple semirings is presented. It will require further research to analyse this system.

1. Introduction

The (generalized) discrete logarithm problem is the basic ingredient of many cryptographic protocols. It asks the following question:

Problem 1.1. (see e.g. [26]). Given a finite group G and elements $g, h \in G$, find an integer $n \in \mathbb{N}$ such that $g^n = h$.

Problem 1.1 has a solution if and only if $h \in \langle g \rangle$, the cyclic group generated by g. If $h \in \langle g \rangle$ then there is a unique integer n satisfying $1 \le n \le ord(g)$ such that $g^n = h$. We call this unique integer the discrete logarithm of h with base g and we denote it by $\log_g h$.

Protocols where the discrete logarithm problem plays a significant role are the Diffie-Hellman key agreement [9], the ElGamal public key cryptosystem [10], the digital signature algorithm (DSA) and ElGamal's signature scheme [26].

The Diffie-Hellman protocol [9] allows two parties, say Alice and Bob, to exchange a secret key over some insecure channel. In order to achieve this goal Alice and Bob agree on a group G and a common base $g \in G$. Alice chooses a random integer $a \in \mathbb{N}$ and Bob chooses a random integer $b \in \mathbb{N}$. Alice transmits to Bob g^a and Bob transmits to Alice g^b. Their common secret key is $k := g^{ab}$.

It is clear that solving the underlying discrete logarithm problem is sufficient for breaking the Diffie-Hellman protocol. For this reason researchers have been searching for groups where the discrete logarithm problem is considered a computationally difficult problem.

In the literature many groups have been proposed as candidates for studying the discrete logarithm problem. Groups which have been implemented in practice are the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$ of integers modulo n, the multiplicative group $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ of nonzero elements inside a finite field $\mathbb{F}$ and subgroups [19, 31] of these groups. In recent time there has been intense study of the discrete logarithm problem in the group over an elliptic curve [3, 16, 27, 26] or more generally the group over an abelian variety [8, 11, 17].

In this paper, we show how the discrete logarithm problem over a group can be seen as a special instance of an action by a semigroup. The interesting thing is that every semigroup action by an abelian semigroup gives rise to a Diffie-Hellman key exchange. With an additional assumption it is also possible to extend the ElGamal protocol.

The idea of using (semi)group actions for the purpose of building one-way trapdoor functions is not a new one and it appeared in one way or the other in several papers. E.g. Yamamura [36] has been considering a group action of $Sl_2(\mathbb{Z})$. Blackburn and Galbraith [2] have been analyzing the system of [36] and they have shown that it is insecure. The key exchange protocol in our paper differs however from [36] and the 'bit by bit' computation of Blackburn and Galbraith [2] does not apply. Other papers where special instances of semigroup actions appear are [1, 15, 33, 34] and we will say more in a moment.

The paper is structured as follows: In the next section we define G-actions on sets, where G is an arbitrary semigroup. Under the assumption that G is abelian we define a general Diffie-Hellman protocol. In Section 3 we consider semigroup actions which can be linearized in the sense that there exists a computable homomorphism which embeds the semigroup G into $\mathrm{Mat}_n(\mathbb{F})$, the ring of $n \times n$ matrices. Section 4 and Section 5 contain the main results of the paper. We show how semirings can be used to build interesting abelian semigroup actions.

A promising practical example which we are describing in Section 5 consists of a two sided action. The idea of such an action originates in the 2003 dissertation of Maze [24]. Later, Shpilrain and Ushakov [33] have described similar two-sided actions in the context of Thompson groups. The semigroups we are studying in Section 5 are built from simple semirings. Simple semirings are of importance as they assure that the induced matrix semiring is simple. In the special case when the semiring is the ring of integers modulo n Slavin [34] filed a patent for the described system citing the work of Maze. Neither [33] nor [34] build general semigroup actions starting from semirings. At this point it is not clear if there exist

parameter ranges where the described twosided action is simultaneously efficient and practically secure.

2. The generalized Diffie-Hellman protocol

Consider a semigroup G, i.e., a set that comes with an associative multiplication '·'. In particular we do not require that G has either an identity element or that each element has an inverse. However, without loss of generality, we will always assume that the semigroup has an identity. We say that the semigroup is abelian if the multiplication · is commutative.

Let S be a finite set and G a a semigroup. A (left) action of G on S is a map

$$\varphi : \quad G \times S \quad \longrightarrow \quad S,$$

satisfying $\phi(g \cdot h, s) = \phi(g, \phi(h, s))$. We will refer to such an action as a G-action on the set S, and when the context is clear, we denote $\phi(g, s)$ simply by gs. Right actions are similarly defined.

We present now the protocols one can define based on semigroup actions:

Protocol 2.1. (Extended Diffie-Hellman Key Exchange) Let S be a finite set, G be an abelian semigroup, and ϕ a G−action on S. The Extended Diffie-Hellman key exchange in (G, S, ϕ) is the following protocol:

1. Alice and Bob publicly agree on an element $s \in S$.
2. Alice chooses $a \in G$ and computes as. Alice's private key is a, her public key is as.
3. Bob chooses $b \in G$ and computes bs. Bob's private key is b, his public key is bs.
4. Their common secret key is then

$$a(bs) = (a \cdot b)s = (b \cdot a)s = b(as).$$

As in the situation of the discrete logarithm problem it is possible to construct ElGamal one-way trapdoor functions which are based on group actions. The interested reader finds more details in [25, 28].
One would build a cryptosystem based on a semigroup action only if the following problem is hard:

Problem 2.2. (Semigroup Action Problem (SAP)): Given a semigroup G acting on a set S and elements $x \in S$ and $y \in Gx$, find $g \in G$ such that $gx = y$.

If an attacker, Eve, can find an $\alpha \in G$ such that $\alpha s = as$, then Eve may find the shared secret by computing $\alpha(bs) = (\alpha \cdot b)s = b(\alpha s) = b(as)$.

Although the semigroup G need not be finite, the finiteness of S is sufficient in order to provide a bound for the size of the data during the communication. Nevertheless, if the action preserves the "size" of s with respect to some fixed representation, finiteness of S is not necessary.

Remark 2.3. The traditional Diffie-Hellman key exchange is a special instance of Protocol 2.1. For this let:

- G be the semigroup $(\mathbb{Z}, \cdot)$ of integers.
- S be a cyclic group H where the discrete logarithm problem is believed to be difficult.

- s is a generator of the group H and the action is defined by

$$\varphi: \quad \begin{array}{ccc} \mathbb{Z} \times H & \longrightarrow & H \\ (n, s) & \longmapsto & s^n. \end{array}$$

The identity $s^{ab} = (s^a)^b$ simply says that φ is a commutative G-action and the reader readily verifies that Protocol 2.1 reduces to the traditional protocol in this case.

Of course, there is an analogue version of the Diffie-Hellman Problem stated in terms of semigroup.

Problem 2.4. (The Diffie-Hellman Semigroup Problem) Given a finite abelian semigroup G acting on a finite set S and elements $x, y, z \in S$ with $y = g \cdot x$ and $z = h \cdot x$ for some $g, h \in G$, find $(gh) \cdot x \in S$.

The security of Protocol 2.1 is equivalent to this problem. The only way we know how to attack Problem 2.4 is to solve SAP. It is unknown if SAP and Problem 2.4 are equivalent.

2.1. GENERIC ATTACKS ON THE SAP. First, we should examine the brute force attack. Suppose Eve intercepts as and bs through an insecure channel and wants to decode the ciphertext $a(bs) = b(as)$. She may want to try the brute force attack to solve Problem 2.2: she computes gs for all possible $g \in G$ until she finds some α with $\alpha s = as$. She is then able to break the system as explained above. To avoid this attack, Bob and Alice must choose G and S sufficiently large and select a good candidate for s. Namely, if

$$G_{Eve} = \{\alpha \in G \mid \alpha s = as\}$$

then the different parameters G, S, s must be chosen such that the size of G_{Eve} is small with respect to the size of G.

If G has the structure of a group (and not just a semigroup) then G_{Eve} is simply a left coset of the stabilizer group

$$\mathrm{Stab}(s) = \{g \in G \mid gs = s\}$$

and in this case we are requiring that the quotient group $G/\mathrm{Stab}(s)$ is large.

For a general abelian semigroup G we observe that $\mathrm{Stab}(s)$ is still a sub-semigroup of G and every element $\alpha \in a\,\mathrm{Stab}(s)$ has the property that $\alpha \in G_{Eve}$, i.e., $a\,\mathrm{Stab}(s) \subset G_{Eve}$. Again in this case we require that $\mathrm{Stab}(s)$ is small in comparison to G.

Note also that every sub-semigroup H of G gives rise to an equivalence relation on S. If one has the ability to efficiently compute canonical representatives for the equivalence classes (among other things), this could potentially be used to an attacker's advantage. But as we will see in Section 4, this is not always an easy task.

It is of course an interesting question if a square root attack exists for general semigroup actions. In the following we explain that for special cases this is possible. In general we do not know how to adapt the known algorithms like e.g. baby step giant step, or the algorithms Pollard rho or Pollard Kangaroo.

Consider an arbitrary instance of the SAP, where one is given a semigroup G (say as a subset of $\{0, 1\}^N$, with N not too much larger than $\log_2 |G|$), a set X (say as a subset of $\{0, 1\}^M$, with M not too much larger than $\log_2 |X|$) and 'black-box' type

functions π and α for quickly computing the semigroup product and the action, respectively:

$$\pi : G \times G \longrightarrow G, \quad \alpha : G \times X \longrightarrow X.$$

In addition, one is given $x \in X$ and an element $y \in Gx$ in the orbit of x. It is also reasonable to assume the availability of oracles for producing elements of G and X uniformly at random. The goal then is to find a $g \in G$ for which $\alpha(g, x) = gx = y$. We do not know a method for solving such an arbitrary instance with $O(\sqrt{|G|})$ operations, except in some special cases.

Situation I: Suppose an element $g \in G$ is known for which $g^k x = y$ for some $k \geq 1$. In this case, one first determines the period and preperiod of g by a method similar to Pollard's rho method, which needs $O(\sqrt{\mathrm{ord}(p)}) = O(\sqrt{|G|})$ operations, where $\mathrm{ord}(p)$ is the period plus the preperiod of g (see the definition in Section 5). Then the baby-step giant-step method can be applied in an obvious way to find k with another $O(\sqrt{\mathrm{ord}(p)}) = O(\sqrt{|G|})$ operations. Note: this applies immediately to the case where G is a cyclic group.

Situation II: G is a group, but not cyclic. For typical groups, inverses are easily computable, but in any case, one may always find inverses with $O(\sqrt{|G|})$ group operations, so it suffices to solve $g_1 x = g_2 y$, from which one obtains $(g_2^{-1} g_1) x = y$. For this, a randomized baby-step giant-step is possible. Compute and store a set $A = \{h_1 x, \ldots, h_m x\}$ for randomly chosen $h_i \in G$ and $m \approx \sqrt{|G|}$. With clever hashing techniques (or, in the worst case, sorting A) it is possible to quickly test if a given element of X is in the set A. One then chooses random values of $h \in G$ until one is found with $hy \in A$. If $hy \in A$, we then have $hy = h_i x$ for some i, and so $g = h^{-1} h_i$.

If the semigroup is neither a group nor the set-theoretic union of a small number of cyclic sub-semigroups we do not know how to adapt the algorithms known for the DLP of abelian groups (see e.g. [4]). In contrast to the DLP problem actions of a semigroup G on a set X can result in a G-orbit Gs, $s \in X$, consisting of many ultimately periodic orbits $\{g^k s \mid k \in \mathbb{N}\}$, $g \in G$. We have observed such phenomena in the action described in Section 5. It is an open research question to come up with a possible square root attack or to show that under certain conditions a square root attack cannot exist for general semigroup actions on sets.

For semigroup actions where a square root attack exists and no other attack is known (like e.g. the DLP over an elliptic curve) it is generally accepted that an orbit size having 160 bits is sufficient for practical security. For cases where no square root attack is known orbit sizes of 80 bits could be sufficient for practical security.

3. Linear abelian semigroup actions over fields

This section is about linearity in the sense that there is a way to see the semigroup action as a matrix action on some vector space. We show that if the correspondence between the two approaches is computationally feasible, then the Diffie-Hellman semigroup problem and the semigroup action problem may be solved easily. Two examples of such action are presented at the end of the section.

Let us describe the situation more specifically. Let $\mathbb{F} = \mathbb{F}_q$ be the field with q elements. Suppose we are given an action $G \times S \longrightarrow S$, with G a finite abelian semigroup and S a finite set, a semigroup homomorphism $\rho : G \longrightarrow \mathrm{Mat}_n(\mathbb{F})$ (with

multiplication as operation) and an embedding $\psi : S \longrightarrow \mathbb{F}^n$ such that for all $g \in G, s \in S$ one has

$$\psi(g \cdot s) = \rho(g)\psi(s).$$

So $\rho(G)$ is a commutative sub-semigroup of $\mathrm{Mat}_n(\mathbb{F})$. Let $\mathbb{F}[G]$ be the commutative subalgebra of $\mathrm{Mat}_n(\mathbb{F})$ generated by the elements of $\rho(G)$.

Suppose there exist polynomial time algorithms that compute the semigroup operation, the semigroup action, the values of the maps ρ and ψ and polynomial time algorithms that compute $\rho^{-1}(M)$ for each $M \in \rho(G)$ and $\psi^{-1}(v)$ for each $v \in \psi(S)$. The next theorem does not take in consideration the speed of these algorithms. It only describes what can be done at the level of the linear algebra without taking consideration of the reduction itself. We also suppose we have access to an oracle Λ that allow us to randomly chose elements in $\mathbb{F}[G]$. This assumption takes into account the desire to capture the situations were the semigroup G is close to a real matrix algebra.

Theorem 3.1. *Let G, S, ψ be arbitrary parameters as above and let $k = \dim_{\mathbb{F}} \mathbb{F}[G]$. Then:*

1. *There exists a probabilistic polynomial time reduction of the Diffie-Hellman semigroup problem to a linear algebra problem over $\mathbb{F}$ that can be solved in an expected $O(k^2 n + n^3)$ number of field operations.*
2. *Let $N = |\mathbb{F}[G]|/|G|$. There exists a probabilistic polynomial time reduction of the SAP to a linear algebra problem over $\mathbb{F}$ that can be solved in an expected $O(N(k^2 n + n^3))$ number of field operations.*

The above O-constants come from the cost of standard linear algebra problems and bounded expected values.

Proof. Let x, $y = g\cdot x$ and $z = h\cdot x$ be three elements of S with u,v and w their images in $\mathbb{F}^n$. We consider the semigroup action problem instance with parameters x and y and the Diffie-Hellman semigroup problem instance with additional parameter z.

1. Suppose we have chosen randomly k different elements $M_1, ..., M_k$ in $\mathbb{F}[G] \subset \mathrm{Mat}_n(\mathbb{F})$ with k call to the oracle Λ. The probability that this family is in fact a basis of the vector space $\mathbb{F}[G]$ over $\mathbb{F}$ is equal to the probability $\mathbb{P}$ that a random matrix chosen in $\mathrm{Mat}_k(\mathbb{F})$ is invertible, which satisfies

$$
\begin{aligned}
\mathbb{P} &= \mathrm{Prob}\,(M_1, ..., M_k \text{ is a basis of } \mathbb{F}[G]) \\
&= \frac{|\mathrm{GL}_k(\mathbb{F})|}{|\mathrm{Mat}_k(\mathbb{F})|} \\
&= \frac{(q^k - 1)(q^k - q)...(q^k - q^{k-1})}{q^{k^2}} \\
&= \left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{q^2}\right)..\left(1 - \frac{1}{q^k}\right) \\
&> \prod_{n \geqslant 1}\left(1 - \frac{1}{2^n}\right) > 0.28 > 1/4.
\end{aligned}
$$
(1)

See [20] for the cardinality of $\mathrm{GL}_k(\mathbb{F})$. Suppose for the moment that $\mathcal{B} = \{M_1, ..., M_k\}$ is a basis of $\mathbb{F}[G]$. If $k \geqslant n$ we extract a sub-family of cardinality n say $M_{i_1}, ..., M_{i_n}$ of $M_1, ..., M_k$ such that

$$\mathrm{Span}_{\mathbb{F}^n}\{M_{i_1}u, ..., M_{i_n}u\} = \mathrm{Span}_{\mathbb{F}^n}\{M_1 u, ..., M_k u\}.$$

Note that this is always possible and can be done in $O(k^2 n)$ field operations (see [7]). If $k < n$ then we may simply complete $\mathcal{B}$ with enough zero matrices to have a family of cardinality n. Let us consider the following equations with unknown $a_1, ..., a_n \in \mathbb{F}$ and $b_1, ..., b_n \in \mathbb{F}$:

$$
\begin{aligned}
(a_1 M_{i_1} + ... + a_n M_{i_n}) u &= v \\
\text{and} \quad (b_1 M_{i_1} + ... + b_n M_{i_n}) u &= w.
\end{aligned}
$$
(2)

If $\mathcal{B}$ is a basis, then both possess at least one solution because of the property of the family $M_{i_1}, ... M_{1_n}$. If $a = [a_1, ..., a_n]^t$ and $b = [b_1, ..., b_n]^t$ then Equations (2) are equivalent to the following :

$$
\begin{aligned}
[M_{i_1} u \,|\, ... \,|\, M_{i_n} u]\, a &= v \\
\text{and} \quad [M_{i_1} u \,|\, ... \,|\, M_{i_n} u]\, b &= w,
\end{aligned}
$$

and therefore both possess a solution that can be found by solving an $n \times n$ system of linear equations in $\mathbb{F}$. If the previous systems do not each have a solution, then we choose another family $M_1, ..., M_k$ and restart the process; the number of trials is expected to be less than 4 by Inequality 1. Therefore we can find the vectors a and b in $O(n^3)$ field operations.

The matrices

$$
\begin{aligned}
M_g &= (a_1 M_{i_1} + ... + a_n M_{i_n}) \\
\text{and} \quad M_h &= (b_1 M_{i_1} + ... + b_n M_{i_n})
\end{aligned}
$$

satisfy

$$
M_g M_h = M_h M_g , \quad M_g u = v \quad \text{and} \quad M_h u = w.
$$

Let $\sigma = M_g M_h u = M_h M_g u$. Since $M_g u = \rho(g) u$ and $M_h u = \rho(h) u$, we have

$$
\sigma = M_g M_h u = \rho(g) \rho(h) u = \psi((gh) \cdot x) \implies \psi^{-1}(\sigma) = (gh) \cdot x
$$

which shows that the Diffie-Hellman semigroup problem instance can be solved after a resolution of a family of problems that take $O(k^2 n + n^3)$ operations over $\mathbb{F}$.

2. The matrix M_g above belongs to $\rho(G)$ with probability $1/N$. Therefore the number of trials before reaching this state is $O(N)$. If $M_g \in \rho(G)$, then $\tilde{g} = \rho^{-1}(M_g)$ is a solution to the semigroup action problem since $\psi(y) = M_g \psi(x) = \psi(\tilde{g} \cdot x)$.

$\square$

Here are some examples where the previous theorem holds or can be used:

Example 3.2. Let M be an $n \times n$ matrix with entries in $\mathbb{F} = \mathbb{F}_q$ and $G = \mathbb{F}[M]$ acting on $\mathbb{F}^n$. If the minimal polynomial of M is $m(x)$ then $\mathbb{F}[M] \cong \mathbb{F}[x]/(m(x))$ (with this isomorphism being efficiently computable) and the latter is a vector space of dimension $k = \deg m \leqslant n$. In such a situation, both the semigroup action problem and Diffie-Hellman semigroup problem are trivial.

Example 3.3. This example comes from invariant theory (see e.g. [35] for an introduction to this classical subject). We will consider a contragradient matrix action on the ring of polynomials. Fix a finite field $\mathbb{F} = \mathbb{F}_q$, an integer d and an abelian sub-semigroup G of $\text{Mat}_n(\mathbb{F})$. Let V_d be the vector space over $\mathbb{F}$ of

polynomials in $\mathbb{F}[x_1, ..., x_n]$ of total degree less or equal to d. The action we are considering is

$$
\begin{aligned}
G \times V_d &\longrightarrow V_d \\
(A, f(x)) &\longmapsto A \cdot f = f((Ax)^t)
\end{aligned}
$$

where $x = [x_1, ..., x_n]^t$ and Ax is the usual matrix multiplication. This action is linear since $A \cdot (f + g) = A \cdot f + A \cdot g$. If $r = \dim_{\mathbb{F}} V_d$ then we can naturally embed V_d in $\mathbb{F}^r$ after having chosen the basis $\mathcal{B} = \{x_1^{e_1}...x_n^{e_n} \mid \sum e_i \leqslant d\}$ of V_d. This makes the map ψ easy to compute and to invert. For sake of clarity, we suppose that $\mathcal{B} = \{v_1 = x_1, ..., v_n = x_n, v_{n+1}, ..., v_r\}$. We define the map $\rho : G \longrightarrow \mathrm{Mat}_r(\mathbb{F})$ as follows:

$$
\rho(A)_{ij} = (A \cdot v_j)_i = \left(\prod_{k=1}^{r} \left(\sum_{l=1}^{n} a_{kl} x_l \right)^{e_k} \right)_i
$$

where $v_j = x_1^{e_1}...x_n^{e_n}$. So ρ gives the matrix representation of the linear map induced by the action since the jth column of $\rho(A)$ is the image of the jth basis vector v_j. Since all the polynomials have degree less or equal to d, the right-hand-side can be computed in $O(rnd \log d)$ field operations (see [32, Chapter 1]). Note that if $M \in \rho(G)$, then we can easily find A such that $\rho(A) = M$ since the ith row of A is contained in the n first components of the ith column of M. Indeed, if $1 \leqslant i \leqslant n$ then

$$
i^{th} \text{ column of } \mathrm{M} = A \cdot v_i = \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} a_{ij} v_j.
$$

Once again the previous theorem holds and makes the Diffie-Hellman semigroup problem as hard as the linear algebra problem in $\mathbb{F}^r$. However note that in that case the semigroup action problem may still be difficult since the ratio $|G|/|\mathbb{F}[G]|$ may take very small values because of the big dimension expansion from n to r.

4. Linear actions of abelian semirings on semi-modules

In this section we construct semigroup actions on finite sets starting from a semimodule defined over a semiring. The setup is general enough that it includes the Diffie-Hellman protocol over a general finite group as a special case. It provides on the other hand the flexibility to construct new protocols where some of the known attacks against the discrete logarithm problem in a finite group do not work anymore.

Let R be a semiring, not necessarily finite. This means that R is a semigroup with respect to both addition and multiplication and the distributive laws hold. It is understood that the semiring is commutative with respect to addition. Some authors assume that a semiring has a neutral element with respect to addition. We will not assume that R has either a zero or a one.

Let M be a finite semimodule over R. With this we mean that M has the structure of a finite semigroup and there is an action:

$$
R \times M \longrightarrow M
$$

such that

$$
r(sm) = (rs)m, \ (r + s)m = rm + sm \text{ and } r(m + n) = rm + rn
$$

for all $r, s \in R$ and $m, n \in M$.

The semigroup action problem in this setting then asks:

  "Given elements $m, n \in M$ find an element $r \in R$ such that $rm = n$".

Before we proceed we would like to explain some of the difficulties in order to derive at a square root algorithm which solves the SAP. For this note that many square root attacks seek in this situation a "collision", e.g. in Pollard's rho method elements $r_1, \ldots, r_4 \in R$ are sought such that

$$(3) \qquad r_1 m + r_2 n = r_3 m + r_4 n.$$

If the semiring is a ring then this results in

$$(r_1 - r_3)m = (r_4 - r_2)n$$

and maybe under benign conditions the semigroup action problem can be solved. If the semiring (like e.g. the ones we describe in the next section) have in general no additive inverses this simple reduction from Equation (3) is not possible. The situation is even worse when R has only a semigroup structure and M is an arbitrary set since in such a situation no addition is at disposal at all.

We proceed now and show how to derive at an abelian semigroup action starting from a semimodule whose coefficient ring is not necessarily multiplicatively commutative.

Let $\mathrm{Mat}_n(R)$ be the set of all $n \times n$ matrices with entries in the semiring R. The semiring structure on R induces a semiring structure on $\mathrm{Mat}_n(R)$. Moreover the semimodule structure on M lifts to a semimodule structure on M^n via the matrix multiplication:

$$
(4) \qquad
\begin{aligned}
\mathrm{Mat}_n(R) \times M^n &\longrightarrow M^n \\
(A, x) &\longmapsto Ax.
\end{aligned}
$$

The action (4) forms a semigroup-action of the multiplicative semigroup of $\mathrm{Mat}_n(R)$ on the set M^n. In general $\mathrm{Mat}_n(R)$ is not commutative with respect to matrix multiplication. However we can easily define a commutative subgroup as follows:

Let $C \subset R$ be the center of R i.e., the subset of R consisting of elements that commute with any other elements. Let $C[t]$ be the polynomial ring in the indeterminant t and let $A \in \mathrm{Mat}_n(R)$ be a fixed matrix. If

$$p(t) = r_0 + r_1 t + \cdots + r_k t^k \in C[t]$$

then we define in the usual way $p(A) = r_0 I_n + r_1 A + \cdots + r_k A^k$, where $r_0 I_n$ is the $n \times n$ diagonal matrix with entry r_0 in each diagonal element.

Consider the semigroup

$$G := C[A] := \{p(A) \mid p(t) \in C[t]\}.$$

Clearly $C[A]$ has the structure of an abelian semigroup. Protocol 2.1 then simply requires that Alice and Bob agree on a vector $s \in M^n$. Then Alice chooses a matrix $X \in C[A]$ and sends to Bob the vector Xs, an element of the module M^n. Bob chooses a matrix $Y \in C[A]$ and sends to Alice the vector Ys. The common key is then the vector XYs which both can compute since X and Y commute.

In the special case when $R = M = \mathbb{F}$ is a finite field one readily reduces the problem to a simple linear algebra problem over the finite field $\mathbb{F}$.

The situation becomes slightly more interesting if we take as a ring $R = \mathbb{Z}$, the integers and as module any finite abelian group $M = H$. The group H is a $\mathbb{Z}$ module

and $\mathrm{Mat}_n(\mathbb{Z})$ operates on $S := H^n = H \times \ldots \times H$ via the formal multiplication:

$$
(5) \qquad \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \longmapsto \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}.
$$

If $l = \mathrm{lcm}\,\{|g_1|, \ldots, |g_n|\}$, and $C \in \mathrm{Mat}_n(\mathbb{Z})$ is a matrix with all entries congruent to zero modulo l, then $(A + C)g = Ag$ for all $A \in \mathrm{Mat}_n(\mathbb{Z})$. Whence, we may simply consider the action of $\mathrm{Mat}_n(\mathbb{Z}/l\mathbb{Z})$ on S.

This problem reduces to a combination of a linear algebra problem and a series of discrete logarithm problems in H as soon as all the elements $\{g_1, \ldots, g_n\} \subset H$ lie in a common cyclic subgroup of H. Such an attack is even possible when the $\mathbb{Z}$-action on the abelian group is more complicated and we refer to the recent system introduced by Climent et. al. [5] and its cryptanalysis [6].

The situation becomes quite a bit more interesting if we consider general finite semirings acting on general semi-modules. In the next section we explain an instance where we do not know how to efficiently attack such a system.

5. A two-sided abelian action based on simple semirings

In this section we describe a particular semigroup action, where we do not know how to solve the SAP once the parameters have been chosen large enough. The idea of such an action originates in the dissertation of Maze [24]. Shpilrain and Ushakov [33] have described a similar two-sided action in the context of Thompson groups and Slavin [34] filed a patent based on such ideas.

Let us fix a finite semiring R, not embeddable in a field and not necessarily commutative. Given such a semiring, consider C, the center of R. Throughout this section, we let n denote an arbitrary positive integer. For $M \in \mathrm{Mat}_n(R)$ we denote by $C[M]$ the abelian sub-semiring generated by M, i.e., the semiring of polynomials in M with coefficients in C. Let $M_1, M_2 \in \mathrm{Mat}_n(R)$ and consider the following action:

$$
\begin{array}{ccl}
(C[M_1] \times C[M_2]) \times \mathrm{Mat}_n(R) & \longrightarrow & \mathrm{Mat}_n(R) \\
((p(M_1), q(M_2)), X) & \longmapsto & p(M_1) \cdot X \cdot q(M_2).
\end{array}
$$

This action is linear since

$$
p(M_1) \cdot (A + B) \cdot q(M_2) = p(M_1) \cdot A \cdot q(M_2) + p(M_1) \cdot B \cdot q(M_2).
$$

Because of this linearity, we avoid the case when R is a finite field (see Theorem 3.1) even if the initial SAP instance related to this semigroup action looks difficult.

The key-exchange algorithm that results from using this semigroup action in Protocol 2.1 explicitly reads as follows.

Protocol 5.1. (Diffie-Hellman with two-sided matrix semiring action)

1. Alice and Bob agree on a finite semiring R with nonempty center C, not embeddable into a field. They choose a positive integer n and matrices $M_1, M_2, S \in \mathrm{Mat}_n(R)$.
2. Alice chooses polynomials $p_a, q_a \in C[t]$ and computes $A = p_a(M_1) \cdot S \cdot q_a(M_2)$. She sends A to Bob.
3. Bob chooses polynomials $p_b, q_b \in C[t]$ and computes $B = p_b(M_1) \cdot S \cdot q_b(M_2)$. He sends B to Alice.

4. Their common secret key is then

$$p_a(M_1)Bq_a(M_2) = p_a(M_1)p_b(M_1)Sq_b(M_2)q_a(M_2) = p_b(M_1)Aq_b(M_2).$$

The corresponding SAP that should be hard is: given $M_1, M_2, S \in \mathrm{Mat}_n(R)$ and $T \in C[M_1]SC[M_2]$ find $U_1 \in C[M_1]$ and $U_2 \in C[M_2]$ so that $T = U_1SU_2$. We do not know if it is necessary for an attacker to solve this problem, but it certainly is sufficient.

The remainder of this section is devoted to describing some necessary conditions on R for this problem to be difficult, and the existence of semirings meeting these necessary conditions.

Definition 5.2. A congruence relation on a semiring R is an equivalence relation $\sim$ such that $a \sim b$ implies that $ac \sim bc$, $ca \sim cb$, $a + c \sim b + c$ and $c + a \sim c + b$ for all possible choice of a, b and c. A semiring R is congruence-free, or *simple*, if the only congruence relations are $R \times R$ and $\{(a, a) \mid a \in R\}$.

Any congruence relation induces a natural semiring structure on the set $R/\sim$ and the quotient map $R \longrightarrow R/\sim$ is a semiring homomorphism. It is also clear that a congruence relation on R induces a congruence relation on $\mathrm{Mat}_n(R)$ for any $n \in \mathbb{N}$.

For cryptographic purposes it is important that the involved semirings are simple to avoid a Pohlig-Hellman type reduction of the SAP. Indeed any congruence relation on R yields a projection of the SAP instance onto a quotient semiring, from which one may gain information about the solution to the original instance. Just as we prefer to work in groups of prime orders to avoid a Pohlig-Hellman attack, we would like to work in simple semirings to avoid such a reduction. Let us mention that Monico [29] provided a partial classification of finite simple semirings in 2002 and that Zumbrägel recently provided in [37] a total classification of non-trivial finite simple semirings together with a method for explicitly constructing such objects. For this we first define:

Definition 5.3. A zero of a semiring R is an element '0' such that $a+0 = 0+a = a$ and $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$. A one of a semiring R is an element '1' such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$.

Next we show how to build large simple semirings from small simple semirings. We start with a technical lemma:

Lemma 5.4. *Let R be an additively commutative semiring with 1 and 0 and let $\sim$ be a congruence relation on $\mathrm{Mat}_n(R)$. Then there exists a congruence relation $\sim_0$ on R such that*

$$A \sim B \in \mathrm{Mat}_n R \Longleftrightarrow a_{ij} \sim_0 b_{ij} \; , \; \forall \, 0 \leqslant i, j \leqslant n.$$

Proof. First, given such a semiring R, and $M \in \mathrm{Mat}_n(R)$, if M' is obtained from M by a permutation of rows and columns, we prove there exist two invertible matrices $S, P \in \mathrm{Mat}_n(R)$ such that $M' = SMP$. Indeed, the statement is true if one consider matrices with entries in $\mathbb{Z}$ and the usual multiplication, i.e., there exist two permutation matrices (therefore with entries in $\{0, 1\}$) such that $M' = S \cdot M \cdot P$ with $\cdot$ being the usual matrix multiplication. It is then straightforward to verify that the same is true with the operation in R because of the properties of 0 and 1. Let us now prove the stated result. Let $f : R \longrightarrow \mathrm{Mat}_n(R)$ be the map that sends $a \in R$ to the diagonal matrix with first diagonal element a and zeros everywhere else. The map f is a semiring homomorphism. Let $\sim_0$ be the relation on R defined

by $a \sim_0 b$ in R if and only if $f(a) \sim f(b)$ in $\mathrm{Mat}_n(R)$. Observe that $\sim_0$ is a congruence relation on R. We prove now that the statement of the lemma is true for $\sim_0$. Let $A, B \in \mathrm{Mat}_n(R)$ and $J = f(1)$. Let $0 \leqslant i, j \leqslant n$ and $S_{ij}, P_{ij} \in \mathrm{Mat}_n(R)$ be permutation matrices such that

$$(S_{ij}AP_{ij})_{11} = a_{ij} \quad \text{and} \quad (S_{ij}BP_{ij})_{11} = b_{ij}.$$

Note that the matrices S_{ij} and P_{ij} exists in $\mathrm{Mat}_n(R)$ by the previous remark. Therefore $JS_{ij}AP_{ij}J = f(a_{ij})$ and $JS_{ij}BP_{ij}J = f(b_{ij})$.
'$\Longrightarrow$': If $A \sim B$ then $JS_{ij}AP_{ij}J \sim JS_{ij}BP_{ij}J$ and therefore $a_{ij} \sim_0 b_{ij}$.
'$\Longleftarrow$': Clearly

$$A = \sum_{i,j} S_{ij}^{-1} f(a_{ij}) P_{ij}^{-1} \text{ and } B = \sum_{i,j} S_{ij}^{-1} f(b_{ij}) P_{ij}^{-1}$$

and since $f(a_{ij}) \sim f(b_{ij})$, $A \sim B$. □

As an immediate consequence of this lemma, we have the following theorem which provides arbitrarily large, finite, simple semirings.

Theorem 5.5. *Let R be an additively commutative semiring with 1 and 0 and let $n \in \mathbb{N}$. Then R is simple if and only if $\mathrm{Mat}_n(R)$ is simple.*

With the help of this Theorem we can readily build large finite simple semirings with 0,1 which are not rings and not embeddable into fields. The following provides several explicit examples of some small finite simple semirings with 0,1 which are not rings and not embeddable into fields.

Example 5.6. Consider the set $S = \{0,1\}$ with the operations max and min for addition and multiplication respectively. One readily verifies that S has the structure of a finite simple semiring. Note that several polynomial time problems over $\mathbb{Z}$, such as polynomial factorization, have been found to be NP-hard when considered over this semiring S [14].

The following example was found by computer search.

Example 5.7. Consider the set $S_{6,1} = \{0,1,2,3,4,5\}$ satisfying the following addition and multiplication rules.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	1	1	1	5
2	2	1	2	1	2	5
3	3	1	1	3	3	5
4	4	1	2	3	4	5
5	5	5	5	5	5	5

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	2	0	0	5
3	0	3	4	3	4	3
4	0	4	4	0	0	3
5	0	5	2	5	2	5

$S_{6,1}$ is a finite simple semiring with 6 elements. This is up to isomorphism the only simple semiring of order 6. This result follows from [29, 37].

Example 5.8. Using the classification of J. Zumbrägel derived in [37] it is possible to derive for many orders addition and multiplication tables. We are grateful to J. Zumbrägel for providing us with the following recently found simple semiring having order 20. Details on how to construct the addition and multiplication table can be found in [37]. Again one can show that this is up to isomorphism the only simple semiring of order 20.

+	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	1
0	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	1
a	a	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	1
b	b	b	b	c	e	e	f	g	h	i	k	k	l	m	n	o	p	q	r	1
c	c	c	c	c	f	f	f	h	h	i	l	l	l	1	n	p	p	q	r	1
d	d	d	e	f	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	1
e	e	e	e	f	e	e	f	g	h	i	k	k	l	m	n	o	p	q	r	1
f	f	f	f	f	f	f	f	h	h	i	l	l	l	1	n	p	p	q	r	1
g	g	g	g	h	g	g	h	g	h	i	m	m	1	m	n	o	p	q	r	1
h	h	h	h	h	h	h	h	h	h	i	1	1	1	1	n	p	p	q	r	1
i	i	i	i	i	i	i	i	i	i	i	n	n	n	n	n	q	q	q	r	n
j	j	j	k	l	j	k	l	m	1	n	j	k	l	m	n	o	p	q	r	1
k	k	k	k	l	k	k	l	m	1	n	k	k	l	m	n	o	p	q	r	1
l	l	l	l	l	l	l	l	1	1	n	l	l	l	1	n	p	p	q	r	1
m	m	m	m	1	m	m	1	m	1	n	m	m	1	m	n	o	p	q	r	1
n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	q	q	q	r	n
o	o	o	o	p	o	o	p	o	p	q	o	o	p	o	q	o	p	q	r	p
p	p	p	p	p	p	p	p	p	p	q	p	p	p	p	q	p	p	q	r	p
q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	r	q
r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
1	1	1	1	1	1	1	1	1	1	n	1	1	1	1	n	p	p	q	r	1

·	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	0	0	a	a	a	a	a	b	b	b	c	a
b	0	0	0	0	a	a	a	b	b	c	a	a	a	b	c	b	b	c	c	b
c	0	a	b	c	a	b	c	b	c	c	a	b	c	b	c	b	c	c	c	c
d	0	0	0	0	0	0	0	0	0	0	d	d	d	d	d	g	g	g	i	d
e	0	0	0	0	a	a	a	b	b	c	d	d	d	e	f	g	g	h	i	e
f	0	a	b	c	a	b	c	b	c	c	d	e	f	e	f	g	h	h	i	f
g	0	0	0	0	d	d	d	g	g	i	d	d	d	g	i	g	g	i	i	g
h	0	a	b	c	d	e	f	g	h	i	d	e	f	g	i	g	h	i	i	h
i	0	d	g	i	d	g	i	g	i	i	d	g	i	g	i	g	i	i	i	i
j	0	0	0	0	0	0	0	0	0	0	j	j	j	j	j	o	o	o	r	j
k	0	0	0	0	a	a	a	b	b	c	j	j	j	k	l	o	o	p	r	k
l	0	a	b	c	a	b	c	b	c	c	j	k	l	k	l	o	p	p	r	l
m	0	0	0	0	d	d	d	g	g	i	j	j	j	m	n	o	o	q	r	m
n	0	d	g	i	d	g	i	g	i	i	j	m	n	m	n	o	q	q	r	n
o	0	0	0	0	j	j	j	o	o	r	j	j	j	o	r	o	r	r	r	o
p	0	a	b	c	j	k	l	o	p	r	j	k	l	o	r	o	p	r	r	p
q	0	d	g	i	j	m	n	o	q	r	j	m	n	o	r	o	q	r	r	q
r	0	j	o	r	j	o	r	o	r	r	j	o	r	o	r	o	r	r	r	r
1	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	1

In order that the two-sided semigroup action described in the beginning of this section is difficult we would like that the sets $C[M_1]$ and $C[M_2]$ are large with regard to the matrix size n. The orders of the matrices M_1 and M_2 chosen to act on the matrix A on the left and on the right are of prime importance. Indeed the cardinality of the commutative semiring $C[M]$ directly depends on the order of M. We study the "sizes" of the orbit of powers of elements in $\mathrm{Mat}_n(S)$ where $S = \{\{0,1\}, \max, \min\}$. We will see that these orders give lower bounds for the maximum orders of elements in any semiring with 0 and 1. Note that since the semiring $\mathrm{Mat}_n(S)$ is finite any sequence $\{M^k\}_{k\in\mathbb{N}}$ will eventually repeat, i.e., create a collision of the form $M^k = M^{k'}$ with $k \neq k'$. Computer experiments also showed that in general the set $C[M]$ is much larger than the set $M^k = M^{k'}$.

Definition 5.9. Let $a = \{a_k\}_{k\in\mathbb{N}}$ be a sequence in a finite set such that $a_n = a_m \implies a_{n+1} = a_{m+1}$. The *order* $\mathrm{ord}(a)$ of a is the least positive integer m for which there exists $k < m$ with $a_k = a_m$. The *preperiod* $p_r(a)$ of a is the largest nonnegative integer m such that for all $k > m$ we have $a_k \neq a_m$. The *period* $\mathrm{per}(a)$ of a is the least positive integer m for which there exists an integer N with $a_{m+k} = a_k$

for all $k > N$. If g is an element of a semigroup, then we set $\text{ord}(g) = \text{ord}(\{g^n\}_{n \in \mathbb{N}})$, $\text{per}(g) = \text{per}(\{g^n\}_{n \in \mathbb{N}})$ and $p_r(g) = p_r(\{g^n\}_{n \in \mathbb{N}})$.

Clearly $\text{ord}(a) = \text{per}(a) + p_r(a)$. Returning to the situation of the multiplicative semigroup of $\text{Mat}_n(S)$, we study the question "How large can the order of $M \in \text{Mat}_n(S)$ be?". There already exist some results in this direction. To describe them, we recall that for a given oriented graph G, a *strongly connected component* (written SCC) of G is a sub-graph H of G inside which any two vertices i and j belong to a common oriented cycle and H is a maximal sub-graph with this property. Such a SCC is written $H \subseteq_{SCC} G$. The period of a strongly connected component is the maximum between the gcd of the length of its cycles and 1. We refer the reader to [21] for the details.

Proposition 5.10. *Let $M \in \text{Mat}_n(S)$ and G be the directed graph whose adjacency matrix is M. Then*

1. *$\text{per}(M) = \text{lcm}\{period\ of\ H \mid H\ is\ a\ SCC\ of\ G\}$,*
2. *The numbers $\text{per}(M), p_r(M)$ and $\text{ord}(M)$ can be computed in $O(n^3)$ time.*

This proposition is essentially in [12]. The algorithm given there computes $\text{per}(M)$ in $O(n^3)$ time and an easy modification of it allows to computes $p_r(M)$ and therefore $\text{ord}(M)$.

We introduce now a function that play a crucial role: Landau's function g. It is defined by

$$
\begin{aligned}
g(n) &= \max\{\text{ord}(\sigma) \mid \sigma \in S_n\} \\
&= \max\{\text{lcm}\{a_1, ..., a_m\} \mid a_i > 0,\ a_1 + ... + a_m = n\}.
\end{aligned}
$$

It was first studied by Landau [18] in 1903 who proved that

$$
(6) \qquad \ln(g(n)) \sim \sqrt{n \ln(n)} \quad \text{as} \quad n \longrightarrow \infty.
$$

In 1984, Massias [22] showed that for sufficiently large n,

$$
(7) \qquad \sqrt{n \ln(n)} \leqslant \ln(g(n)) \leqslant \sqrt{n \ln(n)} \left(1 + \frac{\ln \ln(n)}{2 \ln(n)}\right),
$$

the second inequality in 7 being true for all n. Clearly, the function g is increasing. In any case, we have

$$
\max\{\text{lcm}\{a_1, .., a_m\} : |a_1| + ... + |a_m| = n\} = \exp\left((1 + o(1))\sqrt{n \ln n}\right).
$$

On the other hand, the period of any SCC $H \subset G$ is less or equal to $|H|$ and

$$
\sum_{H \subseteq_{SCC} G} |H| \leqslant n.
$$

Since the function g is increasing, Proposition 5.10 and Equation (6) give

$$
\text{per}(M) \leqslant g\left(\sum_{H \subseteq_{SCC} G} |H|\right) \leqslant g(n) = \exp\left((1 + o(1))n^{1/2} \ln^{1/2} n\right).
$$

Further, it is not difficult to see that there always exists an oriented graph G with period $g(n)$. Indeed if $g(n)$ is reached by a partition $a_1 + ... + a_m = n$, then a graph G built out of cyclic SCCs of order a_i satisfies $\text{per}(M) = g(n)$. Such a matrix $M \in \text{Mat}_n(S)$ that reaches this bound is in fact a permutation matrix, and as such, it can be seen as an element of any semirings with 0 and 1. In other words, in any such semiring, the previous bound is reached:

Proposition 5.11. *Let $n \in \mathbb{N}$ and R be a semiring with 0 and 1. Then*

$$\max\{\operatorname{per}(M) \mid M \in \operatorname{Mat}_n(R)\} \geq g(n) = \exp\left((1 + o(1))n^{1/2}\ln^{1/2} n\right).$$

If $R = S = \{\{0, 1\}, \max, \min\}$, then the above inequality is an equality.

The exact computation of $g(n)$, or more precisely, of the partition $a_1 + \ldots + a_m = n$ that yields the maximum $g(n)$, is necessary in order to build explicitly a matrix $M \in \operatorname{Mat}_n(S)$ such that $\operatorname{per}(M) = g(n)$. Indeed, the integer $g(n)$ is always a product of primes less or equal to $2.86\sqrt{n \ln(n)}$, c.f. [23]. Therefore the factorization of $g(n)$ can be found in polynomial time in n. It is also known that the partition of n that gives the maximum lcm has parts that are all prime powers, c.f. [13], and therefore the factorization of $g(n)$ gives the expected partition directly. The algorithm given in [30] allows one to compute $g(n)$ for large integers n, up to $n = 32,000$, so the exact determination of the matrix M is not a problem. See Table 5.1 for a list of values of $g(n)$ with the associated partition.

TABLE 1. Some values of Landau's function g

n	$g(n)$	Associated partition
256	4243057729190280	8, 9, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 43
512	70373028815644182 \ 5899620	1, 1, 1, 4, 9, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61
1024	855674708268439827 \ 7434193536488991600	1, 1, 1, 16, 27, 25, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89

For a given matrix $M \in \operatorname{Mat}_n(S)$, since $S[M] \supset \{M^k\}_{k \in \mathbb{N}}$, we have

$$|S[M]| \geqslant \operatorname{ord}(M) \geqslant \operatorname{per}(M),$$

and the last inequality can give $|S[M]| \geqslant g(n)$ for a wisely chosen M.

The following corollary shows that the size of the sets $C[M]$ grows exponentially in n for suitable matrices M as soon as the center C contains the elements $0, 1$ of a semiring. Such matrices can even be constructed in an efficient way.

Corollary 5.12. *Let $n \in \mathbb{N}$ and R be a semiring with 0 and 1 and center C. Then there is an $n \times n$ matrix M with entries in R such that the order of M is larger than $g(n)$ in particular the size of $C[M]$ is larger than $g(n)$ as well.*

We conclude the Section with an example to illustrate how finite simple semirings could be used to build a practical semigroup action problem.

Example 5.13. Consider the semiring $R = S_{6,1}$ as defined above. The elements $\{0, 1\}$ form the center C of R. We will consider the matrix ring $\operatorname{Mat}_n(R)$ with $n = 20$. In this situation the key size is $400 \cdot \lg 6 \cong 1033$ bits and the value of Laudau's g function is $g(20) = 1 \cdot 4 \cdot 3 \cdot 5 \cdot 7 = 420$. By the last corollary $\operatorname{Mat}_n(R)$ contains elements M whose multiplicative order $\operatorname{ord}(M)$ is at least 420. For such an element M the abelian semigroup $C[M]$ contains all elements of the

form $\sum_{i=0}^{k} r_i M^i$ with $r_i \in \{0,1\}$. The size of the set $C[M]$ is upper bounded by 2^{k+1}, where $k = \mathrm{ord}(M)$.

The matrices M_1 and M_2 below are chosen to be close to permutation matrices such that the orders are actually more than 420. The matrix S is also chosen sparse as computer experiments with the particular ring $S_{6,1}$ showed that this leads to maximal possible size of the possible matrices

$$C[M_1] \cdot S \cdot C[M_2].$$

Upon using these parameters in Protocol 5.1, Alice chooses polynomials $p, q \in C[t]$ and computes

$$A := p(M_1) \cdot S \cdot q(M_2)$$

$p, q \in C[t]$ were chosen as private keys by Alice in Protocol 5.1.

It is clear that she has more than 2^{420} choices to choose a polynomial $p \in C[t]$ and for such a polynomial $p(M_1)$ can be computed with at most 420 matrix multiplication and addition. - Of course Alice can restrict herself to polynomials of smaller degree, say e.g. $k < 50$ which leaves still 2^{50} choices for p and for q and which reduces the number of matrix multiplications and additions to 100, a task quite easy for an average PC.

Assume Alice has chosen the matrices in the following particular way:

$$M_1 = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0
\end{bmatrix}$$

$$M_2 = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&3&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0
\end{bmatrix}$$

$$S = \begin{bmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&2&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&5&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&1\\
0&0&2&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0
\end{bmatrix}$$

$$A = \begin{bmatrix}
0&1&2&2&2&0&2&4&0&2&2&2&2&4&2&4&0&2&0&0\\
1&2&1&1&2&1&1&1&1&1&1&1&1&4&2&1&1&2&1&4\\
1&2&1&1&2&1&1&1&1&1&1&1&1&4&2&1&1&2&1&4\\
1&2&1&1&2&1&1&1&1&1&1&1&1&4&2&1&1&2&1&4\\
1&2&1&1&2&1&1&1&1&1&1&1&1&4&2&1&1&2&1&4\\
1&2&2&1&1&1&2&1&0&2&2&2&5&1&2&1&1&1&1&1\\
1&2&1&1&1&1&2&0&2&2&1&5&1&1&2&1&1&1&1&1\\
1&2&1&1&1&1&2&0&2&2&2&1&4&1&1&1&1&1&1&1\\
0&2&2&2&2&4&2&4&2&2&1&1&2&0&2&0&0&2&0&0\\
0&2&2&2&2&4&2&4&2&2&1&2&0&2&2&0&2&2&0&0\\
0&2&2&2&2&0&2&0&2&2&2&2&2&0&2&0&0&2&0&0\\
0&2&2&1&2&4&2&4&0&1&1&1&1&4&2&1&0&2&0&0\\
0&2&2&2&2&4&2&4&2&1&1&1&2&4&2&1&0&2&0&0\\
1&2&1&1&1&1&1&1&1&1&1&5&1&1&1&1&1&1&1&1\\
1&2&1&1&1&1&1&1&2&1&1&5&1&1&1&1&1&1&1&1\\
1&2&1&1&1&1&1&1&1&1&1&1&1&4&1&1&1&1&1&1\\
1&2&1&1&1&1&1&1&1&1&1&5&1&2&1&1&1&1&1&1\\
1&2&1&1&1&1&4&0&2&2&1&5&1&1&4&1&1&1&1&1\\
1&2&1&1&1&1&1&1&2&1&2&5&1&1&1&1&1&1&1&1\\
1&2&1&1&1&1&1&1&0&2&2&1&5&1&1&1&1&1&1&1
\end{bmatrix}$$

The only way we know for an attacker to break this system would be to find polynomials $\tilde{p}$ and $\tilde{q}$ such that $\tilde{p}(M_1)S\tilde{q}(M_2) = A$ (or, to solve a similar problem in

terms of the matrix B Bob computes). If the degrees of p, q are in the range of 50 a brute force search will depend on the size of the set:

$$\mathcal{S} := \{p(M_1) \cdot S \cdot q(M_2) \mid \deg p < 50, \deg q < 50\}.$$

An immediate upper bound for the size of the set $\mathcal{S}$ is 2^{100}. We did run extensive computations and could show that $\mathcal{S}$ has size at least 2^{25}, not sufficient to be used as a practical system. It will require further research to estimate better the size of $\mathcal{S}$ and to understand how the sizes grow as we increase both the matrices involved and the simple semirings. E.g. one could run the protocol with the semiring of Example 5.8 and leave the size of the matrices the same.

In order to describe the efficiency of the system assume that Alice and Bob agree on matrices of size n, polynomials p, q of degree at most k and a simple semiring R of cardinality $|R| = \theta$. Then the public key and the data to be transmitted has $O(n^2 \lg \theta)$ bits. The number of required bit operations during encryption is $O(kn^3(\lg \theta))$ and the computation of the common secret key requires $O(n^3(\lg \theta))$ bit operations. If $\tilde{\theta}$ denotes the cardinality of the center C of R then an upper bound for the size of the set $\mathcal{S}$ is $\tilde{\theta}^{2k}$.

These complexity estimates suggest that the system should be further analysed in particular when the sizes of the matrices are small and the sizes of the ring R is large.

6. Conclusion

An abelian group can be viewed in a natural way as $\mathbb{Z}$-module. In this paper we consider the situation when an arbitrary semigroup (instead of just the integers) act on an arbitrary finite set. The generalization of the discrete logarithm problem results in the semigroup action problem which we study in this paper. In the situation when the semigroup is abelian one has a natural Diffie-Hellman secret key exchange and a sufficient condition to break the key exchange is to solve the semigroup action problem.

In the later part of the paper we concentrate on a particular semigroup action. We consider the situation where a simple semiring acts on a semimodule. This generalizes the group situation where G is a cyclic group of prime order p, i.e. where the simple ring $\mathbb{Z}/p\mathbb{Z}$ is acting on G via exponentiation.

Simplicity of the involved semirings is important in order to avoid Pohlig-Hellman type attacks. Using a recently found simple semiring of order 6 we illustrate the techniques in an example. It will require further research to assess the security of such systems.

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