Multiscale analysis of exit distributions for random walks in random environments

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Preliminary version

Abstract

We present a multiscale analysis for the exit measures from large balls in $\mathbb{Z}^d$, $d \geq 3$, of random walks in certain i.i.d. random environments which are small perturbations of the fixed environment corresponding to simple random walk. Our main assumption is an isotropy assumption on the law of the environment, introduced by Bricmont and Kupianien. The analysis is based on propagating estimates on the variational distance between the exit measure and that of simple random walk, in addition to estimates on the variational distance between smoothed versions of these quantities.

1 Introduction

We consider random walks in random environments on $\mathbb{Z}^d$, $d \geq 3$, when the environment is a small perturbation of the fixed environment corresponding to simple random walk. More precisely, let $\mathcal{P}$ be the set of probability distributions on $\mathbb{Z}^d$, charging only neighbors of 0. If $\varepsilon \in (0, 1/2d)$, we set, with $\{e_i\}_{i=1}^d$ denoting the standard basis of $\mathbb{R}^d$,

$$\mathcal{P}_\varepsilon \stackrel{\mathrm{def}}{=} \left\{ q \in \mathcal{P} : \left| q\left(\pm e_i\right) - \frac{1}{2d} \right| \leq \varepsilon, \ \forall i \right\}. \tag{1.1}$$

$\Omega \stackrel{\mathrm{def}}{=} \mathcal{P}^{\mathbb{Z}^d}$ is equipped with the natural product σ-field $\mathcal{F}$. We call an element $\omega \in \Omega$ a *random environment.* For $\omega \in \Omega$, and $x \in \mathbb{Z}^d$, we consider the transition probabilities $p_\omega\left(x, y\right) \stackrel{\mathrm{def}}{=} \omega_x\left(y - x\right)$, if $|x - y| = 1$, and $p_\omega\left(x, y\right) = 0$ otherwise, and construct the random walk $\{S_n\}_{n \geq 0}$ with initial position $x \in \mathbb{Z}^d$ which is, given the environment ω, the Markov chain with $S_0 = x$ and transition probabilities

$$P_{\omega,x}(S_{n+1} = y | S_n = z) = \omega_z(y - z).$$

(By a slight abuse of notation, for consistency with the sequel we also write $P_{\omega,x} = P_{p_\omega,x}$.)

We are mainly interested in the case of a *random* ω. Given a probability measure μ on $\mathcal{P}$, we consider the product measure $\mathbb{P}_\mu \overset{\text{def}}{=} \mu^{\otimes \mathbb{Z}^d}$ on $(\Omega, \mathcal{F})$. We usually drop the index μ in $\mathbb{P}_\mu$. In all that follows we make the following basic assumption

Condition 1.1
μ is invariant under lattice isometries, i.e. $\mu f^{-1} = \mu$ for any orthogonal mapping f which leaves $\mathbb{Z}^d$ invariant, and $\mu(\mathcal{P}_\varepsilon) = 1$ for some $\varepsilon \in (0, 1/2d)$ which will be specified later.

The model of RWRE has been studied extensively. We refer to [7] and [12] for recent surveys. A major open problem is the determination, for $d > 1$, of laws of large numbers and central limit theorems in full generality (the latter, both under the *quenched* measure, i.e. for $\mathbb{P}_\mu$-almost every ω, and under the *annealed* measure $\mathbb{P}_\mu \otimes P_{x,\omega}$). Although much progress has been reported in recent years ([1, 8, 9]), a full understanding of the model has not yet been achieved.

In view of the above state of affairs, attempts have been made to understand the perturbative behavior of the RWRE, that is the behavior of the RWRE when μ is supported on $\mathcal{P}_\varepsilon$ and ε is small. The first to consider such a perturbative regime were [2], who introduced Condition 1.1 and showed that in dimension $d \geq 3$, for small enough ε a quenched CLT holds [1]. Unfortunately, the multiscale proof in [2] is rather difficult, and challenging to follow. This in turns prompted the derivation, in [10], of an alternative multiscale approach, in the context of diffusions in random environments. One expects that the approach of [10] could apply to the discrete setup, as well.

Our goal in this paper is somewhat different: we focus on the exit law of the RWRE from large balls, and develop a multiscale analysis that allows us to conclude that the exit law approaches, in a suitable sense, the uniform measure. Like in [10], the hypothesis propagated involves smoothing. In [10], this was done using certain Hölder norms of (rescalled) transition probabilities. Here, we focus on two ingredients: a propagation of the variational distance between the exit laws of the RWRE and that of simple random walk (which remains small but does not decrease as the scale increases), and the convolution of the exit law of the RWRE with the exit law of a simple random walk from a ball of (random) radius, which decrease to zero as scale increases (a precise statement can be found in Theorem 2.4). This approach is of a different nature than the one in [10] and, we believe, simpler. In future work we hope to combine our exit law approach with suitable exit time estimates in order to deduce a (quenched) CLT for the RWRE.

The structure of the article is the following. In the next section, we introduce our basic notation and state our induction step and our main results. In Section 3, we present our basic perturbation expansion, coarsening scheme for random walks, and auxilliary estimates for simple random walk. The proof of the latter estimates is presented in the appendices. Section 4 is devoted to the propagation of the smoothed estimates, whereas Section 5 is devoted to the propagation of the variation distance

[1] As the examples in [1] demonstrate, for every $\varepsilon > 0$ there are measures μ supported on $\mathcal{P}_\varepsilon$, with $\mathbb{E}_\mu\left[\sum_{i=1}^d (q(e_i) - q(-e_i))\right] = 0$, such that $S_n/n \to_{n\to\infty} v \neq 0$, $\mathbb{P}_\mu$-a.s. One of the goals of Condition 1.1 is to prevent such situations from occuring.

estimate (the non-smooth estimate). Section 6 completes the proof of our main result by using the estimates of Sections 4 and 5.

2 Basic notation and main result

Sets: For $x \in \mathbb{R}^d$, $|x|$ is the Euclidean norm. If $A, B \subset \mathbb{Z}^d$, $x \in \mathbb{Z}^d$, we set $d(x, A) \stackrel{\text{def}}{=} \inf\{|x-y| : y \in A\}$, $d(A, B) \stackrel{\text{def}}{=} \inf\{d(x, B) : x \in A\}$. If $L > 0$, we write $V_L \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^d : |x| \leq L\}$, and for $x \in \mathbb{Z}^d$, $V_L(x) \stackrel{\text{def}}{=} x + V_L$. If $V \subset \mathbb{Z}^d$, ∂V is the outer boundary, i.e. the set of points outside V which have a neighbor point in V. If $x \in V$, we set $d_V(x) \stackrel{\text{def}}{=} d(x, \partial V)$. We also set $d_L(x) = L - |x|$ (note that $d_L(x) \neq d_{V_L}(x)$ with this convention). For $0 \leq a < b \leq L$, we define the "shell"

$$\mathrm{Shell}_L(a, b) \stackrel{\text{def}}{=} \{x \in V_L : a \leq d_L(x) < b\}, \ \mathrm{Shell}_L(b) \stackrel{\text{def}}{=} \mathrm{Shell}_L(0, b). \tag{2.1}$$

Functions: If F, G are functions $\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ we write FG for the (matrix) product: $FG(x, y) \stackrel{\text{def}}{=} \sum_u F(x, u) G(u, y)$, provided the right hand side is absolutely summable. F^k is the k-th power defined in this way, and $F^0(x, y) \stackrel{\text{def}}{=} \delta_{x,y}$. We interpret F also as a kernel, operating from the left on functions $f : \mathbb{Z}^d \to \mathbb{R}$, by $Ff(x) \stackrel{\text{def}}{=} \sum F(x, y) f(y)$. If $W \subset \mathbb{Z}^d$, we use 1_W not only as the indicator function but, by slight abuse of notation, also to denote the kernel $(x, y) \to 1_W(x) \delta_{x,y}$.

For a function $f : \mathbb{Z}^d \to \mathbb{R}$, $\|f\|_1 \stackrel{\text{def}}{=} \sum_x |f(x)|$, and $\|f\|_\infty \stackrel{\text{def}}{=} \sup_x |f(x)|$, as usual. If F is a kernel then, by an abuse of notation, we write $\|F\|_1$ for its norm as operator on L_∞, i.e.

$$\|F\|_1 \stackrel{\text{def}}{=} \sup_x \|F(x, \cdot)\|_1. \tag{2.2}$$

We set $\operatorname{supp} f \stackrel{\text{def}}{=} \{x : f(x) \neq 0\}$. If $f, g : \mathbb{Z}^d \to \mathbb{R}$, we write $f * g$ for the usual convolution.

Transition probabilities: For transition probabilities $p = (p(x, y))_{x,y \in \mathbb{Z}^d}$, not necessarily nearest neighbor, we write $P_{p,x}$ for the law of a Markov chain $S_0, S_1, \ldots$ on $\mathbb{Z}^d$ having p as transition probabilities and $x \in \mathbb{Z}^d$ as a starting point. If $V \subset \mathbb{Z}^d$, $\tau_V \stackrel{\text{def}}{=} \inf\{n \geq 0 : S_n \notin V\}$ is the first exit time from V, and $T_V \stackrel{\text{def}}{=} \tau_{V^c}$ the first entrance time. We set

$$\mathrm{ex}_V(x, z; p) \stackrel{\text{def}}{=} P_{p,x}(S_{\tau_V} = z).$$

For $x \in V^c$, one has $\mathrm{ex}_V(x, z; p) = \delta_{x,z}$. A special case is the standard simple random walk $p(x, \pm e_i) = 1/2d$, where $e_1, \ldots, e_d \in \mathbb{Z}^d$ is the standard base. We abbreviate this as p^{RW}, and set $P_x^{\mathrm{RW}} \stackrel{\text{def}}{=} P_{x,p^{\mathrm{RW}}}$. Also, exit distributions for the simple random walk are written as $\pi_V(x, z) \stackrel{\text{def}}{=} \mathrm{ex}_V\left(x, z; p^{\mathrm{RW}}\right)$.

We will coarse-grain *nearest-neighbor* transition probabilities p in the following way. Given $W \subset \mathbb{Z}^d$, we choose for any $x \in W$ either a fixed subset $U_x \subset W$, $x \in U_x$, or a probability distribution s_x on such sets. Of course, a fixed choice U_x is just a special choice for the distribution s_x, namely the one point distribution on U_x.

Definition 2.1
A collection $\mathcal{S} = (s_x)_{x\in W}$ is called a ***coarse graining scheme*** *on W. Given such a scheme, and nearest neighbor transition probabilities p, we define the coarse grained transitions by*

$$\hat{p}^{\mathrm{CG}}_{\mathcal{S},W}(x,\cdot) \overset{\mathrm{def}}{=} \sum_{U:x\in U\subset W} s_x(U)\,\mathrm{ex}_U(x,\cdot;p). \tag{2.3}$$

In the case of the standard nearest neighbor random walk, we use the notation $\pi_{\mathcal{S},W}$ instead of $\left(\hat{p}^{\mathrm{RW}}\right)^{\mathrm{CG}}_{\mathcal{S},W}$.

Using the Markov property, we have

$$\mathrm{ex}_W(x,\cdot;p) = \mathrm{ex}_W\left(x,\cdot;p^{\mathrm{CG}}_{\mathcal{S},W}\right). \tag{2.4}$$

We will choose the coarse-graining scheme in special ways. Fix once for all a probability density

$$\varphi : \mathbb{R}^+ \to \mathbb{R}^+,\ \varphi \in C^\infty,\ \mathrm{supp}(\varphi) = [1,2]. \tag{2.5}$$

If $m \in \mathbb{R}^+$, the rescaled density is defined by $\varphi_m(t) \overset{\mathrm{def}}{=} (1/m)\,\varphi(t/m)$. The image measure of $\varphi_m(t)\,dt$ under the mapping $t \to V_t(x)\cap W$ defines a probability distribution on subsets of W containing x. We may also choose m to depend on x, i.e. consider a field $\Psi = (m_x)_{x\in W}$ of positive real numbers on W. Such a field then defines via the above scheme coarse grained transition probabilities, which by a slight abuse of notation we denote as $p^{\mathrm{CG}}_{\Psi,W}$. In case $W = \mathbb{Z}^d$, we simply drop W in the notation. In case p is the standard nearest neighbor random walk, we write $\hat{\pi}_\Psi$ instead of p^{CG}_Ψ.

The random environment: We recall from the introduction the notation $\mathcal{P}_\varepsilon$, Ω, $p_\omega(x,y)$, and the natural product σ-field $\mathcal{F}$. For $A \subset \mathbb{Z}^d$, we write $\mathcal{F}_A$ for the σ-field generated by the projections $\omega \to \omega_x,\ x \in A$. We also recall the probability measure μ on $\mathcal{P}$, the product measure $\mathbb{P}_\mu$, and Condition 1.1, which is assumed troughout.

For a random environment $\omega \in \Omega$, we typically write $\Pi_{V,\omega} \overset{\mathrm{def}}{=} \mathrm{ex}_V(\cdot,\cdot;p_\omega)$ and occasionally drop ω in the notation. So Π_V should always be understood as a *random* exit distribution. We will also use $\hat{\Pi}_{\mathcal{S},W}$ for $(p_\omega)^{\mathrm{CG}}_{\mathcal{S},W}$.

For $x \in \mathbb{Z}^d$, $L > 0$, and $\Psi : \partial V_L(x) \to \mathbb{R}^+$, we define the random variables

$$D_{L,\Psi}(x) \overset{\mathrm{def}}{=} \left\| \left(\left[\Pi_{V_L(x)} - \pi_{V_L(x)}\right] \hat{\pi}_\Psi \right)(x,\cdot) \right\|_1, \tag{2.6}$$

$$D^0_L(x) \overset{\mathrm{def}}{=} \left\| \Pi_{V_L(x)}(x,\cdot) - \pi_{V_L(x)}(x,\cdot) \right\|_1, \tag{2.7}$$

and with $\delta > 0$, we set

$$\begin{aligned}
b_1(L,\Psi,\delta) &\overset{\mathrm{def}}{=} \mathbb{P}\left((\log L)^{-9} < D_{L,\Psi}(0) \le (\log L)^{-6.75},\ D^0_L(0) \le \delta \right)\\
b_2(L,\Psi,\delta) &\overset{\mathrm{def}}{=} \mathbb{P}\left((\log L)^{-6.75} < D_{L,\Psi}(0) \le (\log L)^{-4.5},\ D^0_L(0) \le \delta \right)\\
b_3(L,\Psi,\delta) &\overset{\mathrm{def}}{=} \mathbb{P}\left((\log L)^{-4.5} < D_{L,\Psi}(0) \le (\log L)^{-2.25},\ D^0_L(0) \le \delta \right)\\
b_4(L,\Psi,\delta) &\overset{\mathrm{def}}{=} \mathbb{P}\left(\left\{ (\log L)^{-2.25} < D_{L,\Psi}(0) \right\} \cup \left\{ D^0_L(0) > \delta \right\} \right)\\
b(L,\Psi,\delta) &\overset{\mathrm{def}}{=} b_1(L,\Psi,\delta) + b_2(L,\Psi,\delta) + b_3(L,\Psi,\delta) + b_4(L,\Psi,\delta).
\end{aligned}$$

We write $\mathcal{M}_L$ for the set of functions $\Psi : \partial V_L \to [L/2, 2L]$ which are restrictions of functions defined on $\left\{x \in \mathbb{R}^d : L/2 \le |x| \le 2L\right\}$ that have smooth third derivatives bounded by $10L^{-2}$ and fourth derivatives bounded by $10L^{-3}$.

Condition 2.2
Let $L_1 \in \mathbb{N}$, and $\delta > 0$. We say that condition $\mathrm{Cond}\,(\delta, L_1)$ holds provided that for all $L \le L_1$, and for all $\Psi \in \mathcal{M}_L$,

$$b_i\,(L, \Psi, \delta) \le \frac{1}{4} \exp\left[-\left(1 - (4-i)\,/13\right) (\log L)^2\right],\ i = 1,2,3,4. \tag{2.8}$$

In particular, if $\mathrm{Cond}\,(\delta, L_1)$ is satisfied, then for any $L \le L_1$, and any $\Psi \in \mathcal{M}_L$

$$\mathbb{P}\left(\{D_L^0(0) > \delta\} \cup \{D_{L,\Psi}(0) > (\log L)^{-9}\}\right) \le \exp\left[-\frac{10}{13} (\log L)^2\right] \tag{2.9}$$

Our main technical inductive result is

Proposition 2.3
There exist $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$ there exists $\varepsilon_0\,(\delta)$ and $L_0 \in \mathbb{N}$ such that if $\varepsilon \le \varepsilon_0$, $L_1 \ge L_0$, and μ is such that Condition 1.1 holds for ε, then

$$\mathrm{Cond}\,(\delta, L_1) \Longrightarrow \mathrm{Cond}\left(\delta, L_1 (\log L_1)^2\right).$$

Given L_0, δ_0, we can always choose ε_0 so small that if Condition 1.1 is satisfied with ε_0, then $\mathrm{Cond}\,(\delta_0, L_0)$ holds trivially. We therefore see that Proposition 2.3 implies that for any $\delta < \delta_0$, there exists ε_0 small enough such that $\mathrm{Cond}\,(\delta, L)$ holds for all $\varepsilon \le \varepsilon_0$, and all L. In particular, one obtains immediately from Proposition 2.3 the following theorem, which is the main result of this paper.

Theorem 2.4
For each $\delta < \delta_0$ there exists an $\varepsilon_0 > 0$ such that if Condition 1.1 is satisfied with ε_0, then

$$\limsup_{L\to\infty} b\,(L, m_L, \delta) = 0\,,$$

where m_L denotes the element of $\mathcal{M}_L$ that consists of constant smoothing at scale L.

A remark about the wording which is used below. When we say that something holds for "large enough L", we mean that there exists L_0, *depending only on the dimension*, such that the statement holds for $L \ge L_0$. We emphasize that L_0 then *does not depend on ε.*

We write C for a generic positive constant, not necessarily the same at different occurences. C may depend on the dimension d of the lattice, but on nothing else, except when indicated explicitely. Other constants, such as $c_0, c_1, \bar{c}, k_0, K, C_1$ etc., follow the same convention concerning what they depend on (d only, unless explicitely stated otherwise!), but their value is fixed throughout the paper and does not change from line to line.

3 Preliminaries

3.1 The perturbation expansion

Let $p = (p(x,y))_{x,y\in\mathbb{Z}^d}$ be a Markovian transition kernel on $\mathbb{Z}^d$, not necessarily nearest neighbor, but of finite range, and let $V \subset\subset \mathbb{Z}^d$. The Green kernel on V with respect to p is defined by

$$g_V(p)(x,y) \stackrel{\text{def}}{=} \sum_{k\geq 0} (1_V p)^k (x,y).$$

Evidently, if $z \notin V$, then

$$g_V(p)(\cdot,z) = \mathrm{ex}_V(\cdot,z;p). \tag{3.1}$$

If p,q are two transition kernels, the resolvent equation gives for every $n\in\mathbb{N}$

$$\begin{aligned} g_V(p) &= g_V(q) + g_V(q)\,1_V(p-q)\,g_V(p) \\ &= g_V(q) + \sum_{k=1}^{n-1} [g_V(q)\,1_V(p-q)]^k g_V(q) \\ &+ [g_V(q)\,1_V(p-q)]^n g_V(p) \\ &= g_V(q) + \sum_{k=1}^{\infty} [g_V(q)\,1_V(p-q)]^k g_V(q), \end{aligned} \tag{3.2}$$

assuming convergence of the infinite serie, which will always be trivial in cases of interest to us, assuming ellipticity and V finite.

If $V \subset\subset \mathbb{Z}^d$, and $\mathcal{S}$ is any coarse graining scheme on V (as in Definition 2.1), we compare the exit distribution of the RWRE Π_V with the exit distribution π_V of simple random walk through this perturbation expansion, using however coarse grained transitions inside V : Using (3.1) and (2.4) we get for $x\in V$

$$(\Pi_V - \pi_V)(x,\cdot) = \sum_{k=0}^{\infty} \left(\hat{g}_{\mathcal{S},V}\left[\Delta_{\mathcal{S},V}\hat{g}_{\mathcal{S},V}\right]^k \Delta_{\mathcal{S},V}\pi_V\right)(x,\cdot),$$

where

$$\Delta_{\mathcal{S},V} \stackrel{\text{def}}{=} 1_V\left(\hat{\Pi}_{\mathcal{S},V} - \hat{\pi}_{\mathcal{S},V}\right),\ \hat{g}_{\mathcal{S},V} \stackrel{\text{def}}{=} g_V(\hat{\pi}_{\mathcal{S},V}).$$

We will also use the splitting (dropping the $\mathcal{S},V$ indices when no confusion may arise)

$$\hat{g}(x,\cdot) = \delta_{x,\cdot} + \hat{\pi}\hat{g}(x,\cdot),\ x\in V.$$

If we put for $k\geq 0$

$$\zeta^{(k)} = \Delta^k(\Delta\hat{\pi}\hat{g}),$$

we get

$$\Pi_V - \pi_V = \hat{g}\sum_{m=1}^{\infty}\sum_{k_1,\ldots,k_m=0}^{\infty} \zeta^{(k_1)}\cdot\ldots\cdot\zeta^{(k_{m-1})}\Delta^{k_m}\pi_V.$$

Remark that we can replace in $\zeta^{(k)}$ the second part:

$$(\Delta\hat{\pi}\hat{g})(x,y) = \sum_z (\Delta\hat{\pi})(x,z)\,(\hat{g}(z,y) - \hat{g}(x,y)),$$

i.e., we gain a discrete derivative in the Green function.

We will occasionally slightly modify the above expansion, but the basis is always the first equality in (3.2).

3.2 The smoothing scheme on V_L

We next make an explicit choice of smoothing schemes that we will use. Set

$$r(L) \stackrel{\text{def}}{=} L/(\log L)^{10}, \ s(L) \stackrel{\text{def}}{=} L/(\log L)^{3}, \tag{3.3}$$

and

$$\gamma \stackrel{\text{def}}{=} \min\left(\frac{1}{10}, \frac{1}{2}\left(1-\left(\frac{2}{3}\right)^{1/(d-1)}\right)\right). \tag{3.4}$$

We fix a C^∞-function $h : \mathbb{R}^+ \to \mathbb{R}^+$, which satisfies $h(u) = u$ for $u \leq 1/2$, $h(u) = 1$ for $u \geq 2$, and is strictly monotone and concave on $(1/2, 2)$. For $x \in V_L$, we set

$$h_L(x) \stackrel{\text{def}}{=} \gamma s(L) h\left(\frac{d_L(x)}{s(L)}\right). \tag{3.5}$$

Remark that for $d_L(x) \geq 2s(L)$, we have $h_L(x) = \gamma s(L)$.

Lemma 3.1
Fix $\delta_1 > 0$. Then, there is a constant $\bar{k}_0 = \bar{k}_0(\delta_1)$ such that if $k \geq \bar{k}_0(\delta_1)$ then for all L large, if for some $\delta > 0$, $d_L(x) \leq r(L)$, $D^0_{kr(L)}(x) \leq \delta$, then

$$\sum_{y \in V_L : d_L(y) \leq r(L)} |\Delta(x,y)| \leq \delta + \delta_1. \tag{3.6}$$

Proof. Fix k. We have

$$\begin{aligned}
&\sum_{y \in V_L : d_L(y) \leq r(L)} |\Delta(x,y)| \\
&\leq \Pi_{V_{kr(L)}(x)}(x, V_L \cap \mathrm{Shell}_L(r(L))) + \pi_{V_{kr(L)}(x)}(x, V_L \cap \mathrm{Shell}_L(r(L))) \\
&\leq \delta + 2\pi_{V_{kr(L)}(x)}(x, V_L \cap \mathrm{Shell}_L(r(L))).
\end{aligned}$$

Choosing k large enough completes the proof. ■

We will work with two smoothing schemes on V_L. The first will depend on a constant $k_0 > 1$ that will be chosen below, see (3.22).

Definition 3.2

a) *The smoothing scheme $\mathcal{S}_1 = \mathcal{S}_{1,L,k_0} = (s_x)_{x \in V_L}$ is defined as follows. For $d_L(x) \leq r(L)$ we set $s_x = \delta_{k_0 r(L)}$, i.e. for such an x, the coarse graining is done by choosing the exit distribution from $V_{k_0 r(L)}(x) \cap V_L$. For $d_L(x) > r(L)$, we take $s_x(dt) \stackrel{\text{def}}{=} \varphi_{h_L(x)}(t)\, dt$.*

b) *The smoothing scheme $\mathcal{S}_2 = \mathcal{S}_{2,L} = (s_x)_{x \in V_L}$ is simply defined by $s_x(dt) \stackrel{\text{def}}{=} \varphi_{h_L(x)}(t)\, dt$ for all x.*

We will need the second scheme only in the propagation of the part of the estimate $b_4(L,\Psi,\delta)$ involving the expression $D_L^0(x)$ of (2.7). Up to Section 5, we therefore only work with $\mathcal{S}_1$.

We write $B_L^{(i)}$, $i=1,2,3,4$, for the collection of points which are bad on level i, and in the right scale: For $i=1,2,3$, $B_L^{(i)}$, are the set of points $x\notin \mathrm{Shell}_L(r(L))$ such that for *some* $r\in[h_L(x),2h_L(x)]$, one has $D_{r,h_L(x)}(x)>(\log L)^{-11.25+2.25i}$, but for all $r\in[h_L(x),2h_L(x)]$, $D_{r,h_L(x)}(x)\le(\log L)^{-9+2.25i}$, and $D_r^0(x)\le\delta$. $B_L^{(4)}$ is the set of points x which for $d_L(x)>r(L)$ have the property that for some r, $h_L(x)\le r\le 2h_L(x)$, $D_{r,h_L(x)}(x)>(\log L)^{-2.25}$, or $D_r^0(x)>\delta$, and for $d_L(x)\le r(L)$ satisfy $D^0_{r(L)}(x)\ge\delta$. We also write

$$B_L\overset{\mathrm{def}}{=}B_L^{(1)}\cup B_L^{(2)}\cup B_L^{(3)}\cup B_L^{(4)}, \tag{3.7}$$

and

$$\mathrm{Good}_L\overset{\mathrm{def}}{=}\{B_L=\emptyset\}. \tag{3.8}$$

We write $\rho_{i,L}(x)$, $i=1,2$, for the range of the coarse graining scheme at x in scheme i, i.e.

$$\rho_{1,L}(x)\overset{\mathrm{def}}{=}\begin{cases}k_0r(L) & \text{for } d_L(x)\le r(L)\\ 2h_L(x) & \text{for } r(L)<d_L(x)\end{cases} \tag{3.9}$$

and similarly for the second scheme.

If $L_1\le L\le L_1(\log L_1)^2$ then all the radii involved in the definition of badness are smaller than L_1, if L_1 is chosen large enough. Remark that if $d_L(x)>r(L)$, then $h_L(x+\cdot)\in\mathcal{M}_r$ for $h_L(x)\le r\le 2h_L(x)$, and therefore, if $\mathrm{Cond}(\delta,L_1)$ holds, and $L_1\le L\le L_1(\log L_1)^2$, then

$$\mathbb{P}(x\in B_L)\le 2\gamma s(L)\exp\left[-\frac{10}{13}\left(\log\frac{\gamma L}{(\log L)^{10}}\right)^2\right]\le\exp\left[-0.7(\log L)^2\right], \tag{3.10}$$

if L_1 is large enough.

The points y whose random environment ω_y can influence the badness of x are evidently within radius $\rho_{1,L}(x)$ from x. If $|x-y|>\rho_{1,L}(x)+\rho_{1,L}(y)$, then $\{x\in B_L\}$ and $\{y\in B_L\}$ are independent. Therefore, if we define

$$\mathrm{TwoBad}_L\overset{\mathrm{def}}{=}\bigcup_{x,y\in V_L:|x-y|>\rho_{1,L}(x)+\rho_{1,L}(y)}\{x\in B_L\}\cap\{y\in B_L\}, \tag{3.11}$$

then:

Lemma 3.3
Assume L_1 large enough, (2.8) for L_1, and $L_1\le L\le L_1(\log L_1)^2$. Then

$$\mathbb{P}(\mathrm{TwoBad}_L)\le\exp\left[-1.2(\log L)^2\right].$$

3.3 Estimates on exit distributions and the Green's function

For notational convenience, we write π_L instead of π_{V_L}, and similarly in other expressions. For instance, we write τ_L instead of τ_{V_L}.

Lemma 3.4

a)

$$\frac{1}{C} L^{-d+1} \leq \pi_L(x) \leq C L^{-d+1}.$$

b) Let x be a vector of unit length in $\mathbb{R}^d$, let $0 < \theta < 1$, and define the cone $C_\theta(x) \stackrel{\text{def}}{=} \left\{ y \in \mathbb{Z}^d : \langle y, x \rangle \geq (1-\theta) |y| \right\}$. For any θ, there exists $\delta(\theta) > 0$, such that for all L large enough, and all x

$$\pi_L(0, C_\theta(x)) \geq \delta(\theta). \tag{3.12}$$

c) Let $0 < l < L$, and $x \in \mathbb{Z}^d$ satisfy $l < |x| < L$. Then

$$P_x^{\text{RW}}(\tau_L < T_{V_l}) = \frac{l^{-d+2} - |x|^{-d+2} + O\left(l^{-d+1}\right)}{l^{-d+2} - L^{-d+2}}$$

Proof. a) is Lemma 1.7.4 of [5]. b) is immediate from a). c) is Proposition 1.5.10 of [5]. ∎

We will repeatedly make use of the following lemma.

Lemma 3.5

Assume $x, y \in V_L$, $1 \leq a \leq 5 d_L(y)$, $x \notin V_{2a}(y)$. Then

$$P_x\left(T_{V_a(y)} < \tau_{V_L}\right) \leq C \frac{a^{d-2} d_L(y)\, d_L(x)}{|x-y|^d} \tag{3.13}$$

The proof will be given in Appendix A.

We will need a corresponding result for the Brownian motion. We write $\pi_L^{\text{BM}}(y, dy')$ for the exit distribution of the Brownian motion from the ball C_L of radius L in $\mathbb{R}^d$. The following lemma is an easy consequence of the Poisson formula, see [5, (1.43)].

Lemma 3.6

For any $y \in C_L$, it holds that

$$\frac{C^{-1} d(y, \partial C_L)}{|y-y'|^d} \leq \frac{\pi_L^{\text{BM}}(y, dy')}{dy'} \leq \frac{C d(y, \partial C_L)}{|y-y'|^d}, \tag{3.14}$$

where dy' is the surface measure on ∂C_L.

We will need a comparison between smoothed exit distribution of the random walk, and that of Brownian motion. Given $L > 0$, and $\Psi \in \mathcal{M}_L$, let

$$\phi_{L,\Psi} \stackrel{\text{def}}{=} \pi_{V_L} \hat{\pi}_\Psi. \tag{3.15}$$

We consider also the corresponding Brownian kernel on $\mathbb{R}^d$,

$$\phi_{L,\Psi}^{\text{BM}}(y, dz) \stackrel{\text{def}}{=} \int_{\partial C_L(0)} \pi_{C_L(0)}^{\text{BM}}(y, dw) \int \pi_{C_t(w)}^{\text{BM}}(w, dz)\, \varphi_{m_w}(t)\, dt,$$

where $\Psi = (m_w)$, and where we write $\phi_{L,\Psi}^{\text{BM}}(y, z)$ for the density with respect to d-dimensional Lebesgue measure.

Lemma 3.7
There exists a constant C such that for $L>0$, and $\Psi\in\mathcal{M}_L$, we have

$$\sup_{y\in V_L}\sup_{z\in\mathbb{Z}^d}\left|\phi_{L,\Psi}(y,z)-\phi^{\mathrm{BM}}_{L,\Psi}(y,z)\right|\le CL^{-d-1/5}$$

Lemma 3.8
There exists a constant C such that for $L>0$ and $\Psi\in\mathcal{M}_L$, we have

$$\sup_{y,z}\left\|\partial_y^3\phi^{\mathrm{BM}}_{L,\Psi}(y,z)\right\|\le CL^{-d-3}$$

The proofs of these two lemmas are again in Appendix A.

We can draw two immediate conclusions from these results:

Proposition 3.9
a) *Let y,y' be in V_L, and $\Psi\in\mathcal{M}_L$. Then*

$$\left|\phi_{L,\Psi}(y,z)-\phi_{L,\Psi}(y',z)\right|\le C\left(L^{-d-1/5}+\left|y-y'\right|L^{-d-1}\right). \tag{3.16}$$

b) *Let $x\in V_L$, and l be such that $V_l(x)\subset V_L$. Consider a signed measure μ on V_l with total mass 0 which is invariant under lattice isometries. Then*

$$\left|\sum\nolimits_y\mu(y-x)\,\phi_{L,\Psi}(y,z)\right|\le C\left|\mu\right|\left(L^{-d-1/5}+\left(\frac{l}{L}\right)^3L^{-d}\right), \tag{3.17}$$

where $|\mu|$ denotes the total variation norm of μ.

Proof of Proposition 3.9. a) is immediate from Lemmas 3.7 and 3.8. As for b), we get from the same lemmas

$$\left|\sum\nolimits_y\mu(y-x)\,\phi_{L,\Psi}(y,z)-\sum\nolimits_y\mu(y-x)\,\phi^{\mathrm{BM}}_{L,\Psi}(y,z)\right|\le C\left|\mu\right|L^{-d-1/5}.$$

$$\begin{aligned}\sum\nolimits_y\mu(y-x)\,\phi^{\mathrm{BM}}_{L,\Psi}(y,z)&=\sum\nolimits_y\mu(y-x)\left[\phi^{\mathrm{BM}}_{L,\Psi}(y,z)-\phi^{\mathrm{BM}}_{L,\Psi}(x,z)\right]\\&=\sum\nolimits_y\mu(y-x)\,\partial_x\phi^{\mathrm{BM}}_{L,\Psi}(x,z)\left[y-x\right]\\&+\frac{1}{2}\sum\nolimits_y\mu(y-x)\,\partial_x^2\phi^{\mathrm{BM}}_{L,\Psi}(x,z)\left[y-x,y-x\right]\\&+R(\mu,x,z),\end{aligned} \tag{3.18}$$

where

$$\left|R(\mu,x,z)\right|\le C\left|\mu\right|\left(\frac{l}{L}\right)^3L^{-d} \tag{3.19}$$

uniformly in x and z. $\partial^kF[u_1,\dots,u_k]$ denotes the k-th derivative in directions $u_1,\dots,u_k$. The first summand on the right hand side of (3.18) vanisches because μ has mean 0. The second vanishes because by the invariance under lattice isometry of μ, the summand involves only the Laplacian of $\phi^{\mathrm{BM}}_{L,\Psi}(\cdot,z)$, which vanishes because of harmonicity of $\pi^{\mathrm{BM}}_{C_L(0)}(x,\cdot)$ in the x-variable. The estimate (3.19) follows from Lemma 3.8. The proof of the proposition is complete. ■

The next lemma gives a-priori estimates for coarse-grained walks. We use $\hat{\pi}_L^{(i)}$, $i = 1, 2$, to denote the transitions of the coarse grained random walk that uses the smoothing scheme $\mathcal{S}_i$, and $\hat{g}_L^{(i)}$ to denote the corresponding Green's function. Note that these quantitites all depend on L and k_0, but we supress these from the notation. We set $S_L \stackrel{\text{def}}{=} \text{Shell}_L\left(r\left(L\right)\right)$.

Lemma 3.10
There exists a constant C (independent of k_0!) such that:

a)

$$\sup_{x \in V_L} \hat{g}_L^{(1)}\left(x, S_L\right) \leq C.$$

b) *If $i = 1$ and $r\left(L\right) \leq a \leq 3s\left(L\right)$ or $i = 2$ and $a \leq 3s\left(L\right)$ then,*

$$\sup_{x \in V_L} \hat{g}_L^{(i)}\left(x, \text{Shell}_L\left(a, 2a\right)\right) \leq C.$$

c) *For all $x, y \in V_L \setminus \text{Shell}_L(s(L))$, and $i = 1, 2$,*

$$\hat{g}_L^{(i)}\left(x, y\right) \leq C \begin{cases} \frac{1}{s(L)^2 [|x-y| \vee s(L)]^{d-2}}, & y \neq x \\ 1, & y = x\,. \end{cases}$$

d) *For $i = 1, 2$,*

$$\sup_{x \in V_L} \hat{g}_L^{(i)}\left(x, V_L\right) \leq C \left(\log L\right)^6.$$

e) *For $i = 1, 2$,*

$$\sup_{x, x' \in V_L : |x - x'| \leq s(L)} \sum_{y \in V_L} \left| \hat{g}_L^{(i)}\left(x, y\right) - \hat{g}_L^{(i)}\left(x', y\right) \right| \leq C \left(\log L\right)^3$$

The proof is presented in Appendix B.

Lemma 3.10 plays a crucial role in our smoothing procedure. As a preparation, for $k \geq 1$, set

$$B_1\left(k\right) \stackrel{\text{def}}{=} \text{Shell}_L\left(\left(4/3\right)^k r\left(L\right)\right).$$

$B_1\left(k\right) \subset \text{Shell}_L\left(s\left(L\right)\right)$ if $k \leq 20 \log \log L$. By Lemma 3.10, we get that there exists a constant $\bar{c} \geq 1$ such that

$$\sup_{x \in V_L} \hat{g}_L^{(1)}\left(x, B_1\left(k\right)\right) \leq \bar{c} \begin{cases} k\,, & \text{if } k \leq 20 \log \log L \\ \left(\log L\right)^6 & \text{if } k > 20 \log \log L \end{cases}. \tag{3.20}$$

and, for any ball $V_{rs(L)}(z) \subset V_{L - s(L)}$,

$$\sup_{x \in V_L} \hat{g}_L^{(1)}\left(x, B_{rs(L)}(z)\right) \leq \bar{c} r^d\,. \tag{3.21}$$

With $\bar{c}$ as in (3.20) and (3.21), we fix the constant k_0 large enough such that:

$$\begin{aligned} k_0 &\geq \bar{k}_0(1/200\bar{c}), \\ \sup_{x \in S_L} P_x^{\text{RW}}\left(\tau_{V_L} < \tau_{V_{k_0 r(L)}(x)}\right) &\geq 9/10, \\ \sup_{x \in S_L} \pi_{V_{k_0 r(L)}(x)}\left(x, V_L\right) &\leq 17/32. \end{aligned} \tag{3.22}$$

That the two last estimates in (3.22) hold for k_0 large is obvious, for example from Donsker's invariance principle.

4 Smoothed exits

Throughout this section, we consider the coarse graining scheme $\mathcal{S} = \mathcal{S}_1$ as in Definition 3.2, and we write ρ_L for $\rho_{1,L}$. We regard $\hat{\Pi}_{\mathcal{S},L}$ as a field $\left(\hat{\Pi}_{\mathcal{S},L}(x,\cdot)\right)_{x\in V_L}$ of random transition probabilities. We defined the "goodified" transition probabilities

$$\mathrm{gd}\left(\hat{\Pi}_{\mathcal{S},L}\right)(x,\cdot) \stackrel{\mathrm{def}}{=} \begin{cases} \hat{\Pi}_{\mathcal{S},L}(x,\cdot) & \text{if } x \notin B_L \\ \hat{\pi}_{\mathcal{S},L}(x,\cdot) & \text{if } x \in B_L \end{cases} .$$

This field might no longer come from an i.i.d. RWRE, but nevertheless, we have the property that $\mathrm{gd}\left(\hat{\Pi}_{\mathcal{S},L}\right)(x,\cdot)$ and $\mathrm{gd}\left(\hat{\Pi}_{\mathcal{S},L}\right)(y,\cdot)$ are independet provided $|x-y| > \rho_L(x) + \rho_L(y)$. If X is a random variable depending on ω only trough the $\hat{\Pi}_{\mathcal{S},L}$ we define $\mathrm{gd}(X)$ by replacing $\hat{\Pi}_{\mathcal{S},L}$ by $\mathrm{gd}\left(\hat{\Pi}_{\mathcal{S},L}\right)$.

In the sequel, we keep L fixed and typically drop it from the notation. We set $\hat{g} \stackrel{\mathrm{def}}{=} \hat{g}_{\mathcal{S}_1,L}$, $\Delta \stackrel{\mathrm{def}}{=} \Delta_{\mathcal{S}_1,L}$, $\pi \stackrel{\mathrm{def}}{=} \pi_{V_L}$. We take $\Psi \in \mathcal{M}_L$, and set $\phi \stackrel{\mathrm{def}}{=} \phi_{L,\Psi}$, as in (3.15). An easy consequence of our definitions and Lemma 3.1 is the following.

Lemma 4.1
If $\delta \le (1/800\bar{c})$ then, for all $x \in V_L$ and $k \ge 2$,

$$\mathbf{1}_{\{B_L=\emptyset\}} \|\Delta^k(x,\cdot)\|_1 \le \frac{1}{\bar{c}}\left(\frac{1}{8}\right)^k . \tag{4.1}$$

Proof. Since $\max_{x\in V_L} \|\Delta(x,\cdot)\|_1 \le 2$ and $\bar{c} \ge 1$, it is enough to prove that

$$\mathbf{1}_{\{B_L=\emptyset\}} \sum_{z\in V_L} |\Delta^2(x,z)| \le \left(\frac{1}{64\bar{c}}\right) .$$

If $d_L(x) > r(L)$ then, on the event $\{B_L = \emptyset\}$, $\|\Delta(x,\cdot)\|_1 \le \delta$ and hence $\|\Delta^2(x,\cdot)\|_1 \le 2\delta \le 1/64\bar{c}$ due to our choice of δ. On the other hand, if $d_L(x) \le r(L)$ then on the event $\{B_L = \emptyset\}$,

$$\begin{aligned} \sum_{z\in V_L} |\Delta^2(x,z)| &= \sum_{z\in V_L} \left| \sum\nolimits_{y\in V_L} \Delta(x,y)\Delta(y,z) \right| \qquad (4.2) \\ &\le 2\left| \sum_{y\in \mathrm{Shell}_L(r(L))} \Delta(x,y) \right| + \left| \sum_{y\in V_L \setminus \mathrm{Shell}_L(r(L))} \Delta(x,y) \right| \max_{y\in V_L\setminus \mathrm{Shell}_L(r(L))} \sum_{z\in V_L} |\Delta(y,z)| \\ &\le 2(\delta + \frac{1}{200\bar{c}}) + 2\delta = 4\delta + \frac{1}{100\bar{c}} < \frac{1}{64\bar{c}} , \end{aligned}$$

where Lemma 3.1 and $k_0 \ge \bar{k}_0(1/200\bar{c})$ were used in the next to last inequality. ■

In what follows, we will always consider $\delta \le 1/800\bar{c}$.

4.1 The linear part

For $x \in V_L,\ B \subset V_L$, set

$$\begin{aligned} \xi_x^{(k)}(B,z) &= \sum_{y\in B} \hat{g}(x,y)\left(\Delta^k \pi \hat{\pi}_\Psi\right)(y,z) \qquad (4.3) \\ &= \sum_{y\in B}\sum_{y'\in V_L} \hat{g}(x,y)\,\Delta^k(y,y')\left(\phi(y',z) - \phi(y,z)\right) \end{aligned}$$

We write $\xi_x^{(k)}(z)$ for $\xi_x^{(k)}(V_L, z)$, and define

$$G_L \stackrel{\text{def}}{=} \left\{ \sup_{x \in V_L} \sum\nolimits_{k \geq 1} \left\| \xi_x^{(k)} \right\|_1 \leq (\log L)^{-37/4} \right\}.$$

Proposition 4.2
If L is large enough, then

$$\mathbb{P}\left((G_L)^c \cap \mathrm{Good}_L\right) \leq \exp\left[-(\log L)^{17/8}\right].$$

Proof. It suffices to prove that

$$\sup_{x \in V_L} \mathbb{P}\left(\sum\nolimits_{k \geq 1} \left\| \xi_x^{(k)} \right\|_1 \geq (\log L)^{-37/4},\ \mathrm{Good}_L\right) \leq \exp\left[-(\log L)^{9/4}\right].$$

Note that

$$\begin{aligned}
&\mathbb{P}\left(\sum\nolimits_{k \geq 1} \left\| \xi_x^{(k)} \right\|_1 \geq (\log L)^{-37/4},\ \mathrm{Good}_L\right) \\
&= \mathbb{P}\left(\sum\nolimits_{k \geq 1} \left\| \mathrm{gd}\left(\xi_x^{(k)}\right) \right\|_1 \geq (\log L)^{-37/4},\ \mathrm{Good}_L\right) \\
&\leq \mathbb{P}\left(\sum\nolimits_{k \geq 1} \left\| \mathrm{gd}\left(\xi_x^{(k)}\right) \right\|_1 \geq (\log L)^{-37/4}\right).
\end{aligned}$$

For notation convenience, we drop the notation $\mathrm{gd}(\cdot)$, and just use the fact that all $\hat{\Pi}_{\mathcal{S},L}$ involved satisfy the appropriate "goodness" properties. (Remark that after "goodifications", the distribution of $\hat{\Pi}_{\mathcal{S},L}(x, x+\cdot)$ remains invariant under lattice isometries, provided $d_L(x) > 2s(L)$.)

We split $\xi_x^{(k)}$ into different parts. If $d_L(y) > r(L)$ and $\Delta(y, y') > 0$, we have, assuming $\gamma \leq 1/8$, $|y - y'| \leq d_L(y)/4$, i.e. $d_L(y) \leq (4/3)\, d_L(y')$. Therefore, if $d_L(y') \leq r(L)$ and $\Delta^k(y, y') > 0$, then $d_L(y) \leq (4/3)^k r(L)$. Recall the set $B_1(k)$ and the estimate (3.20). If $y \in B_1(k)$, and $\Delta^k(y, y') > 0$, we have

$$\begin{aligned}
|y - y'| &\leq k k_0 r(L) + 3^k \max\left(r(L), d_L(y)\right) \\
&\leq \left(k k_0 + 4^k\right) r(L),
\end{aligned}$$

and applying (3.16), we see that for $y \in B_1(k)$, and y' such that $\Delta^k(y, y') > 0$, we have

$$|\phi(y, z) - \phi(y', z)| \leq C\left(k k_0 + 4^k\right) L^{-d} (\log L)^{-10}.$$

By Lemma 4.1, we have $\left\| \Delta^k(y, \cdot) \right\|_1 \leq 16 \cdot 8^{-k}$. Combining all these estimates, we have

$$\left\| \xi_x^{(k)}\left(B_1(k)\right) \right\|_1 \leq C \begin{cases} k 8^{-k}\left(k k_0 + 4^k\right)(\log L)^{-10}, & \text{if } k \leq 20 \log\log L, \\ 8^{-k}\left(k k_0 + 4^k\right)(\log L)^{-4}, & \text{if } k > 20 \log\log L. \end{cases} \tag{4.4}$$

(We emphasize our convention regarding constants, and in particular the fact that C does not depend on x.) Hence,

$$\sup_x \sum_{k \geq 1} \left\| \xi_x^{(k)}\left(B_1(k)\right) \right\|_1 \leq C (\log L)^{-10} \leq (\log L)^{-37/4}/3. \tag{4.5}$$

Next, let

$$B_2(k) \overset{\text{def}}{=} \mathrm{Shell}_L\left(\left(4/3\right)^k r\left(L\right), \left(5/4\right)^k 2s\left(L\right)\right).$$

If $y \in B_2(k)$ and $\Delta^k(y, y') > 0$, we have $d_L(y') > r(L)$, and we get, using the fact that for $d_L(x) > r(L)$, one can write $\pi(x, \cdot) = (\hat{\pi}\pi)(x, \cdot)$,

$$\xi_x^{(k)}\left(B_2(k), z\right) = \sum_{y \in B_2(k)} \sum_{y' \in V_L} \hat{g}(x, y) D_k(y, y') \left(\phi(y, z) - \phi(y', z)\right),$$

where

$$D_k \overset{\text{def}}{=} \Delta^k \hat{\pi}, \tag{4.6}$$

and

$$\begin{aligned} \sup_{y \in B_2(k)} \left\| D_k(y, \cdot) \right\|_1 &\leq \sup_{y \in B_2(k)} \left\| \Delta^{k-1}(y, \cdot) \right\|_1 \sup_{x : d_L(x) > r(L)} \left\| \Delta\hat{\pi}(x, \cdot) \right\|_1 \\ &\leq C 8^{-k} (\log L)^{-9}. \end{aligned} \tag{4.7}$$

Using Lemma 3.10 b), we have $\sup_x \hat{g}(x, \mathrm{Shell}_L(3s(L))) \leq C \log\log L$. Put

$$A_j \overset{\text{def}}{=} \mathrm{Shell}_L\left(\left(2 + (j-1)/4\right) s(L), \left(2 + j/4\right) s(L)\right), j \geq 1.$$

Starting from a point in A_j, $j \geq 3$, the coarse grained random walk has a probability $\geq 1/C$ to reach A_{j-2} in one step. Starting from A_{j-2}, an ordinary random walk has a probability $\geq 1/C$ to leave $V_{L+k_0 r(L)}$ before reaching A_j, and therefore, the coarse grained walk leaves V_L before reaching A_j with at least the same probability. Therefore $\sup_x \hat{g}(x, A_j) \leq Cj$, and thus,

$$\sup_x \hat{g}(x, B_2(k)) \leq C\left(\left(\frac{5}{4}\right)^{2k} + \log\log L\right) \leq C\left(2^k + \log\log L\right).$$

If $y \in B_2(k)$, and $\Delta^k(y, y') > 0$, then $|y - y'| \leq 2ks(L)$, and therefore

$$\left|\phi(y, z) - \phi(y', z)\right| \leq CkL^{-d} (\log L)^{-3},$$

again by (3.16). Therefore, we get,

$$\begin{aligned} \left\| \xi_x^{(k)}\left(B_2(k), \cdot\right) \right\|_1 &\leq Ck (\log L)^{-12} \left[4^{-k} + 8^{-k} \log\log L\right], \\ \sup_x \sum_{k \geq 1} \left\| \xi_x^{(k)}\left(B_2(k), \cdot\right) \right\|_1 &\leq (\log L)^{-37/4} / 3. \end{aligned} \tag{4.8}$$

Let $B_3(k) \overset{\text{def}}{=} V_L \setminus \left(B_1(k) \cup B_2(k)\right)$. Given $j \in \mathbb{Z}$, let

$$I_j \overset{\text{def}}{=} \left\{jks(L) + 1, \ldots, (j+1) ks(L)\right\}.$$

Then for $\mathbf{j} \in \mathbb{Z}^d$, put $W_{\mathbf{j}} \overset{\text{def}}{=} B_3(k) \cap I_{j_1} \times \cdots \times I_{j_d}$. Let J be the set of $\mathbf{j}$'s for which these sets are not empty. We subdivide J into subsets $J_1, \ldots, J_{K(d)}$ such that for any $1 \leq r \leq K(d)$,

$$\mathbf{j}, \mathbf{j}' \in J_r,\ \mathbf{j} \neq \mathbf{j}' \Longrightarrow d\left(W_{\mathbf{j}}, W_{\mathbf{j}'}\right) > ks(L). \tag{4.9}$$

We also have $\operatorname{diam}(W_{\mathbf{j}}) \leq \sqrt{d} k s(L)$.

We set, recalling (4.6),

$$\xi_{x,\mathbf{j}}^{(k)}(z) \stackrel{\mathrm{def}}{=} \sum_{y \in W_{\mathbf{j}}} \sum_{y' \in V_L} \hat{g}(x,y) D_k(y,y') (\phi(y,z) - \phi(y',z)). \tag{4.10}$$

We fix for the moment k and x. If $t > 0$, and

$$\sum\nolimits_{\mathbf{j}} \mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z) \leq t/2, \tag{4.11}$$

and we have

$$\begin{aligned}
\mathbb{P}\left(\left\|\xi_x^{(k)}(B_3(k),\cdot)\right\|_1 \geq t\right) &\leq \mathbb{P}\left(\left|\sum\nolimits_{\mathbf{j}}\left(\xi_{x,\mathbf{j}}^{(k)}(z) - \mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z)\right)\right| \geq t/2\right) \\
&\leq K(d) \max_{1 \leq r \leq K(d)} \mathbb{P}\left(\left|\sum\nolimits_{\mathbf{j} \in J_r}\left(\xi_{x,\mathbf{j}}^{(k)}(z) - \mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z)\right)\right| \geq t/(2K(d))\right).
\end{aligned}$$

The random variables $\xi_{x,\mathbf{j}}^{(k)}(z) - \mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z)$, $\mathbf{j} \in J_r$, are independent and centered, due to (4.9), and we are going to estimate their sup-norm. $|\phi(y,z) - \phi(y',z)|$ is again $\leq C\left(k(\log L)^{-3} L^{-d} + L^{-d-1/5}\right)$ for y, y' for which $D_k(y,y') \neq 0$. According to Lemma 3.10 c), we have

$$\hat{g}(x, W_{\mathbf{j}}) \leq Ck^d \left(1 + \frac{d(x, W_{\mathbf{j}})}{s(L)}\right)^{-d+2}.$$

Implementing that into (4.10), we get

$$\left\|\xi_{x,\mathbf{j}}^{(k)}(z)\right\|_\infty \leq Ck^{d+1} 8^{-k} \left(1 + \frac{d(x, W_{\mathbf{j}})}{s(L)}\right)^{-d+2} L^{-d} (\log L)^{-12}.$$

By Hoeffding's inequality (see e.g. [6, (1.23)]), we have for $1 \leq r \leq K(d)$

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum\nolimits_{\mathbf{j} \in J_r}\left(\xi_{x,\mathbf{j}}^{(k)}(z) - \mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z)\right)\right| \geq \frac{2^{-k} L^{-d}}{2K(d) (\log L)^{37/4}}\right) \\
&\leq 2\exp\left[-\frac{1}{C} \frac{(\log L)^{-37/2}}{k^{2d+2} 4^{-2k} (\log L)^{-24} \sum_{r=1}^{C(\log L)^3} r^{-d+3}}\right] \leq 2\exp\left[-\frac{1}{C} \frac{(\log L)^{5/2}}{k^{2d+2} 4^{-2k}}\right],
\end{aligned}$$

where we used $d \geq 3$ in the last inequality. The upshot of this estimate is that provided (4.11) holds true with $t = 2^{-k} L^{-d} (\log L)^{-37/4}$, we have

$$\begin{aligned}
\sup_x \mathbb{P}\left(\sum\nolimits_{k \geq 1}\left\|\xi_x^{(k)}(B_3)\right\|_1 \geq (\log L)^{-37/4}\right) &\leq 2\sum_{k \geq 1} \exp\left[-\frac{1}{C} \frac{(\log L)^{5/2}}{k^{2d+2} 4^{-2k}}\right] \\
&\leq \exp\left[-(\log L)^{17/8}\right],
\end{aligned}$$

It remains to prove (4.11) with this t. Write

$$\sum_{\mathbf{j}} \mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z) = \sum_{y \in B_3} \sum_{y' \in V_L} \hat{g}(x,y) \mathbb{E}(D_k(y,y')) (\phi(y,z) - \phi(y',z)).$$

For every y, $y' \mapsto \mathbb{E}(D_k(y,y'))$ is a signed measure with total mass 0, which is invariant unter lattice isometries. Furthermore

$$\sum_{y'} |\mathbb{E}(D_k(y,y'))| \le C8^{-k} (\log L)^{-9}.$$

Applying (3.17), we get

$$\begin{aligned}
&\left|\sum\nolimits_{y'} \mathbb{E}(D_k(y,y'))(\phi(y,z)-\phi(y',z))\right| \\
&\le C8^{-k} (\log L)^{-9} \left(L^{-d-1/4} + \left(\frac{Lk(\log L)^{-3}}{L}\right)^3 L^{-d}\right) \le C4^{-k} (\log L)^{-18} L^{-d},
\end{aligned}$$

uniformly in $y \in B_3(k)$, and k. By Lemma 3.10 d), we have

$$\sup_x \sum_{y \in B_3(k)} \hat{g}(x,y) \le C(\log L)^6.$$

From this (4.11) follows. ■

4.2 The non-linear part, no bad boxes

Recall the random variable $D_{L,\Psi}(0)$, c.f. (2.6).

Proposition 4.3
If L is large enough and $\Psi \in \mathcal{M}_L$, then

$$\mathbb{P}\left(D_{L,\Psi}(0) \ge (\log L)^{-9}; \mathrm{Good}_L\right) \le \exp\left[-(\log L)^{17/8}\right].$$

Proof. We recall the abbreviation $S_L \stackrel{\text{def}}{=} \mathrm{Shell}_L(r(L))$. By Proposition 4.2, it suffices to estimate the rest of the perturbation expansion

$$R_L \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \sum_{k_1,\dots,k_n=0}^{\infty} \left(\hat{g}\Delta^{k_1}\Delta\hat{\pi}\right) \cdot \ldots \cdot \left(\hat{g}\Delta^{k_n}\Delta\hat{\pi}\right) \sum_{k=1}^{\infty} \left(\hat{g}\Delta^k \phi\right), \tag{4.12}$$

on $G_L \cap \mathrm{Good}_L$. The last factor is $\sum_{k=1}^{\infty} \xi^{(k)}$ of the last section, and therefore, we only have to show that on Good_L, the other factors are staying below 1, for instance

$$\sup_x \sum_{k \ge 0} \left\| \left(\hat{g}\Delta^k \Delta\hat{\pi}\right)(x,\cdot)\right\|_1 \le 15/16. \tag{4.13}$$

First, we observe that

$$\sup_{y \notin S_L} \|(\Delta\hat{\pi})(y,\cdot)\|_1 \le C(\log L)^{-9},$$

$$\sum_{k\ge 0} \sup_x \left\|\left(\hat{g}\Delta^k\right)(x,\cdot)\right\|_1 \le C(\log L)^6.$$

Therefore, we have

$$\sum_{k \ge 0} \sup_x \left\| \sum\nolimits_{y \notin S_L} \left(\hat{g}\Delta^k\right)(x,y)(\Delta\hat{\pi})(y,\cdot)\right\|_1 \le 1/16,$$

if L is large enough, and in order to prove (4.13) it therefore suffices to prove

$$\sum_{k\geq 0}\sup_x \left\|\sum\nolimits_{y\in S_L}\left(\hat{g}\Delta^k\right)(x,y)\left(\Delta\hat{\pi}\right)(y,\cdot)\right\|_1 \leq 7/8.$$

As in the proof of proposition (4.2), if $\Delta^k(z,y) > 0$ for $y \in S_L$ then $z \in B_1(k)$. Hence, using (3.20) and Lemma 4.1 together with our choice of k_0 in the second inequality,

$$\begin{aligned}
&\sum_{k\geq 1}\sup_x \left\|\sum\nolimits_{y\in S_L}\left(\hat{g}\Delta^k\right)(x,y)\left(\Delta\hat{\pi}\right)(y,\cdot)\right\|_1\\
&\leq \sum_{k\geq 1}\sup_x \hat{g}(x,B_1(k))\sup_{z\in B_1(k)}\left\|\Delta^{k+1}(z,\cdot)\right\|_1\\
&\leq \sum_{k=2}^{20\log\log L-1} k\left(\frac{1}{8}\right)^k + \sum_{k\geq 20\log\log L}(\log L)^6\left(\frac{1}{8}\right)^k < \frac{1}{8}
\end{aligned}$$

Therefore, it suffices to prove

$$\sup_x \left\|\sum\nolimits_{y\in S_L}\hat{g}\left(x,y\right)\left(\Delta\hat{\pi}\right)(y,\cdot)\right\|_1 \leq 3/4. \tag{4.14}$$

From the choice (3.22) it follows that

$$\sup_{x\in V_L}\hat{g}\left(x,S_L\right)\leq 10/9.$$

Furthermore we have assumed $\delta \leq 1/32$, so that by using the third part in (3.22), we have

$$\sup_{x\in S_L}\Pi_{V_{k_0 r(L)}(x)}\left(x,V_L\right)\leq 9/16.$$

Combining that, we get

$$\begin{aligned}
&\sup_x \left\|\sum\nolimits_{y\in S_L}\hat{g}\left(x,y\right)\left(\Delta\hat{\pi}1_{V_L}\right)(y,\cdot)\right\|_1\\
&=\sup_x\sum\nolimits_{y\in S_L}\hat{g}\left(x,y\right)\Pi_{V_{k_0 r(L)}(y)}\left(y,V_L\right)+\sup_x\sum\nolimits_{y\in S_L}\hat{g}\left(x,y\right)\pi_{V_{k_0 r(L)}(y)}\left(y,V_L\right)\\
&\leq\frac{5}{8}+\frac{1}{9}<\frac{3}{4},
\end{aligned}$$

as required.

We therefore get

$$\sup_{x\in V_L}\left\|R_L\left(x,\cdot\right)\right\|_1 \leq C\left(\log L\right)^{-37/4}$$

on $G_L \cap \mathrm{Good}_L$. ■

4.3 Presence of bad points

On $\left(\mathrm{Good}_L \cup \mathrm{TwoBad}_L\right)^c$, it is clear that for some $x \in V_L$, we have

$$B_L \subset V_{5\rho(x)}\left(x\right). \tag{4.15}$$

We write $\mathcal{D}_L$ for the collection of balls $V_{5\rho_L(x)}(x)$, $x \in V_L$, and for $D \in \mathcal{D}_L$, we write $\mathrm{Bad}_L(D)$ for the event that $\{B_L \subset D\}$, and $\mathrm{Bad}_L^{(i)}(D)$ for the event that $\left\{B_L^{(i)} \subset D\right\}$, $i = 1, 2, 3, 4$.

The main aim of this section is to prove the following result.

Proposition 4.4
There exists a $\delta_0 \leq 1/800\bar{c}$ such that if $\delta < \delta_0$, and if $\mathrm{Cond}(L_1, \delta)$ holds for a given L_1, and if $L \leq L_1 (\log L_1)^2$ and $\Psi \in \mathcal{M}_L$, then, for $i = 1, 2, 3, 4$,

$$\sup_{D \in \mathcal{D}_L} \mathbb{P}\left(\left\| \left([\Pi_{V_L} - \pi_{V_L}] \hat{\pi}_\Psi \right)(0, \cdot) \right\|_1 \geq (\log L)^{-11.25 + 2.25 i},\ \mathrm{Bad}_L^{(i)}(D) \right)$$
$$\leq \frac{1}{100} \exp\left[-(\log L)^2 \right].$$

The proof of Proposition 4.4 relies on an auxilliary construction. To fix the constants in the construction, we need the following lemma. Write $\tilde{G}_L$ for the Green function of the coarse-grained RWRE in a "goodified" environment.

Lemma 4.5
There exists a constant c_0 such that for all L large, and $D \in \mathcal{D}_L$, $D \subset \mathrm{Shell}_L(L/2)$,

$$\tilde{G}_L(0, D) \leq c_0 \left[\frac{\mathrm{diam}(D)^{d-2} \left(\max_{y \in D} d_L(y) \vee s(L) \right)}{L^{d-1}} \right]. \tag{4.16}$$

Further, there exists a constant $c_1 \geq 1$ such that

$$\sup_{y \in V_L} \tilde{G}_L(y, D) \leq c_1. \tag{4.17}$$

We postpone for a moment the proof of Lemma 4.5 and turn to the

Proof of Proposition 4.4. We start with the case when D is "not near" the boundary, meaning that $D \subset V_{L/2}$. We write $D = V_{5\rho_L(x_0)}(x_0) = V_{5\gamma s(L)}(x_0)$. By Lemma 3.10 c), we can find a constant K (not depending on L, x_0), such that for any point $x \notin \widetilde{D} \stackrel{\mathrm{def}}{=} V_{5K\gamma s(L)}(x_0)$, and all L large, one has $\hat{g}(x, D) \leq 1/10$. We modify now the transition probabilities $\hat{\Pi}, \hat{\pi}$ slightly, when starting in $x \in D$, by defining

$$\widetilde{\Pi}(x, \cdot) \stackrel{\mathrm{def}}{=} \begin{cases} \mathrm{ex}_{\widetilde{D}}\left(x, \cdot; \hat{\Pi}\right) & \text{for } x \in D \\ \hat{\Pi}(x, \cdot) & \text{for } x \notin D \end{cases}, \tag{4.18}$$

and similarly we define $\widetilde{\pi}$. (Remark that this destroys somewhat the symmetry, when $x \neq x_0$, but this is no problem below). Clearly, these transition probabilities have the same exit distribution from V_L as the one used before. If we write $\widetilde{g}$ for the Green's function on V_L of $\widetilde{\pi}$, we have $\widetilde{g}(x, y) = \hat{g}(x, y)$ for $y \notin \widetilde{D}$, and all x, whereas $\widetilde{g}(x, y) \leq \hat{g}(x, y)$ for $y \in \widetilde{D}$. In particular, we have

$$\sup_{x \notin \widetilde{D}} \widetilde{g}(x, D) \leq 1/10. \tag{4.19}$$

Writing down our perturbation expansion, we have

$$\left([\Pi_{V_L} - \pi_{V_L}] \hat{\pi}_\Psi \right) = \sum_{m=1}^{\infty} \sum_{k_1, \ldots, k_m = 0}^{\infty} \left(\widetilde{g} \Delta^{k_1} \Delta \hat{\pi} \right) \cdot \ldots \cdot \left(\widetilde{g} \Delta^{k_{m-1}} \Delta \hat{\pi} \right) \left(\widetilde{g} \Delta^{k_m} \Delta \phi \right),$$

where Δ now uses the modified transitions, that is $\Delta(x,y) = \tilde{\Pi}(x,y) - \tilde{\pi}(x,y)$, but remark that for $x \notin D$, $\Delta(x,\cdot)$ is the same as before. Also, ϕ is modified accordingly.

We first estimate the part with $m = 1$. In anticipation of what follows, we consider an arbitrary starting point $x \in V_L$. Put $k = k_1 + 1$. The part of the sum

$$\sum_y \sum_{x_1,\ldots,x_k} \widetilde{g}(x,x_1)\,\Delta(x_1,x_2)\cdot\ldots\cdot\Delta(x_k,y)\,\phi(y,\cdot)$$

where all $x_j \notin D$, is estimated in Section 4.1, and the probability that it exceeds $(\log L)^{-9}/3$ is bounded by $\exp\left[-(\log L)^2\right]/100$. If an $x_j \in D$, then the sum over x_{j+1} extends only to points outside $\widetilde{D}$, and therefore, the sum over $x_{j+1}, x_{j+2}, \ldots, x_{j+K}$ is running only over points outside D. Therefore

$$\sup_{x_j \in D} \sum_{x_{j+1},\ldots x_{j+K}} \left|\Delta(x_j,x_{j+1})\cdot\ldots\cdot\Delta(x_{j+K},x_{j+K+1})\right| \le 2\delta^K. \tag{4.20}$$

Further, let j denote the smallest index such that $x_j \in D$. Let

$$\mathcal{X}_j := \{x_1 : \Delta(x_1,x_2)\cdots\Delta(x_{j-1},x_j)\} > 0\,.$$

Then $\max_{x_1\in\mathcal{X}_j} d(x_1,D) \le 5j\gamma s(L)$. For $j < (\log L)^2$ it follows that $\mathcal{X}_j \subset V_{L-s(L)}$ and therefore, by (3.21), $\max_{x\in V_L} \tilde{g}(x,\mathcal{X}_j) \le Cj^d$. Thus,

$$\left|\sum\nolimits_{x_1,\ldots,x_j} \widetilde{g}(x,x_1)\,\Delta(x_1,x_2)\cdots\Delta(x_{j-1},x_j)\right| \le C\delta^{j-1}j^d\,. \tag{4.21}$$

On the other hand, for $j \ge (\log L)^2$ one has

$$\Big|\sum_{x_1,\ldots,x_j} \widetilde{g}(x,x_1)\,\Delta(x_1,x_2)\cdots\Delta(x_{j-1},x_j)\Big| \le C(1/8)^j(\log L)^6\,.$$

Therefore, using (4.20),

$$\sum_{j=1}^{\infty}\left|\sum_{x_1,\ldots,x_{j-1}\notin D, x_j\in D} \widetilde{g}(x,x_1)\,\Delta(x_1,x_2)\cdots\Delta(x_{j-1},x_j)\right| \le C\,.$$

If $x_k \notin D$, then $\left\|\sum_y \Delta(x_k,y)\,\phi(y,\cdot)\right\|_1 \le C(\log L)^{-12}$. On the other hand, if $x_k \in D$, then

$$\left\|\sum\nolimits_y \Delta(x_k,y)\,\phi(y,\cdot)\right\|_1 \le C\gamma K(\log L)^{-12+2.25i}. \tag{4.22}$$

Combining all the above, we conclude that for some constant c_2 it holds that

$$\Big|\sum_{y,z}\sum_{x_1,\ldots,x_k} \widetilde{g}(x,x_1)\,\Delta(x_1,x_2)\cdot\ldots\cdot\Delta(x_k,y)\,\phi(y,z)\Big| \le c_2\gamma K(\log L)^{-12+2.25i}\,.$$

It follows that

$$\left\|\sum\nolimits'_{x_1,\ldots,x_k} \widetilde{g}(0,x_1)\,\Delta(x_1,x_2)\cdot\ldots\cdot(\Delta\phi)(x_k,\cdot)\right\|_1 \le (\log L)^{-11.5+2.25i}, \tag{4.23}$$

where $\sum'$ denotes summation where at least one x_j is in D.

(We note that for $i = 1, 2, 3$, one would not need to use the K-enlargement and modification of the transition probabilities, as we catch in any case a factor δ for any Δ).

The case $m \geq 2$ is handled with an evident modification of the above procedure, using the estimate (4.19). Indeed, let $D' = \{z \in V_L : d(z, \tilde{D}) \leq 2\gamma s(L)\}$. A repeat of the previous argument shows that

$$\sup_x \sum_{k=4}^{\infty} \sum_{x_k} \Big| \sum_{\substack{x_1,\ldots,x_{k-1}: \\ \exists j \leq k, x_j \in D'}} \widetilde{g}(x, x_1)\, \Delta(x_1, x_2) \cdot \ldots \cdot \Delta(x_{k-2}, x_{k-1})\, \hat{\pi}(x_{k-1}, x_k) \Big| \leq C\delta$$

while

$$\sup_x \sum_{x_3} \Big| \sum_{\substack{x_1,x_2: \\ \exists j \leq 3, x_j \in D'}} \widetilde{g}(x, x_1)\, \Delta(x_1, x_2)\, \hat{\pi}(x_2, x_3) \Big| \leq \begin{cases} \frac{2}{10}, & x \notin D' \\ C, & x \in D', \end{cases}$$

and, by the computation in Section 4.2, c.f. (4.13),

$$\sup_x \sum_{k=3}^{\infty} \sum_{x_k \notin D'} \Big| \sum_{\substack{x_1,\ldots,x_{k-1}: \\ x_j \notin D'}} \widetilde{g}(x, x_1)\, \Delta(x_1, x_2) \cdot \ldots \cdot \Delta(x_{k-2}, x_{k-1})\, \hat{\pi}(x_{k-1}, x_k) \Big| \leq \frac{15}{16}.$$

Hence, we conclude that always,

$$\sup_x \sum_{k \geq 0} \left\| \left(\hat{g}\Delta^k \Delta\hat{\pi}\right)(x, \cdot) \right\|_1 \leq C, \tag{4.24}$$

and for all δ small,

$$\sup_x \sum_{k_1, k_2 \geq 0} \left\| \left(\hat{g}\Delta^{k_1} \Delta\hat{\pi}\right) \left(\hat{g}\Delta^{k_2} \Delta\hat{\pi}\right)(x, \cdot) \right\|_1 \leq \frac{16}{17}. \tag{4.25}$$

Together with the computation for $m = 1$, c.f. (4.23) when D' is visited, and Proposition 4.2 when it is not, this completes the proof of Proposition 4.4 in case $D \subset V_{L/2}$.

We next turn to $D \cap V_{L/2}^c \neq \emptyset$. Recall the Green function $\tilde{G}_L$ of the goodified environment, introduced above Lemma 4.5. Let $\Pi_{V_L}^g$ denote the exit measure Π_{V_L} with the environment replaced by the goodified environment. Let $\Delta^g = 1_D(\Pi_{\mathcal{S},V_L} - \Pi_{\mathcal{S},V_L}^g)$. The perturbation expansion then gives

$$[\Pi_{V_L} - \Pi_{V_L}^g](z) = \sum \tilde{G}_L(0, y)\Delta^g(y, y')\Pi_{V_L}(y', z),$$

and thus

$$\|\Pi_{V_L} - \Pi_{V_L}^g\| \leq 2\tilde{G}_L(0, D) \leq 3 \cdot 10^d c_0 \frac{s(L)^{d-1}}{2L^{d-1}} \leq C(\log L)^{3(1-d)}, \tag{4.26}$$

This completes the proof in case $i = 3, 4$ (and also $i = 1, 2$ if $d \geq 4$).

Consider next the case $i = 1, 2$ (and $d = 3$). Rewrite the perturbation expansion as

$$[\Pi_{V_L} - \Pi^g_{V_L}](z) = \sum_{k\geq 1}\sum_y \tilde{G}_L\,(\Delta^g)^k\,(0,y)\left(\hat{\Pi}_{\mathcal{S},V_L}\tilde{G}_L\Delta^g\Pi^g_{V_L}\right)(y,z) \tag{4.27}$$

In particular, using Lemma 4.1,

$$\begin{aligned}\|\Pi_{V_L} - \Pi^g_{V_L}\| &\leq 32\tilde{G}_L(0,D)\sum_{k\geq 1}(1/8)^{k-1}(\log L)^{-9+2.25i}\sup_{y'\in V_L}\tilde{G}_L(y',D)\\ &\leq C(\log L)^{3(2-d)}(\log L)^{-9+2.25i} \leq (\log L)^{-11.5+2.25i}\,. \end{aligned}\tag{4.28}$$

■

Proof of Lemma 4.5. We begin by establishing some auxilliary estimates for the unperturbed Green function $\hat{g}$. We first show that

$$\sup_{y\in V_L}\hat{g}_L(y,D) \leq C\,. \tag{4.29}$$

Indeed, in proving (4.29), it is enough to consider $y \in D$. Fix a constant β to be chosen below (see (4.30)). For D such that $D\cap \mathrm{Shell}_L(\beta s(L)) \neq \emptyset$, the estimate (4.29) (with $C = C(\beta)$ depending on the choice of β) is an immediate consequence of parts a) and b) of Lemma 3.10. If $D\cap\mathrm{Shell}_L(\beta s(L)) = \emptyset$ and $y \in D$, then, using Lemma 3.5 in the second inequality, and the choice of γ implying $10\gamma \leq 1$, see (3.4), we may find a constant C_1 independent of β such that

$$\begin{aligned}\max_{y\in D}\hat{g}_L(y,D) &\leq 1 + \max_{y\in D}G_{\gamma s(L)}(y,D) +\\ &\qquad \sum_{x\in\mathrm{Shell}_L(2s(L))} P^{\mathrm{RW}}_y(S_{\sigma'} = x)P^{\mathrm{RW}}_x(T_D < T_{V_L})\max_{y\in D}\hat{g}_L(y,D)\\ &\leq C + C_1\frac{(\beta+3)s(L)^2 s(L)^{d-2}}{((\beta-3)s(L))^d}\max_{y\in D}\hat{g}_L(y,D)\,.\end{aligned}$$

Choosing $\beta > 3$ large enough such that

$$C_\beta := C_1(\beta+3)/(\beta-3)^d < 1\,, \tag{4.30}$$

we find that

$$\max_{y\in D}\hat{g}_L(y,D) \leq C + C_\beta\max_{y\in D}\hat{g}_L(y,D)\,,$$

from which the conclusion (4.29) follows.

We next note that, for any $z \in V_L$,

$$\hat{g}_L(z,D) \leq P^{\mathrm{RW}}_z(T_D < \tau_{V_L})\max_{z\in D}\hat{g}_L(z,D)\,.$$

Applying (4.29) and Lemma 3.5, we deduce that for some constant C_0,

$$\hat{g}_L(z,D) \leq C_0\left[\frac{\mathrm{diam}(D)^{d-2}d_L(z)\max_{y\in D}d_L(y)}{d(z,D)^d}\vee 1\right]\,. \tag{4.31}$$

We next write the perturbation expansion

$$\tilde{G}_L(z,D) - \hat{g}_L(z,D) = \sum_{k\geq 1}\sum_{y,y',w}\hat{g}_L(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}_L(w,D) + \mathrm{NL}\,, \tag{4.32}$$

where NL denotes the nonlinear term in the perturbation expansion, that is

$$\mathrm{NL} = \sum_{n=2}^{\infty} \sum_{k_1,\ldots,k_n=0}^{\infty} \left(\hat{g}_L \Delta^{k_1} \Delta\hat{\pi}\right) \cdot \ldots \cdot \left(\hat{g}_L \Delta^{k_{n-1}} \Delta\hat{\pi}\right) \left(\hat{g}_L \Delta^{k_n} \Delta\hat{g}_L(\cdot, D)\right), \quad (4.33)$$

We first handle the linear term in (4.32). Recall that $\sup_{w\in V_L} \hat{g}_L(w,D) \le C$. Thus, in a goodified environment,

$$|\sum_{k\ge 1} \sum_{y,y',w: d_L(y')\ge k_0 r(L)} \hat{g}_L(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}_L(w,D)| \le C(\log L)^6 (1/8)^k (\log L)^{-9}, \quad (4.34)$$

and

$$|\sum_{y,y',w} \hat{g}_L(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}_L(w,D)| \le C(\log L)^6 (1/8)^k . \quad (4.35)$$

From (4.35) it follows that

$$|\sum_{k\ge 20\log\log L} \sum_{y,y',w} \hat{g}_L(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}_L(w,D)| \le C(\log L)^{-9} . \quad (4.36)$$

On the other hand, if $d_L(y') \le k_0 r(L)$ and $\Delta^k(y,y') > 0$ then, as in the proof of Proposition 4.2, $d_L(y) \le (4/3)^k k_0 r(L)$. Using parts a),b) of Lemma 3.10, we get that for $k \le 20\log\log L$,

$$|\sum_{y,y',w: d_L(y')\le k_0 r(L)} \hat{g}_L(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}_L(w,D)| \le Ck(1/8)^k \quad (4.37)$$

Combining (4.34), (4.36) and (4.37), we conclude that

$$\sup_{z\in V_L} \sum_{k\ge 1} \sum_{y,y',w} \hat{g}_L(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}_L(w,D) \le C .$$

The term involving NL is handled by recalling that

$$\sup_x \sum_{k\ge 0} \left\| \left(\hat{g}\Delta^k \Delta\hat{\pi}\right)(x,\cdot) \right\|_1 \le 15/16, ,$$

see (4.13).We then conclude, using (4.29), that (4.17) holds.

To prove (4.16), our starting point is the perturbation expansion (4.32). Again, the main contribution is the linear term. One has

$$\sum_{y,y',w} \hat{g}_L(0,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}_L(w,D) \le C(\log L)^6 (1/8)^k .$$

Hence, there exists a constant c_d such that for all L large,

$$\sum_{k\ge c_d \log\log L} \sum_{y,y',w} \hat{g}_L(0,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}_L(w,D) \le \left(\frac{r(L)}{L}\right)^{d-1} . \quad (4.38)$$

We next divide the sum in the linear term according to the location of w with respect to D, writing

$$\sum_{y,y',w} \hat{g}_L(0,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}_L(w,D) = \sum_{y,y'} \hat{g}_L(0,y)\Delta^k(y,y')\sum_{j=1}^{2}\sum_{w\in B_j^{(k)}} \hat{\pi}(y',w)\hat{g}_L(w,D)\,, \tag{4.39}$$

where

$$B_1^{(k)} = \{z \in V_L : d(z,D) \le L/8\}\,, \quad B_2^{(k)} = \{z \in V_L : d(z,D) > L/8\}$$

Considering the term involving $B_1^{(k)}$, for $k < c_d \log\log L$ the summation over y extends over a subset of V_L that is covered by at most Ck^d elements of $\mathcal{D}_L$, all inside $\mathrm{Shell}_L(L/2)$. Thus, for such k, using (4.32),

$$\sum_{y,y'} \hat{g}_L(0,y)\Delta^k(y,y') \sum_{w\in B_1^{(k)}} \hat{\pi}(y',w)\hat{g}_L(w,D) \le C\left(\frac{1+\gamma}{8}\right)^k k^d \frac{\mathrm{diam}(D)^{d-2}\max_{y\in D} d_L(y)}{L^{d-1}}$$

and hence

$$\sum_{k\le c_d \log\log L}\sum_{y,y'} \hat{g}_L(0,y)\Delta^k(y,y') \sum_{w\in B_1^{(k)}} \hat{\pi}(y',w)\hat{g}_L(w,D) \le C\frac{\mathrm{diam}(D)^{d-2}\max_{y\in D} d_L(y)}{L^{d-1}}\,. \tag{4.40}$$

The term involving $w \in B_2^{(k)}$ is simpler: indeed, one has in that case that $\hat{g}(w,D)$ satisfies, by (4.31), the required bounds, whereas

$$\sum_{y:\exists y' \text{ with } \Delta^k(y,y')\hat{\pi}(y',w)>0} \hat{g}_L(0,y) \le Ck^d\,,$$

yielding

$$\begin{aligned}&\sum_{k\le c_d \log\log L}\sum_{y,y'} \hat{g}_L(0,y)\Delta^k(y,y') \sum_{w\in B_2^{(k)}} \hat{\pi}(y',w)\hat{g}_L(w,D)\\ &\le C\sum_{k\le c_d \log\log L} k^d(1/8)^k \frac{\mathrm{diam}(D)^{d-2}\max_{y\in D} d_L(y)}{L^{d-1}}\,.\end{aligned} \tag{4.41}$$

Combining (4.38), (4.40) and (4.41) results in the required control on the linear term in (4.32). The nonlinear term is even simpler and similar to the handling of the nonlinear term when estimating $\hat{g}(z,D)$. ■

5 The non-smoothed exit estimate

The aim of this section is to prove the following result.

Proposition 5.1
There exists $0 < \delta_0 \le 1/2$ such that for $\delta \le \delta_0$, there exist $L_0(\delta)$ and $\varepsilon_0(\delta)$ such that if $L_1 \ge L_0$ and $\varepsilon \le \varepsilon_0$, then $\mathrm{Cond}(L_1,\delta)$, *and $L \le L_1(\log L_1)^2$ imply*

$$\mathbb{P}\left(\|\Pi_L(0,\cdot) - \pi_L(0,\cdot)\|_1 \ge \delta\right) \le \frac{1}{10}\exp\left[-(\log L)^2\right]$$

Proof. We use the coarse graining scheme $\mathcal{S}_2$ from Definition 2.1, but we stick to the notations before, so $\hat{\pi} = \hat{\pi}_{\mathcal{S}_2,L}$, etc. Using $\mathcal{S}_2$ means that we refine the smoothing scale up to the boundary. In particular, the smoothing scale is $h_L(x) = \gamma d_L(x)$ for all x with $d_L(x) \leq s(L)/2$, and $\hat{\pi}(x,\cdot)$ is obtained by averaging exit distributions from balls with radii between $\gamma d_L(x)$ and $2\gamma d_L(x)$. (γ from (3.4)). If $d_L(x) < 1/2\gamma$, then there is no smoothing at all, and $\hat{\pi}(x,\cdot) = p^{\mathrm{RW}}(x,\cdot)$.

The drawback of this smoothing scheme is that the presence of many bad regions close to the boundary is unavoidable. We will however show that they cannot be too frequent.

We consider layers

$$\Lambda_j \stackrel{\mathrm{def}}{=} \mathrm{Shell}_L\left(2^{j-1}, 2^j\right),$$

for $j = 1, \ldots, J_1(L) \stackrel{\mathrm{def}}{=} \left[\frac{\log r(L)}{\log 2}\right] + 1$, so that

$$\mathrm{Shell}_L(r(L)) \subset \bigcup\nolimits_{j \leq J} \Lambda_j \subset \mathrm{Shell}_L(2r(L)). \tag{5.1}$$

We subdivide each Λ_j into subsets $D_1^{(j)}, D_2^{(j)}, \ldots, D_{N_j}^{(j)}$ of diameter $\leq \sqrt{d}2^j$, where

$$N_j \leq C\left(L2^{-j}\right)^{d-1}. \tag{5.2}$$

The set of these subsets is denoted by $\mathcal{L}_j$. $\mathcal{L}_j$ is split into disjoint $\mathcal{L}_j^{(1)}, \ldots, \mathcal{L}_j^{(R)}$, such that for any m one has

$$d(D, D') > 5\gamma 2^j, \ \forall D, D' \in \mathcal{L}_j^{(m)}, \tag{5.3}$$

$$N_j^{(m)} \stackrel{\mathrm{def}}{=} \left|\mathcal{L}_j^{(m)}\right| \geq N_j/2R. \tag{5.4}$$

We can do that in such a way that $R \in \mathbb{N}$ depends only on the dimension d.

If $B \in \mathcal{L}_j$, we write $\mathrm{Bad}(B)$ for the event $\{B \subset \mathrm{Good}_L\}^c$. Remark that

$$\mathbb{P}(\mathrm{Bad}(B)) \leq C2^{(d+1)j} \exp\left[-\log^2\left(\gamma 2^{j-1}\right)\right] \leq \exp\left[-j^{5/3}\right] \stackrel{\mathrm{def}}{=} p_j.$$

for $j \geq J_0$, J_0 approriately chosen (depending on γ).

We set

$$X_j^{(m)} \stackrel{\mathrm{def}}{=} \sum_{D \in \mathcal{L}_j^{(m)}} 1_{\mathrm{Bad}(D)}, \ X_j \stackrel{\mathrm{def}}{=} \sum_{m=1}^{R} X_j^{(m)}.$$

Due to (5.3), the events $\mathrm{Bad}(D)$, $D \in \mathcal{L}_j^{(m)}$, are independent. Remark that $p_j < j^{-3/2} \leq 1/2$ for all $j \geq 2$. From a standard coin tossing estimate, we get

$$\mathbb{P}\left(X_j^{(m)} \geq j^{-3/2} N_j^{(m)}\right) \leq \exp\left[-N_j^{(m)} I\left(j^{-3/2} \mid p_j\right)\right].$$

with $I(x \mid p) \stackrel{\mathrm{def}}{=} x\log(x/p) + (1-x)\log((1-x)/(1-p))$

$$I\left(j^{-3/2} \mid p_j\right) \geq -\frac{3}{2} j^{-3/2}\log j + j^{-3/2} j^{5/3} - \log 2 \geq 2Rj^{1/7}$$

if J_0 is large enough. Therefore

$$\begin{aligned}\mathbb{P}\left(X_j \geq j^{-3/2}N_j\right) &\leq R \max_{1\leq m\leq R} \mathbb{P}\left(X_j^{(m)} \geq j^{-3/2}N_j^{(m)}\right) \\ &\leq R\exp\left[-\left(L2^{-j}\right)^{d-1} j^{1/7}\right] \leq R\exp\left[-\frac{1}{C}\left(\log L\right)^{20} j^{1/7}\right]\end{aligned}$$

for $J_0(\gamma) \leq j \leq J(L)$, L large enough (implied by L_0 large enough). Using this, we get

$$\sum_{J_0(\gamma)\leq j\leq J(L)} \mathbb{P}\left(X_j \geq j^{-3/2}N_j\right) \leq \frac{1}{20}\exp\left[-\left(\log L\right)^2\right]$$

by choosing J_0 large enough. Setting

$$\mathrm{ManyBad}_L \overset{\mathrm{def}}{=} \bigcup\nolimits_{J_0(\gamma)\leq j\leq J(L)} \left\{X_j \geq j^{-3/2}N_j\right\} \cup \mathrm{TwoBad}_L,$$

we get

$$\begin{aligned}\mathbb{P}\left(\mathrm{ManyBad}_L\right) &\leq \frac{1}{20}\exp\left[-\left(\log L\right)^2\right] + \exp\left[-1.2\left(\log L\right)^2\right] \\ &\leq \frac{1}{10}\exp\left[-\left(\log L\right)^2\right],\end{aligned} \tag{5.5}$$

again by choosing L_0 approriate. We now choose $\varepsilon_0(\gamma) > 0$ small enough such that for $\varepsilon \leq \varepsilon_0$, one has $X_j = 0$, deterministically, for $j < J_0(\gamma)$. (As γ is completely fixed in (3.4), we usually don't explicitly indicate it in the notation).

We will show that if $\omega \notin \mathrm{ManyBad}_L$, then $\left\|\Pi_{L,\omega} - \pi_L\right\|_1 \leq \delta$. This proves Proposition 5.1.

We disinguish between two (disjoint) bad regions $B_1, B_2 \subset V_L$. We set $\widetilde{B}_L \overset{\mathrm{def}}{=} B_L \backslash \mathrm{Shell}_L(r(L))$, (for B_L, see (3.7)). Set

$$B_2' \overset{\mathrm{def}}{=} \bigcup \left\{D_i^{(j)} : \omega \in \mathrm{Bad}\left(D_i^{(j)}\right),\ j = 1, \ldots, R;\ i \leq N_j\right\}.$$

On the complement of TwoBad_L there exists x_0, $|x_0| > r(L)$, such that $\widetilde{B}_L \subset V_{5\rho(x_0)}(x_0)$. (see (4.15). There is some ambiguity in choosing x_0, but this of no importance.) If $|x_0| \leq L/2$, we define $B_1 \overset{\mathrm{def}}{=} V_{5\rho(x_0)}(x_0) = V_{5\gamma s(L)}(x_0)$, and $B_2 \overset{\mathrm{def}}{=} B_2'$. If $|x_0| > L/2$, we put $B_1 \overset{\mathrm{def}}{=} \emptyset$, and $B_2 \overset{\mathrm{def}}{=} B_2' \cup V_{5\rho(x_0)}(x_0)$. Of course, if $\widetilde{B}_L = \emptyset$, then $B_1 \overset{\mathrm{def}}{=} \emptyset$, and $B_2 \overset{\mathrm{def}}{=} B_2'$. Remark that B_1 and B_2 are disjoint. We put $B \overset{\mathrm{def}}{=} B_1 \cup B_2$, and $G \overset{\mathrm{def}}{=} V_L \backslash B$.

In case $B_1 = V_{5\gamma s(L)}(x_0)$, $|x_0| \leq L/2$, we use the same (slight) modification of $\hat{\Pi}(y,\cdot)$, $\hat{\pi}(y,\cdot)$ for $y \in V_{5\gamma s(L)}(x_0)$ as used in Section 4.3, i.e. we replace $\hat{\pi}, \hat{\Pi}$ by $\widetilde{\pi}, \widetilde{\Pi}$ as defined in (4.18), but we retain the $\hat{}$-notation for convenience.

We use a slightly modified perturbation expansion. Again with $\Delta \overset{\mathrm{def}}{=} \hat{\Pi} - \hat{\pi}$, we have

$$\Pi_L = \pi_L + \hat{g}1_B\Delta\Pi_L + \hat{g}1_G\Delta\Pi_L.$$

Set $\gamma_k \overset{\mathrm{def}}{=} \hat{g}\left(1_G\Delta\right)^k$. Then

$$\begin{aligned}\gamma_k\Pi_L &= \hat{g}\left(1_G\Delta\right)^k \Pi_L \\ &= \hat{g}\left(1_G\Delta\right)^k \pi_L + \hat{g}\left(1_G\Delta\right)^k \hat{g}\Delta\Pi_L \\ &= \hat{g}\left(1_G\Delta\right)^k \pi_L + \hat{g}\left(1_G\Delta\right)^k 1_B\Delta\Pi_L + \hat{g}\left(1_G\Delta\right)^k \hat{\pi}\hat{g}\Delta\Pi_L + \gamma_{k+1}\Pi_L\end{aligned}$$

Therefore, iterating, we get

$$\begin{aligned}\Pi_L &= \pi_L + \hat{g}\sum_{k=0}^{\infty}\left(1_G\Delta\right)^k 1_B\Delta\Pi_L + \hat{g}\sum_{k=1}^{\infty}\left(1_G\Delta\right)^k \hat{\pi}\hat{g}\Delta\Pi_L + \hat{g}\sum_{k=1}^{\infty}\left(1_G\Delta\right)^k \pi_L \\ &= \pi_L + \hat{g}\overline{\Gamma}1_B\Delta\Pi_L + \hat{g}\Gamma\hat{\pi}\Pi_L.\end{aligned}$$

where $\Gamma \overset{\text{def}}{=} \sum_{k=1}^{\infty}\left(1_G\Delta\right)^k$, $\overline{\Gamma} \overset{\text{def}}{=} I+\Gamma$. With the partition $B = B_1 \cup B_2$, we get with the setting $\Xi_1 \overset{\text{def}}{=} \hat{g}\overline{\Gamma}1_{B_1}\Delta$, $\Xi_2 \overset{\text{def}}{=} \hat{g}\overline{\Gamma}1_{B_2}\Delta$

$$\Pi_L = \pi_L + \Xi_1\Pi_L + \Xi_2\Pi_L + \hat{g}\Gamma\hat{\pi}\Pi_L,$$

and by induction on $m \in \mathbb{N}$, replacing successively Π_L in the second summand

$$\Pi_L - \pi_L = \left(\sum_{r=1}^{m}\Xi_1^r\right)\pi_L + \left(\sum_{r=0}^{m}\Xi_1^r\right)\Xi_2\Pi_L + \left(\sum_{r=0}^{m}\Xi_1^r\right)\hat{g}\Gamma\hat{\pi}\Pi_L + \Xi_1^{m+1}\Pi_L$$

i.e. with $m \to \infty$

$$\begin{aligned}\Pi_L - \pi_L &= \sum_{r=1}^{\infty}\Xi_1^r\pi_L + \left(\sum_{r=0}^{\infty}\Xi_1^r\right)\Xi_2\Pi_L + \left(\sum_{r=0}^{\infty}\Xi_1^r\right)\hat{g}\Gamma\hat{\pi}\Pi_L \\ &:= A_1 + A_2 + A_3\,.\end{aligned}\tag{5.6}$$

For $D \subset V_L$, we write

$$U_k(D) \overset{\text{def}}{=} \left\{y \in V_L : \exists x \in D \text{ with } \Delta^k(y,x) > 0\right\}.$$

We now prove that each of the three parts A_1, A_2, A_3 is bounded by $\delta/3$.

First summand A_1 : This does not involve the bad regions near the boundary, and we can apply the estimates from Section 4.3. There is nothing to prove if $B_1 = \emptyset$, so we assume $B_1 = V_{5\gamma s(L)}(x_0)$, $|x_0| \le L/2$. From Lemma 3.10 we have

$$\sup_{x\in V_L}\left|\hat{g}\left(1_G\Delta\right)^k(x,B_1)\right| \le \delta^k\hat{g}\left(x,U_k(B_1)\right) \le C\delta^k k^d,\tag{5.7}$$

and therefore.

$$\sum_{k=0}^{\infty}\left\|\hat{g}\left(1_G\Delta\right)^k 1_{B_1}\right\|_1 \le C.\tag{5.8}$$

In the same way, we obtain, with K from Section 4.3,

$$\sum_{k=0}^{\infty}\sup_{x\notin V_{5K\gamma s(L)}}\left\|\hat{g}\left(1_G\Delta\right)^k 1_{B_1}(x,\cdot)\right\|_1 \le \frac{1}{2},\tag{5.9}$$

by using (5.7) for $k \ge 1$, and (4.19) for $k = 0$. Furthermore ,

$$\begin{aligned}\left\|\Xi_1\pi_L\right\|_1 &\le \sum_{k=0}^{\infty}\left\|\hat{g}\left(1_G\Delta\right)^k 1_{B_1}\Delta\pi_L\right\|_1 \le C\sum_{k=0}^{\infty}\left\|\hat{g}\left(\cdot,U_k(B_1)\right)\right\|_\infty 2^{-k}\sup_{x\in B_1}\left\|\Delta\pi_L(x,\cdot)\right\|_1 \\ &\le C\sum_{k=0}^{\infty}k^d 2^{-k}\sup_{x\in B_1}\left\|\Delta\pi_L(x,\cdot)\right\|_1 \le C\left(\log L\right)^{-3}.\end{aligned}\tag{5.10}$$

Using these inequalities, we get $\|A_1\|_1 \leq C(\log L)^{-3} \leq C(\log L_0)^{-3} \leq \delta/3$ by choosing $L_0(\delta)$ large enough: When estimating $\|\Xi_1^r \pi_L\|_1$ for $r \geq 2$, we use (5.8) for the first factor Ξ_1, (5.10) for the last $\Xi_1 \pi_L$, and (5.9) for the middle Ξ_1^{r-2}. The point is that $(1_{B_1}\Delta)(x,y)$ is $\neq 0$ only if $y \notin V_{5K\gamma s(L)}(x_0)$, and so we can use (5.9) for this part.

Second summand A_2: We drop here the Π_L-factor, using the trivial estimate $\|\Pi_L(x,\cdot)\|_1 \leq 2$. If $r=0$, one has to estimate $\|\Xi_2(0,\cdot)\|_1$ where B_2 consists of the bad regions in the layers $\mathcal{L}_j$, and the possible one bad ball from $\widetilde{B}_L$ which is outside $V_{L/3}$. In case $r \geq 1$, when $B_1 \neq \emptyset$, we have $B_2 = B_2'$, which is at distance $\geq L/3$ from B_1. Therefore, in case $r=0$, we have to estimate

$$\left\|\hat{g}\,(1_G\Delta)^k\, 1_{B_2}(0,\cdot)\right\|_1 \tag{5.11}$$

(the last Δ is of no help, and we drop it), and in case $r \geq 1$, using (5.8) and (5.9)

$$C2^{-r} \sup_{|x| \leq 2L/3} \left\|\hat{g}\,(1_G\Delta)^k\, 1_{B_2}(x,\cdot)\right\|_1,$$

but in this case, we have $B_2 \subset \mathrm{Shell}_L(2r(L))$. The estimate of the second case is entirely similar to the estimate of (5.11), and we therefore provide the details only of the proof of the latter.

We split the parts coming from the different bad regions. For a bad region $D_i^{(j)}$ in layer $\mathcal{L}_j$, we have

$$\left\|\hat{g}\,(1_G\Delta)^k\, 1_{D_i^{(j)}}(0,\cdot)\right\|_1 \leq C2^{-k}\hat{g}\left(0, U_k\left(D_i^{(j)}\right)\right).$$

It suffices to estimate $\hat{g}\left(0, U_k\left(D_i^{(j)}\right)\right)$ very crudely. Points in $U_k\left(D_i^{(j)}\right)$ are at distance of most $2^j(1-2\gamma)^{-k}$ from the $D_i^{(j)}$. We first consider k's only such that $\mathrm{Shell}_L(s(L))^c$ is not touched, which is the case if $k \leq 20\log\log L$ (L large enough). The number of layers touched is bounded by $1+k$, and for each Λ_r which intersects $U_k\left(D_i^{(j)}\right)$, a very crude estimate gives

$$\left|U_k\left(D_i^{(j)}\right) \cap \Lambda_r\right| \leq C2^r 2^{(d-1)j}(1-2\gamma)^{-(d-1)k} \leq C2^r 2^{(d-1)j}\left(\frac{3}{2}\right)^k \tag{5.12}$$

where in the last inequality, we have used (3.4). Using Lemma 3.5, we see that

$$\hat{g}\left(0, U_k\left(D_i^{(j)}\right)\right) \leq C(1+k)\,2^{(d-1)j}\left(\frac{3}{2}\right)^k L^{-d+1}.$$

Therefore, using $\omega \notin \bigcup_{J_0(\gamma) \leq j \leq J(L)} \left\{X_j \geq j^{-3/2} N_j\right\}$, we have the estimates

$$\sum_{k \leq 10\log\log L} \left\|\hat{g}\,(1_G\Delta)^k\, 1_{B_2' \cap \Lambda_j}(0,\cdot)\right\|_1 \leq Cj^{-3/2},$$
$$\sum_{k \leq 10\log\log L} \left\|\hat{g}\,(1_G\Delta)^k\, 1_{B_2'}(0,\cdot)\right\|_1 \leq CJ_0^{-1/2}.$$

For the sum over $k > 20 \log\log L$, we simply estimate $\hat{g}(0, U_k(B_2')) \le \hat{g}(0, V_L) \le C(\log L)^6$ and we therefore get

$$\sum_k \left\| \hat{g}(1_G\Delta)^k 1_{B_2\cap \mathrm{Shell}_L(r(L))}(0,\cdot)\right\|_1 \le C\left(J_0^{-1/2} + (\log L)^6 2^{-20\log\log L}\right) \tag{5.13}$$
$$\le C\left(J_0^{-1/2} + (\log L)^{-7}\right) \le \delta/6$$

by choosing $J_0(\delta)$ and $L_0(\delta)$ large enough.

It remains to add the part of B_2 outside B_2'. This is (contained in) a ball $V_{5\gamma\rho(x_0)}(x_0)$ with $|x_0| > L/2$.

$$\hat{g}\left(0, U_k\left(V_{5\gamma\rho(x_0)}(x_0)\right)\right) \le \hat{g}\left(0, U_k\left(V_{5\gamma s(L)}(x_0)\right)\right) \le \hat{g}\left(0, V_{(5+2k)\gamma s(L)}(x_0)\right).$$

As $|x_0| \ge L/2$, we have $V_{(5+2k)\gamma s(L)}(x_0) \cap V_{L/3} = \emptyset$ provided $k \le (\log L)^3/C$, and $V_{(5+2k)\gamma s(L)}(x_0)$ can be covered by $\le Ck^d$ balls $V_{s(L)}(y)$, $|y| \ge L/3$. By Lemma 3.5, one has $\hat{g}\left(0, V_{s(L)}(y)\right) \le C(\log L)^{-3}$. (This remains true also if $V_{s(L)}(y)$ intersects $\mathrm{Shell}_L(s(L))$, as is easily checked). Therefore, for $k \le (\log L)^3/C$, we have

$$\hat{g}\left(0, U_k\left(V_{5\gamma\rho(x_0)}(x_0)\right)\right) \le Ck^d(\log L)^{-3},$$

and therefore,

$$\sum_k \left\| \hat{g}(1_G\Delta)^k 1_{V_{5\gamma\rho(x_0)}(x_0)}(0,\cdot)\right\|_1$$
$$\le C\sum_{k\le(\log L)^3/C} 2^{-k}k^d(\log L)^{-3} + C\sum_{k>(\log L)^3/C} 2^{-k}(\log L)^6 \le \delta/6,$$

provided L_0 is large enough. Combining this with (5.13) proves $\|A_2\|_1 \le \delta/3$.

Third summand A_3. By the same argument as in the discussion of A_2, it suffices to consider $r = 0$, and we drop Π_L.

$$\sum_{k\ge1}\left\| \sum_{x\notin \mathrm{Shell}_L(r(L))} \hat{g}(1_G\Delta)^{k-1}(0,x)(1_G\Delta\hat{\pi})(x,\cdot)\right\|_1$$
$$\le \sum_{k\ge1}^{\infty} 2^{-k+1}\hat{g}(0,V_L) \sup_{x\notin \mathrm{Shell}_L(r(L))} \|1_G\Delta\hat{\pi}(x,\cdot)\|_1 \tag{5.14}$$
$$\le C(\log L)^{-3} \le \delta/9$$

if L_0 is large enough. For $J_0(\gamma) \le j \le J_1(L)$

$$\left\| \sum_{x\in *_j} \hat{g}(1_G\Delta)^{k-1}(0,x)(1_G\Delta\hat{\pi})(x,\cdot)\right\|_1 \le 2^{-k+1}\hat{g}(0,U_k(\Lambda_j)) \sup_{x\in\Lambda_j}\|1_G\Delta\hat{\pi}(x,\cdot)\|_1$$
$$\le Cj^{-9}2^{-k+1}\hat{g}(0,U_k(\Lambda_j)),$$

and it is evident from Lemma 3.5 that $\sum_{k\ge1} 2^{-k+1}\hat{g}(0,U_k(\Lambda_j)) \le C$. Therefore

$$\sum_{k\ge1}\left\| \sum_{J_0(\gamma)\le j\le J_1(L)} \sum_{x\in\Lambda_j} \hat{g}(1_G\Delta)^{k-1}(0,x)(1_G\Delta\hat{\pi})(x,\cdot)\right\|_1 \le CJ_0(\gamma)^{-8} \le \delta/9, \tag{5.15}$$

if J_0 is chosen large enough. Put $\hat{\Lambda} \overset{\mathrm{def}}{=} \bigcup_{j \leq J_0(\gamma)} \Lambda_j$

$$\begin{aligned}\sum_{k\geq 1}\left\|\sum\nolimits_{x\in\hat{\Lambda}}\hat{g}\left(1_G\Delta\right)^{k-1}(0,x)\left(1_G\Delta\hat{\pi}\right)(x,\cdot)\right\|_1 &\leq C\sum_{k\geq 1}2^{-k+1}\hat{g}\left(0,U_k\left(\hat{\Lambda}\right)\right)\sup_{x\in\hat{\Lambda}}\left\|\Delta\left(x,\cdot\right)\right\|_1\\ &\leq C\left(J_0\right)\sup_{x\in\hat{\Lambda}}\left\|\Delta\left(x,\cdot\right)\right\|_1\leq\delta/9\end{aligned}$$

if $\varepsilon \leq \varepsilon_0(\delta)$. Combining this with (5.14) and (5.15) proves $\|A_3\|_1 \leq \delta/3$. ■

6 Proof of Proposition 2.3

We just have to collect the estimates we have obtained so far. We take δ_0 small enough according to Proposition 4.4, and Proposition 5.1, and for $\delta \leq \delta_0$, we choose L_0 large enough, also according these propositions.

For $L_1 \geq L_0$ we assume $\mathrm{Cond}(\delta, L_1)$, and take and $L \leq L_1 (\log L_1)^2$. For $i = 1,2,3$, and $\Psi \in \mathcal{M}_L$, we have according to Lemma 3.3 and Proposition 4.3

$$\begin{aligned}b_i\left(L,\Psi,\delta\right) &\leq \mathbb{P}\left(D_{L,\Psi}\left(0\right)>\left(\log L\right)^{-11.25-2.25i}\right)\\ &\leq \mathbb{P}\left(D_{L,\Psi}\left(0\right)>\left(\log L\right)^{-11.25-2.25i},\left(\mathrm{TwoBad}_L\right)^c\cap\left(\mathrm{Good}_L\right)^c\right)\\ &+\mathbb{P}\left(D_{L,\Psi}\left(0\right)>\left(\log L\right)^{-9},\mathrm{TwoBad}_L\cap\mathrm{Good}_L\right)+\mathbb{P}\left(\mathrm{TwoBad}_L\right)\\ &\leq \mathbb{P}\left(D_{L,\Psi}\left(0\right)>\left(\log L\right)^{-11.25-2.25i},\left(\mathrm{TwoBad}_L\right)^c\cap\left(\mathrm{Good}_L\right)^c\right)\\ &+\exp\left[-1.2\left(\log L\right)^2\right]+\exp\left[-\left(\log L\right)^{17/8}\right].\end{aligned}$$

We therefore only have to estimate the first summand.

$$\begin{aligned}&\mathbb{P}\left(D_{L,\Psi}\left(0\right)>\left(\log L\right)^{-11.25-2.25i},\left(\mathrm{TwoBad}_L\right)^c\cap\left(\mathrm{Good}_L\right)^c\right)\\ &\leq \sum_{D\in\mathcal{D}_L}\sum_j\mathbb{P}\left(\left\|\left(\left[\Pi_{V_L}-\pi_{V_L}\right]\hat{\pi}_\Psi\right)(0,\cdot)\right\|_1\geq\left(\log L\right)^{-11.25+2.25i},\ \mathrm{Bad}_L^{(j)}\left(D\right)\right)\\ &\leq \sum_{D\in\mathcal{D}_L}\sum_{j\leq i}\mathbb{P}\left(\left\|\left(\left[\Pi_{V_L}-\pi_{V_L}\right]\hat{\pi}_\Psi\right)(0,\cdot)\right\|_1\geq\left(\log L\right)^{-11.25+2.25j},\ \mathrm{Bad}_L^{(j)}\left(D\right)\right)\\ &\quad+\sum_{D\in\mathcal{D}_L}\sum_{j>i}\mathbb{P}\left(\mathrm{Bad}_L^{(j)}\left(D\right)\right)\\ &\leq \frac{4\left|\mathcal{D}_L\right|}{100}\exp\left[-\left(\log L\right)^2\right]+\left|\mathcal{D}_L\right|\exp\left[-\left[1-\left(4-i-1\right)/13\right]\left(\log\frac{L}{\left(\log L\right)^{10}}\right)^2\right]\\ &\leq \frac{1}{8}\exp\left[-\left[1-\left(4-i\right)/13\right]\left(\log L\right)^2\right].\end{aligned}$$

Combining these estimates, we get

$$b_i\left(L,\Psi,\delta\right)\leq\frac{1}{4}\exp\left[-\left[1-\left(4-i\right)/13\right]\left(\log L\right)^2\right],$$

L large enough, for $i = 1,2,3$.

For $i = 4$, we have

$$b_4(L, \Psi, \delta) \leq \mathbb{P}\left(D_{L,\Psi}(0) > (\log L)^{-2.25}\right) + \mathbb{P}\left(\|\Pi_L(0,\cdot) - \pi_L(0,\cdot)\|_1 \geq \delta\right).$$

The second summand is estimated by Proposition 5.1, and the first in the same way as the $b_i,\ i \leq 3$.

This completes the proof of Proposition 2.3.

A Proofs of the random walk results

We begin by stating and proving some auxilliary estimates. If $A \subset\subset \mathbb{Z}^d$, $x \in A,\ y \in \partial A$, then by the usual time reversal, one gets

$$\begin{aligned} P_x(S_{\tau_A} = y) &= \sum_{y' \in A,\ |y-y'|=1} (2d)^{-1} g_A(y', x) \\ &\leq \sum_{y' \in A,\ |y-y'|=1} (2d)^{-1} P_{y'}(T_x < \tau_A). \end{aligned} \tag{A.1}$$

Throughout this appendix, we write $\tau \stackrel{\text{def}}{=} \tau_{V_L}$.

Lemma A.1
Let $x \in V_L,\ y \in \partial V_L$. Then, for some $\bar{c}_1 \geq 1$,

$$P_x(S_\tau = y) \leq \bar{c}_1 d_L(x)^{-d+1}.$$

Proof. Let $r \stackrel{\text{def}}{=} d_L(x)$. We may assue that $r \geq 4$. Put $r' \stackrel{\text{def}}{=} [r/2] - 1$. Then $V_{r'}(x) \subset V_{L-r'}$. If y' is any neighbor of y in V_L then

$$P_{y'}\left(T_{\partial V_{r'}(x)} < \tau_A\right) \leq P_{y'}\left(T_{V_{L-r'}} < \tau_A\right) \leq \frac{C}{r}.$$

Furthermore uniformly in $z \in \partial V_{r'}(x)$,

$$P_z(T_x < \tau_A) \leq P_z(T_x < \infty) \leq Cr'^{-d+2} \leq Cr^{-d+2}.$$

Using the Markov property and (A.1) proves the claim. ∎

Lemma A.2
Let $x \in V_L,\ y \in \partial V_L$ and set $t \stackrel{\text{def}}{=} |x-y|$. Then for some $\bar{c}_2 \geq 1$

$$P_x(S_\tau = y) \leq \bar{c}_2 \frac{d_L(x)}{t} \inf_{x' \in \partial V_{t/3}(y) \cap V_L} P_{x'}(S_\tau = y).$$

Proof. The bound is evident if $r \stackrel{\text{def}}{=} d_L(x) \geq t/10$. Therefore, we assume $r < t/10$. We choose a point $x' \in \partial V_{L+r}$ with $|x - x'| \leq 3r$. Then

$$P_x\left(T_{V_r(x')} < \tau_{V_{t/2}(x')}\right) \geq \frac{1}{C}\frac{r}{t}.$$

Remark that $V_{t/2}(x') \cap V_{t/3}(y) = \emptyset$, and $V_r(x') \cap V_L = \emptyset$. Therefore, by the Markov property,

$$P_x(S_\tau = y) \leq \frac{Cr}{t} \inf_{z \in \partial V_{t/2}(x') \cap V_L} P_z(S_\tau = y) \leq \frac{Cr}{t} \inf_{x' \in \partial V_{t/3}(y) \cap V_L} P_{x'}(S_\tau = y),$$

which completes the proof. ∎

Lemma A.3
With x, y, r as above, and $\bar{c}_1, \bar{c}_2$ from the previous lemmas,

$$P_x\left(S_\tau = y\right) \leq \bar{c}_1 \bar{c}_2^d 3^{(d-1)^2} \frac{d_L\left(x\right)}{r^d}.$$

Proof. Put $\eta \overset{\text{def}}{=} 3^{-d+1}\bar{c}_2^{-1}$, and set $\bar{K} \overset{\text{def}}{=} \bar{c}_1 \eta^{-d+1} = \bar{c}_1 \bar{c}_2^{d-1} 3^{(d-1)^2}$. Using Lemma A.2, it suffices to prove

$$\inf_{x \in \partial V_r \cap V_L} P_x\left(S_\tau = y\right) \leq \bar{K} r^{-d+1}. \tag{A.2}$$

As $\bar{K} \geq r^{(d-1)^2}$, there is nothing to prove if $r \leq 9$. Assume that we have proved (A.2) for all $r \leq r_0$, and assume $r_0 < r \leq 2r_0$. Then for $d_L\left(x\right) > \eta r$, we have by Lemma A.1 that

$$P_x\left(S_\tau = y\right) \leq c_1 \eta^{-d+1} r^{-d+1} = \bar{K} r^{-d+1},$$

and for $d_L\left(x\right) \leq \eta r$, by Lemma A.2 and the fact that $r/3 \leq r_0$,

$$P_x\left(S_\tau = y\right) \leq c_2 \eta \bar{K} \left(\frac{r}{3}\right)^{-d+1} \leq \bar{K} r^{-d+1}.$$

Therefore, the lemma is proved by induction. ∎

Proof of Lemma 3.5. If $|x - y| \leq d_L\left(y\right)/2$, then $d_L\left(x\right) \geq d_L\left(y\right)/2$, and in this case, we can simply use

$$P_x\left(T_{V_a(y)} < \tau\right) \leq P_x\left(T_{V_a(y)} < \infty\right) \leq C\left(\frac{a}{|x-y|}\right)^{d-2} \leq C \frac{a^{d-2} d_L\left(y\right) d_L\left(x\right)}{|x-y|^d}.$$

Therefore, we may assume $|x - y| > d_L\left(y\right)/2$. Furthermore, it suffices to consider the case $1 \leq a \leq d_L\left(y\right)/5$, simply because for $d_L\left(y\right)/5 < a \leq 5 d_L\left(y\right)$, we get an upper bound with replacing a by $d_L\left(y\right)/5$. Assume that we have proved the bound for $a = d_L\left(y\right)/5$. Then we get for $a < d_L\left(y\right)/5$

$$P_x\left(T_{V_a(y)} < \tau\right) \leq C \frac{d_L\left(y\right)^{d-1} d_L\left(x\right)}{|x-y|^d} \left(\frac{a}{d_L\left(y\right)}\right)^{d-2} \leq C \frac{a^{d-2} d_L\left(y\right) d_L\left(x\right)}{|x-y|^d}.$$

We therefore see that it suffices to prove the bound for $a = d_L\left(y\right)/5$.

Let $y' \in \partial V_L$ be a point closest to y. There exists $\delta > 0$, such that

$$\inf_{x' \in V_a(y)} P_{x'}\left(S_\tau \in V_a\left(y'\right)\right) \geq \delta.$$

Evidently, $\inf_{z \in V_a(y') \cap \partial V_L} |x - z| \geq |x - y|/2$, and therefore, by Lemma A.3,

$$\sup_{z \in V_a(y') \cap \partial V_L} P_x\left(S_\tau = z\right) \leq C \frac{d_L\left(x\right)}{|x-y|^d}.$$

Consequently

$$\begin{aligned}
\frac{d_L\left(x\right) a^{d-1}}{|x-y|^d} &\geq \frac{1}{C} P_x\left(S_\tau \in V_a\left(y'\right)\right) \geq \frac{1}{C} P_x\left(S_\tau \in V_a\left(y'\right),\ T_{V_a(y)} < \tau\right) \\
&= \frac{1}{C} \sum_{x' \in V_a(y)} P_x\left(T_{V_a(y)} < \tau,\ S_{T_{V_a(y)}} = x'\right) P_{x'}\left(S_\tau \in V_a\left(y'\right)\right) \\
&\geq \frac{\delta}{C} P_x P_x\left(T_{V_a(y)} < \tau\right).
\end{aligned}$$

This proves the claim. ■

Before presenting the proofs of Lemmas 3.7 and 3.8, we introduce some notation and state and prove some additional auxilliary estimates. For $M = (m_x) \in \mathcal{M}_L$, set

$$\hat{\pi}_M(x,z) \stackrel{\text{def}}{=} \int_{\mathbb{R}^+} \frac{1}{m_x} \varphi\left(t/m_x\right) \mathrm{ex}_{V_t}\left(x,z;p^{\mathrm{RW}}\right) dt,$$

and the corresponding Brownian quantity

$$\hat{\pi}_M^{\mathrm{BM}}(x,dz) \stackrel{\text{def}}{=} \int \frac{1}{m_x} \varphi\left(t/m_x\right) \pi_{C_t}^{\mathrm{BM}}(x,dz) dt.$$

$\hat{\pi}_M^{\mathrm{BM}}(x,dz)$ has a density with respect to Lebesgue measure which, by an abuse of notation, we write as $\hat{\pi}_M^{\mathrm{BM}}(x,z)$.

Lemma A.4
There is a constant C such that for any L large enough, any $M \in \mathcal{M}_L$, any $x, x', z, z' \in \mathbb{Z}^d$, it holds that

$$\hat{\pi}_M(x,z) \le CL^{-d}, \quad \hat{\pi}_M^{\mathrm{BM}}(x,z) \le CL^{-d}. \tag{A.3}$$

$$|\hat{\pi}_M(x,z) - \hat{\pi}_M(x',z)| \le C|x-x'|L^{-(d+1)} \log L, \tag{A.4}$$

$$|\hat{\pi}_M^{\mathrm{BM}}(x,z) - \hat{\pi}_M^{\mathrm{BM}}(x',z)| \le C|x-x'|L^{-(d+1)} \log L, \tag{A.5}$$

$$|\hat{\pi}_M(x,z) - \hat{\pi}_M(x,z')| \le C|z-z'|L^{-(d+1)} \log L, \tag{A.6}$$

$$|\hat{\pi}_M^{\mathrm{BM}}(x,z) - \hat{\pi}_M^{\mathrm{BM}}(x,z')| \le C|x-x'|L^{-(d+1)} \log L. \tag{A.7}$$

Further, for $1 < a < b < 2$, and $aL \le |x-z| \le bL$,

$$\hat{\pi}_M(x,z) \ge C(a,b)^{-1} L^{-d}. \tag{A.8}$$

Proof of Lemma A.4. The estimates (A.4) and (A.8) are immediate from Lemmas 3.4 and 3.6, and the definition of $\hat{\pi}_M$.

We turn to the proof of (A.3) and (A.6). It clearly suffices to consider only the cases $|x-x'| = 1$ or $|z-z'| = 1$. Note first that

$$\begin{aligned} |\hat{\pi}_M(x,z) - \hat{\pi}_M(x',z)| &= \left[1 - \frac{m_x}{m_{x'}}\right] \hat{\pi}_M(x,z) \\ &+ \frac{1}{m_{x'}} \int_{\mathbb{R}^+} \left[\varphi\left(\frac{t}{m_x}\right) - \varphi\left(\frac{t}{m_{x'}}\right)\right] \pi_{V_t(x)}(x,z) dt \\ &+ \frac{1}{m_{x'}} \int_{\mathbb{R}^+} \varphi\left(\frac{t}{m_{x'}}\right) \left[\pi_{V_t(x)}(x,z) - \pi_{V_t(x')}(x',z)\right] dt \stackrel{\text{def}}{=} I_1 + I_2 + I_3. \end{aligned}$$

Since $M \in \mathcal{M}_L$, it holds that $\left[1 - \frac{m_x}{m_{x'}}\right] \le CL^{-1}|x-x'|$, and hence, using (A.4), it holds that

$$I_1 \le CL^{-d} \frac{|x-x'|}{L}. \tag{A.9}$$

Similarly, using the smoothness of φ and the estimates $m_{x'} \ge L/2$ and $\pi_{V_t(x)}(x,z) \le CL^{1-d}$, see Lemma 3.4 a), one gets

$$I_2 \le CL^{-d} \frac{|x-x'|}{L}. \tag{A.10}$$

By translation invariance of simple random walk, we have that $\pi_{V_r(x)}(x,z) = \pi_{V_r}(0, z-x)$. Thus, both (A.3) and (A.6) will follow if we can show, for $|x-x'| = 1$ and $y = x$ or x', the estimate

$$\left| \int_{\mathbb{R}^+} \varphi\left(\frac{t}{m_y}\right) \left[\pi_{V_t}(0, z-x) - \pi_{V_t}(0, z-x')\right] dt \right| \leq CL^{-d}. \tag{A.11}$$

Of course, we may assume that $|x-z|$ is of order L. Note that the integration in (A.11) is over the union of two intervals, each of length at most $\sqrt{d}$. Hence, due to the smoothness of φ, (A.11) will follow if we can show that

$$\left| \int_{\mathbb{R}^+} \left[\pi_{V_t}(0, z-x) - \pi_{V_t}(0, z-x')\right] dt \right| \leq CL^{-d}. \tag{A.12}$$

Let $J \stackrel{\mathrm{def}}{=} \{t > 0 : x - z \in \partial V_t\}$. J is an interval of length at most $\sqrt{d}$. For $t \in J$, we set

$$t' = t'(t) \stackrel{\mathrm{def}}{=} \left| x' - t\frac{z-x}{|z-x|} \right|.$$

Evidently, $dt'/dt = 1 + O\left(L^{-1}\right)$, and if we set $J' \stackrel{\mathrm{def}}{=} \{t > 0 : x' - z \in \partial V_{t'}\}$, then J' is an interval of the same lenght as J, up to $O\left(L^{-1}\right)$, and further $|J \Delta J'| = O\left(L^{-1}\right)$. Therefore, if we prove

$$\left| \int_{J \cap J'} \left[\pi_{V_t(x)}(x,z) - \pi_{V_{t'}(x')}(x',z)\right] dt \right| \leq CL^{-d}\log L, \tag{A.13}$$

the estimate (A.11) will follow. To abbreviate notation, we write V for $V_t(x)$, and V' for $V_{t'}(x')$. A first exit decomposition yields

$$\pi_V(x,z) \leq \pi_{V'}(x,z) + \sum_{y \in V \setminus V'} P_x^{\mathrm{RW}}\left(T_y < \tau_V\right) \pi_V(y,z). \tag{A.14}$$

We have two simple geometric facts:

- $$\bigcup\nolimits_{t \in J \cap J'} (V \setminus V') \subset x + \mathrm{Shell}_L(C).$$
- For any $y \in x + \mathrm{Shell}_L(C)$
$$\int_{J \cap J'} 1_{\{y \in V \setminus V'\}} dt \leq C\frac{|y-z|}{L}.$$

Using this together with $\pi_{V'}(x,z) = \pi_{V'}(x',z) + O\left(L^{-d}\right)$, see [5, Theorem 1.7.1], we deduce from (A.14) that

$$\begin{aligned}\int_{J \cap J'} \pi_{V_t(x)}(x,z) dt &\leq \int_{J \cap J'} \pi_{V_{t'}(x')}(x',z) dt + O\left(L^{-d}\right) + CL^{-d} \sum_{y \in x + \mathrm{Shell}_L(C)} |y-z|^{-d} \frac{|y-z|}{L} \\ &\leq \int_{J \cap J'} \pi_{V_{t'}(x')}(x',z) dt + O\left(L^{-d}\log L\right)\end{aligned}$$

The inequality in the opposite direction is proved in the same way. This proves (A.12) and completes the proof of (A.3) and (A.6).

The estimates (A.5) and (A.7) can be obtained either by repeating the argument above, replacing the random walk by Brownian motion, or by applying the Poisson formula [5, (1.43)]. We ommit further details. ■

In order to prove Lemma 3.7 we need also the following technical result:

Lemma A.5
There exists a constant $C = C(\beta, \epsilon)$ such that for any $A \in \partial V_L$, $\beta > 6\epsilon > 0$, $y \in V_L$ with $d(y, \partial V_L) > L^\beta$ and $L > L_0$,

$$\sum_{y' \in A} \pi_{V_L}(y, y') \leq \int_{d(y',A) \leq L^\beta} \pi_L^{\mathrm{BM}}(y, dy') \left(1 + \frac{C(\beta, \epsilon))}{L^{\beta - 5\epsilon}}\right) + \frac{C(\beta,\epsilon)}{L^{d+1}} . \tag{A.15}$$

and for any $A' \in \partial C_L$ and $z \in V_L$ with $d(z, \partial C_L) > L^\beta$,

$$\int_{A'} \pi_L^{\mathrm{BM}}(z, dy') \leq \sum_{y' : d(y', A) \leq L^\beta} \pi_{V_L}(z, y') \left(1 + \frac{C(\beta, \epsilon)}{L^{\beta - 5\epsilon}}\right) + \frac{C(\beta,\epsilon)}{L^{d+1}} . \tag{A.16}$$

Finally, for any $x, z \in \mathbb{Z}^d$ and $M \in \mathcal{M}_L$,

$$\left|\hat{\pi}_M(x,z) - \hat{\pi}_M^{\mathrm{BM}}(x,z)\right| \leq \frac{C}{L^{d+1/4}}. \tag{A.17}$$

Proof of Lemma A.5. We first prove (A.15). Set $A_\beta = \{y' \in \partial C_L : d(y', A) \leq L^\beta\}$. Pick $\epsilon \in (0, \beta)$ and set $L' = L + L^\epsilon$ and $L'' = L + L^{2\epsilon}$. Let A'_β be the image of A_β in $\partial C_{L'}$ under the map $x \mapsto (L'/L)x$. Then, one has (with $\hat{y} = L'y/L$),

$$\int_{A_\beta} \pi_L^{\mathrm{BM}}(y, dy') = \int_{A'_\beta} \pi_{L'}^{\mathrm{BM}}(\hat{y}, dy') . \tag{A.18}$$

Note further, using the Poisson formula [5, (1.43)], that

$$\begin{aligned} \int_{A'_\beta} \pi_{L'}^{\mathrm{BM}}(\hat{y}, dy') &= \int_{A'_\beta} \frac{d\pi_{L'}^{\mathrm{BM}}(\hat{y}, \cdot)}{d\pi_{L'}^{\mathrm{BM}}(y, \cdot)} \pi_{L'}^{\mathrm{BM}}(y, dy') \\ &= \int_{A'_\beta} \frac{\left((L')^2 - |\hat{y}|^2\right)|y' - y|^d}{\left((L')^2 - |y|^2\right)|y' - \hat{y}|^d} \pi_{L'}^{\mathrm{BM}}(y, dy') \end{aligned} \tag{A.19}$$

An explicit computation, using that $|y| \leq L - L^\beta$ and that $1 > \beta > \epsilon > 0$, reveals that

$$\left|\log \frac{\left((L')^2 - |\hat{y}|^2\right)|y' - y|^d}{\left((L')^2 - |y|^2\right)|y' - \hat{y}|^d}\right| \leq CL^{\epsilon - \beta} .$$

Substituting in (A.19) one finds that

$$\int_{A_\beta} \pi_L^{\mathrm{BM}}(y, dy') \geq \int_{A'_\beta} \pi_{L'}^{\mathrm{BM}}(y, dy') \left(1 - C(\beta, \epsilon) L^{-\beta + 2\epsilon}\right) . \tag{A.20}$$

Recall that π_L^{BM} is unchanged if one replaces the Brownian motion by a Brownian motion of covariance $I_d/\sqrt{d}$. Let W_t^y be such a Brownian motion started at y, and recall that by [11, Corollary 1], there exists a constant C_0 such that for every integer n, one may construct $\{W_t^x\}$ in the same space as $\{S_n\}$ such that

$$P_x(\max_{0\le m\le n} |S_m - W_m^x| > C_0 \log n) \le \frac{C_0}{n^{d+1}}\,. \tag{A.21}$$

Standard estimates involving the maximum of the increments of the Brownian motion, imply that one may construct the Brownian motion W_t^y and the random walk S_n on the same space such that, with

$$D \stackrel{\text{def}}{=} \{ \sup_{0\le t\le L^{2+\epsilon/100}} \left|S_{[t]} - W_t^y\right| \le 4C_0 \log L\}\,,$$

one has

$$P_y(D^c) \le \frac{2C_0}{n^{d+1}}\,. \tag{A.22}$$

Set $\tau \stackrel{\text{def}}{=} \min\{n : S_n \in \partial V_L\}$, $\tau' \stackrel{\text{def}}{=} \inf\{t : W_t^y \in \partial C_{L'}\}$, $\tau'' \stackrel{\text{def}}{=} \min\{n : S_n \in \partial V_{L''}\}$, and $B \stackrel{\text{def}}{=} \left\{(\tau' \vee \tau'') \le L^{2+\epsilon/100}\right\}$. Standard estimates imply that if $S_0 = y$ then $P(B^c)$ decays like a stretched exponential, and in particular $P(B^c) \le L^{-d-1}$ for large L. Note that on $D \cap B$, one has that $\tau < \tau' < \tau''$. Now, defining $G'_\beta = \{z \in \mathbb{Z}^d : d(z, (A'_\beta)^c \cap \partial C_L) < 4C_0 \log L\}$, and setting $T_{G'_\beta} = \inf\{n : S_n \in G'_\beta\}$,

$$\begin{aligned}
P\left(W^y_{\tau'} \in A'_\beta\right) &\ge P_y(S_\tau \in A, W_{\tau'} \in A'_\beta) \qquad\qquad\qquad\qquad (\text{A.23})\\
&\ge P_y(S_\tau \in A, W_{\tau'} \in A'_\beta, B \cap D) - \frac{1}{L^{d+1}}\\
&\ge P_y(S_\tau \in A) - P_y(S_\tau \in A, W_{\tau'} \notin A'_\beta, B \cap D) - \frac{2}{L^{d+1}}\\
&\ge P_y^{\text{RW}}(S_\tau \in A) - P_y^{\text{RW}}(S_\tau \in A, T_{G'_\beta} < \tau'') - \frac{2}{L^{d+1}}
\end{aligned}$$

Using the Markov property, one has

$$\begin{aligned}
P_y^{\text{RW}}(S_\tau \in A, T_{G'_\beta} < \tau'') &\le P_y^{\text{RW}}(S_\tau \in A) \sup_{z\in A} P_z^{\text{RW}}(T_{G'_\beta} < \tau'')\\
&\le \sup_{z\in A} \sum_{z'\in G'_\beta} P_z^{\text{RW}}(T_{z'} < \tau'')\\
&\le \sup_{z\in A} C \sum_{z'\in G'_\beta} \frac{L^{3\epsilon}\log^{d+2} L}{|z'-z|^d}\\
&\le C L^{5\epsilon-\beta}\,,
\end{aligned}$$

where the next to last inequality is due to Lemma 3.5. Substituting in (A.23), one completes the proof of (A.15). The reverse inequality (A.16) is proved similarly.

It remains to prove (A.17). Fix $\alpha = 2/3$, $\beta = 1/3$, and $\epsilon = 1/60$. Note that with $\mathcal{D} = C_{L^\alpha}(z)$, using (A.6),

$$\hat\pi_M\,(x,z) \le \frac{1}{|\mathcal{D}|} \sum_{z'\in\mathcal{D}} \hat\pi_M\,(x,z') + CL^{-d-1+\alpha} \log L\,. \tag{A.24}$$

Next, note that

$$
\begin{aligned}
\sum_{z'\in\mathcal{D}} \hat{\pi}_M(x,z') &= \int dt\varphi_{m_x}(t) \sum_{z'\in\mathcal{D}} \pi_{V_t(x)}(x,z') \\
&\le \int dt\varphi_{m_x}(t) \int_{C_{L^\alpha+L^\beta}(z)} \pi_t^{\mathrm{BM}}(x,dz')\left(1+\frac{C}{L^{\beta-5\epsilon}}\right)+\frac{C|\mathcal{D}|}{L^{d+1}} \\
&\le \hat{\pi}_M^{\mathrm{BM}}(x,\mathcal{D})\left(1+\frac{C}{L^{\beta-5\epsilon}}\right)+CL^{-d}|C_{L^\alpha+L^\beta}(z)\setminus C_{L^\alpha}(z)|+\frac{C|\mathcal{D}|}{L^{d+1}} \\
&\le |\mathcal{D}|\hat{\pi}_M^{\mathrm{BM}}(x,z)\left(1+\frac{C}{L^{\beta-5\epsilon}}\right)+\frac{C|\mathcal{D}|\log L}{L^{d+1-\alpha}}+\frac{C|\mathcal{D}|}{L^{\alpha-\beta-d}}\,.
\end{aligned}
$$

Substituting in (A.24), one gets

$$
\hat{\pi}_M(x,z) \le \hat{\pi}_M^{\mathrm{BM}}(x,z)+CL^{-d-1/4}\,.
$$

The reverse equality is proved similarly. This completes the proof of (A.17) and of the lemma ■

Proof of Lemma 3.7. Fix $\alpha=2/3,\beta=1/3$. Set $\eta \stackrel{\mathrm{def}}{=} d(y,\partial V_L)$, and let $y_1\in\partial V_L$ be such that $\eta=|y-y_1|$. Consider first $\eta\le L^{\beta+1/15}$. Then, using (3.13) and (A.4) in the first inequality and (A.3) in the second,

$$
\begin{aligned}
\phi_{L,M}(y,z) &\le \sum_{y'\in\partial V_L:|y'-y_1|<L^\alpha} \pi_{V_L}(y,y')\,\hat{\pi}_M(y',z)+\frac{C\log L}{L^{d+\alpha-\beta}} \\
&\le \hat{\pi}_M(y_1,z)\sum_{y'\in\partial V_L:|y'-y_1|<L^\alpha} \pi_{V_L}(y,y')+\frac{C}{L^{d+1/5}}\,.
\end{aligned}
$$

Consequently,

$$
\phi_{L,M}(y,z)\le\hat{\pi}_M(y_1,z)+\frac{C}{L^{d+1/5}}\,.
$$

Applying now (3.14) in the first inequality and (A.3) in the second, we conclude that

$$
\begin{aligned}
\phi_{L,M}(y,z) &\le \hat{\pi}_M(y_1,z)\int_{y'\in\partial V_L:|y'-y_1|<L^\alpha}\pi_L^{\mathrm{BM}}(y,dy')+\frac{C}{L^{d+1/5}} \\
&\le \int_{y'\in\partial V_L:|y'-y_1|<L^\alpha}\hat{\pi}_M(y',z)\,\pi_L^{\mathrm{BM}}(y,dy')+\frac{C}{L^{d+1/5}}\,.
\end{aligned}
$$

An application of (A.17) then implies that for $\eta\le L^{\beta+1/15}$,

$$
\phi_{L,M}(y,z)\le\phi_{L,M}^{\mathrm{BM}}(y,z)+CL^{-d-1/5}
$$

where, as in our convention, the constant C is uniform in the choice of y,z. The reverse inequality is obtained using the same steps.

Consider next $\eta>L^{\beta+1/15}$. Fix strictly positive constants c_j, $j=1,\ldots,4$, depending on d,α only, and a sequence of disjoint sets $A_i\subset\partial V_L$, $i=1,\ldots,k_L$ with $\cup_{i=1}^{k_L}A_i=\partial V_L$, $c_1L^{\alpha(d-1)}\le|A_i|\le c_2L^{\alpha(d-1)}$, $\mathrm{diam}(A_i)\le c_3L^\alpha$, $d(y_1,\partial A_1\cap\partial V_L)\ge \mathrm{diam}(A_1)/4$, and $|\partial A_i|\cap\partial V_L\le c_4L^{\alpha(d-2)}$ (such a collection of "cube-like" A_i can clearly be found). We also set $A_i^\beta = \{y \in \mathbb{R}^d : d(y, A_i) \le L^\beta\}$ and for $i \ge 2$, fix an arbitrary $y_i \in A_i$. We then have

$$\begin{aligned}\phi_{L,M}(y,z) &= \sum_{i=1}^{k_L} \sum_{y' \in A_i} \pi_{V_L}(y,y')\, \hat{\pi}_M(y',z) \\ &\le \sum_{i=1}^{k_L} \hat{\pi}_M(y_i,z) \sum_{y' \in A_i} \pi_{V_L}(y,y') + \frac{C \log L}{L^{d+1-\alpha}}\,,\end{aligned}$$

where (A.3) was used in the last inequality. Consequently, using (A.15),

$$\phi_{L,M}(y,z) \le \sum_{i=1}^{k_L} \hat{\pi}_M(y_i,z) \int_{A_i^\beta} \pi_L^{\mathrm{BM}}(y,dy') \left(1 + \frac{C}{L^{1/4}}\right) + \frac{C}{L^{d+1/5}}\,. \tag{A.25}$$

Let $\{\tilde{A}_i \subset \partial C_L\}_{i=1}^{k_L}$ be a collection of measurable disjoint sets with $\cup \tilde{A}_i = \partial C_L$, $\tilde{A}_1 = A_1^\beta \cap \partial C_L$, and $\tilde{A}_i \subset A_i^\beta$. Using (3.14) and $d(y, \partial C_L) \ge L^{\beta+1/15}/2$, one gets

$$\int_{A_i^\beta} \pi_L^{\mathrm{BM}}(y,dy') \le \int_{\tilde{A}_i} \pi_L^{\mathrm{BM}}(y,dy') \left(1 + C\frac{|(A_i^\beta \cap \partial C_L) \setminus \tilde{A}_i|}{|A_i^\beta \cap \partial C_L|}\right)\,.$$

Substituting in (A.25) we get

$$\phi_{L,M}(y,z) \le \sum_{i=1}^{k_L} \hat{\pi}_M(y_i,z) \int_{\tilde{A}_i} \pi_L^{\mathrm{BM}}(y,dy') \left(1 + CL^{-1/5}\right) + \frac{C}{L^{d+1/5}}\,.$$

Hence, recalling (A.4), (A.3), and (A.17), we get

$$\begin{aligned}\phi_{L,M}(y,z) &\le \sum_{i=1}^{k_L} \int_{\tilde{A}_i} \hat{\pi}_M(y',z)\, \pi_L^{\mathrm{BM}}(y,dy') + \frac{C}{L^{d+1/5}} \\ &\le \sum_{i=1}^{k_L} \int_{\tilde{A}_i} \hat{\pi}_M^{\mathrm{BM}}(y',z)\, \pi_L^{\mathrm{BM}}(y,dy') + \frac{C}{L^{d+1/5}} = \phi_{L,M}^{\mathrm{BM}}(y,z) + \frac{C}{L^{d+1/5}}\,.\end{aligned}$$

The reverse inequality is obtained by a similar argument. ■

Proof of Lemma 3.8. We write $\pi_t^{\mathrm{BM}}(w,z)$ as the density with respect to Lebesgue's measure of the measure $\pi_{C_t(w)}^{\mathrm{BM}}(w,dz)$. Set $g(w,z) = \int \pi_t^{\mathrm{BM}}(w,z)\, \varphi_{m_w}(t)\, dt$. Then,

$$\phi_{L,M}^{\mathrm{BM}}(y,z) = \int_{\partial C_L(0)} \pi_{C_L(0)}^{\mathrm{BM}}(y,dw)\, g(w,z)\,.$$

For each fixed z, $u(y,z) = \phi_{L,M}^{\mathrm{BM}}(y/L,z)$ satisfies the equation

$$\begin{cases} \frac{1}{2}\Delta_y u(y,z) = 0, & y \in C_1(0)\,, \\ u(y,z) = g(y/L,z), & y \in \partial C_1(0)\,. \end{cases}$$

Thus, by [4, Theorem 6.3.2],

$$\left\|\partial_y^3 \phi_{L,M}^{\mathrm{BM}}(y,z)\right\| \le C\left[\left\|\partial_w^3 g(w,z)\right\| + L\left\|\partial_w^4 g(w,z)\right\|\right] \tag{A.26}$$

By the smoothness of φ and the translation invariance and scaling properties of the Brownian motion, one gets that

$$\begin{aligned}\left\|\partial_w^3 g(w,z)\right\| + L\left\|\partial_w^4 g(w,z)\right\| &\le CL^{-d-3}\left(\left\|\partial_w^3 \pi^{\mathrm{BM}}_{C_1(0)}(w,z/L)\right\| + \left\|\partial_w^4 \pi^{\mathrm{BM}}_{C_1(0)}(w,z/L)\right\|\right)\\ &\le CL^{-d-3},\end{aligned}$$

where the last inequality is due to [3, Theorem 2.10], and the constant C does not depend on z. Substituting in (A.26), the lemma follows. ■

B A local CLT and proof of Lemma 3.10

We need a number of properties for simple random walk, and spread out random walks which can readily obtained from known results. We keep L and V_L fixed through this section, and don't emphasize them in the notation. π is π_{V_L}, the exit distribution of simple random walk from V_L. Since the proofs are very similar, and for concreteness, we prove all results for the smoothing scheme $\mathcal{S}_1$ and only sketch the necessary changes for the scheme $\mathcal{S}_2$. That is, we take:

$$s_x \stackrel{\mathrm{def}}{=} \begin{cases} \delta_{k_0 r(L)} & \text{if } x \in \mathrm{Shell}_L\left(r(L)\right) \\ \varphi_{h_L(x)}\left(t\right)dt & \text{if } d_L\left(x\right) > r\left(L\right) \end{cases}.$$

$(r(L) = L/\left(\log L\right)^{10})$. Remark here that $h_L\left(x\right) = \gamma s(L) = \gamma L/\left(\log L\right)^3$ for $d_L\left(x\right) \ge 2s(L)$, and $h_L\left(x\right) \le \left(\gamma/2\right)s(L)$ for $x \in \mathrm{Shell}_L\left(r, 2s(L)\right)$. We then write $\hat{\pi}_{\mathcal{S}}$ for the corresponding transition probabilities. By a slight abuse of notation, we write $\hat{\pi}_m$ for the transition probabilities on $\mathbb{Z}^d$ with a smoothing scheme (s_x) which is constant in x, and given by $\varphi_m\left(t\right)dt$. We also write $\hat{\pi}_m\left(x\right)$ for $\hat{\pi}_m\left(0,x\right)$. For $x \in V_{L-2s(L)}$, $\hat{\pi}_{\mathcal{S}}\left(x,\cdot\right) = \hat{\pi}_{\gamma s(L)}\left(x,\cdot\right)$.

Let $m \in \mathbb{R}^+$. $\hat{\pi}_m$ is centered, and the covariances satisfy

$$\sum_x x_i x_j \hat{\pi}_m\left(x\right) = \alpha\left(m\right)\delta_{ij},$$

where for some $0 < \alpha_1 < \alpha_2$

$$\alpha_1 m^2 \le \alpha\left(m\right) \le \alpha_2 m^2.$$

(It is evident that $\alpha\left(m\right)/m^2$ converges as $m \to \infty$).

Using Lemma 3.4 a), one sees that for $1 < a < b < 2$, one has for some δ (which may depend on a, b)

$$\inf_{am \le |x| \le bm} \hat{\pi}_m\left(x\right) \ge \delta m^{-d}. \tag{B.1}$$

Furthermore, by definition, we have $\hat{\pi}_m\left(x\right) = 0$ for $|x| \ge 2m$.

We will also use the following fact, proved in Lemma A.4.

$$\left|\hat{\pi}_m\left(x\right) - \hat{\pi}_m\left(y\right)\right| \le Cm^{-d}\left|\frac{x-y}{m}\right|^{1/15}.$$

Proposition B.1

$$\hat{\pi}_m^{*n}\left(x\right) = \frac{1}{\left(2\pi m^2\sigma_m^2 n\right)^{d/2}}\exp\left[-\frac{|x|^2}{2m^2\sigma_m^2 n}\right] + O\left(m^{-d}n^{-(d+2)/2}\left(\log n\right)^4\right)$$

Proof of Proposition B.1. The proof is standard, but we need to keep track of the m-dependence, and we are not aware of a reference for that in the literature. Let

$$\chi_m(z) \stackrel{\text{def}}{=} \sum_x \mathrm{e}^{iz\cdot x/m} \hat{\pi}_m(x), \ z \in B_m \stackrel{\text{def}}{=} [-m\pi, m\pi]^d$$

By Fourier inversion, we have

$$\hat{\pi}_m^{*n}(x) = (2\pi)^{-d} m^{-d} \int_{B_m} \mathrm{e}^{-iz\cdot x/m} \chi_m(z)^n \, dz.$$

We will choose $0 < a < A$, $b > 0$, and $\alpha \in (0,1)$ (not depending on n, m) and split

$$\begin{aligned}\int_{B_m} \mathrm{e}^{-iz\cdot x/m} \chi_m(z)^n \, dz &= \int\limits_{|z| \le \frac{b \log n}{\sqrt{n}}} + \int\limits_{\frac{b \log n}{\sqrt{n}} < |z| \le a} + \int\limits_{a < |z| \le A} + \int\limits_{A < |z| \le m^\alpha} + \int\limits_{m^\alpha < |z|, \ z \in B_m} \\ &= A_1 + A_2 + A_3 + A_4 + A_5, \text{ say.}\end{aligned}$$

From Taylors formula, we get

$$\chi_m(z) = 1 - \frac{|z|^2 \sigma_m^2}{2} + O\left(|z|^4\right),$$

and therefore, for $|z| \le 1/C$

$$\log \chi_m(z) = -\frac{|z|^2 \sigma_m^2}{2} + O\left(|z|^4\right).$$

From that we get for b sufficiently large and $n \ge C(b)$

$$\begin{aligned} A_1 &= \left(1 + O\left(\frac{(\log n)^4}{n}\right)\right) \int\limits_{|z| \le \frac{b \log n}{\sqrt{n}}} \exp\left[-i\frac{z\cdot x}{m} - \frac{n|z|^2 \sigma_m^2}{2}\right] dz \\ &= \left(1 + O\left(\frac{(\log n)^4}{n}\right)\right) \int \exp\left[-i\frac{z\cdot x}{m} - \frac{n|z|^2 \sigma_m^2}{2}\right] dz + O\left(n^{-d/2-1}\right) \qquad \text{(B.2)} \\ &= \frac{(2\pi)^{d/2}}{n^{d/2}\sigma_m^d} \exp\left[-\frac{|x|^2}{2m^2\sigma_m^2 n}\right] + O\left(n^{-d/2-1}(\log n)^4\right). \end{aligned}$$

In order to prove the proposition, it therefore suffices to prove that $A_2, \ldots, A_5$ are of order $O\left(n^{-d/2-1}\right)$, uniformly in L.

To handle A_2, we choose a such that $\log \chi_m(z) \le -|z|^2 \sigma_m^2/3$ for $|z| \le a$. Then

$$|A_2| \le \int_{\frac{b \log n}{\sqrt{n}} < |z|} \exp\left[-|z|^2 n\sigma_m^2/3\right] dz = O\left(n^{-d/2-1}\right).$$

if we choose b sufficiently large.

For A_3, we use the following fact, which is an easy consequence of (B.1): for any $a < A$, one has

$$\sup_{m, a \le |z| \le A} |\chi_m(z)| < 1. \qquad \text{(B.3)}$$

Using this, we immediately get

$$|A_3| \leq CA^d (1 - 1/C)^n . \tag{B.4}$$

We come now to A_4 which is more difficult. First remark that by the assumed symmetry under lattice isomorphisms, we only have the consider z-values with all components positive. Put $|z|_\infty \stackrel{\mathrm{def}}{=} \max(z_1, \ldots, z_d)$. For simplicity, we assume that z_1 is the biggest component of z, so that $|z|_\infty = z_1$. Let $M \stackrel{\mathrm{def}}{=} [2\pi m/z_1]$, and $K \stackrel{\mathrm{def}}{=} [(2m+1)/M]$. We may assume that $M < m$ by choosing A large enough. We write

$$\begin{aligned}\chi_m(z) = &\sum_{(x_2,\ldots,x_d)} \exp\left[\frac{i}{m}\sum\nolimits_{s=2}^d x_s z_s\right] \\ &\times \left\{\sum_{j=1}^{K}\sum_{x_1=-m+(j-1)M}^{-m+jM-1} \mathrm{e}^{ix_1z_1/m}\hat{\pi}_m(x) + \sum_{x_1=-m+KM}^{m} \mathrm{e}^{ix_1z_1/m}\hat{\pi}_m(x)\right\}.\end{aligned}$$

In the first summand, inside the x_1-summation, we write for each j separately, $\hat{\pi}_m(x) = \hat{\pi}_m(x) - \hat{\pi}_m(x') + \hat{\pi}_m(x')$, where $x' = (-m + (j-1)M, x_2, \ldots, x_d)$. Then we estimate

$$|\hat{\pi}_m(x) - \hat{\pi}_m(x')| \leq Cm^{-d}\left(\frac{x_1 + m - (j-1)M}{m}\right)^{1/15}.$$

Therefore

$$\left|\sum\nolimits_{x_1=-m+(j-1)M}^{-m+jM-1} \mathrm{e}^{ix_1z_1/m}\left(\hat{\pi}_m(x) - \hat{\pi}_m(x')\right)\right| \leq Cm^{-d+1}\frac{1}{z_1^{16/15}},$$

and therefore

$$\left|\sum_{j=1}^{K}\sum_{x_1=-m+(j-1)M}^{-m+jM-1} \mathrm{e}^{ix_1z_1/m}\left(\hat{\pi}_m(x) - \hat{\pi}_m(x')\right)\right| \leq Cm^{-d+1}|z|^{-1/15}.$$

$$\left|\sum_{j=1}^{K}\sum_{x_1=-m+(j-1)M}^{-m+jM-1} \mathrm{e}^{ix_1z_1/m}\hat{\pi}_m(x')\right| \leq K\hat{\pi}_m(x')\left|\frac{1-\exp[iz_1M/m]}{1-\exp[iz_1/m]}\right| \leq C|z|m^{-d},$$

$$\left|\sum_{x_1=-m+KM}^{m} \mathrm{e}^{ix_1z_1/m}\hat{\pi}_m(x)\right| \leq m^{-d+1}|z|^{-1}.$$

Therefore, we get the estimate

$$|\chi_m(z)| \leq C_1\left(|z|^{-1/15} + \frac{|z|}{m}\right).$$

From this, we get

$$|A_4| \leq C_1^n \int_{A\leq|z|\leq m^\alpha}\left(|z|^{-1/15} + \frac{|z|}{m}\right)^n dz \leq 2^{-n} \tag{B.5}$$

for large enough A and m.

For A_5, we need a slight modification. Let again $z_1 > 0$ be the largest of the z-components. Then we write

$$\hat{\pi}_m(x) = \sum_{y=-m}^{x_1} \left(\hat{\pi}_m(y, x_2, \ldots, x_d) - \hat{\pi}_m(y-1, x_2, \ldots, x_d)\right),$$

$$\begin{aligned}\chi_m(z) &= 2i \sum_{x_2,\ldots,x_d} \exp\left[\frac{i}{m}\sum\nolimits_{s=2}^{d} x_s z_s\right] \\ &\quad\times \sum_{y=-m}^{m} \left(\hat{\pi}_m(y, x_2, \ldots, x_d) - \hat{\pi}_m(y-1, x_2, \ldots, x_d)\right) \\ &\quad\times \frac{\mathrm{e}^{i(z_1/m)(y-1/2)} - \mathrm{e}^{i(z_1/m)(m+1/2)}}{\sin(z_1/2m)}.\end{aligned}$$

Therefore

$$|\chi_m(z)| \le Cm^{d-1}\frac{m}{z_1}\sum_{y=-m}^{m} \left|\hat{\pi}_m(y, x_2, \ldots, x_d) - \hat{\pi}_m(y-1, x_2, \ldots, x_d)\right| \le C\frac{m^{14/15}}{|z|},$$

and if $\alpha > 1-\gamma$

$$\begin{aligned}|A_5| &\le m^{-d}\int_{m^\alpha \le |z|} |\chi_m(z)|^n \, dz \le C^n m^{-d} m^{14n/15}\int_{m^\alpha}^{\infty} r^{d-1} r^{-n} dz \qquad \text{(B.6)} \\ &\le C^n m^{-d} m^{14n/15} m^{\alpha(d--n)} \le 2^{-n},\end{aligned}$$

if m and n are large enough.

Combining (B.2)-(B.6), we have proved the Proposition. ■

We next need a simple large deviation estimate

Lemma B.2
There exists $\delta > 0$, such that for $|x| \ge 2m$

$$\hat{\pi}_m^{*n}(x) \le Cm^{-d}\exp\left[-\frac{|x|^2}{Cnm^2}\right].$$

Proof of Lemma B.2. If $|x| \ge r$, then one of the d components of x satisfies $|x_i| \ge r/\sqrt{d}$. By rotational symmetry, we get

$$\sum_{x:|x|\ge r} \hat{\pi}_m^{*n}(x) = dP\left(\left|\sum\nolimits_{j=1}^{n} \xi_j\right| \ge r/\sqrt{d}\right),$$

where the ξ_j are i.i.d. with the one-dimensional marginal of $\hat{\pi}$ as its distribution. Then

$$P\left(\left|\sum\nolimits_{j=1}^{n} \xi_j\right| \ge r/\sqrt{d}\right) \le 2\exp\left[-nI\left(\frac{r}{\sqrt{d}n}\right)\right]$$

where

$$I(t) = \sup\left\{\lambda t - \log E\left(\mathrm{e}^{\lambda t}\right)\right\}.$$

By symmetry $I'(0) = 0$, and from our assumptions, we have $I''(0) \geq 1/Cm^2$. Furthermore, $I(t) = \infty$ if $|t| > 2$. By convexity of I, we therefore have $I(t) \geq t^2/Cm^2$. Implementing gives

$$\sum_{x:|x|\geq r} \hat{\pi}_m^{*n}(x) \leq C \exp\left[\frac{r^2}{Cnm^2}\right].$$

From this, we get

$$\begin{aligned}
\hat{\pi}^{*n}(x) &= \sum_y \hat{\pi}_m^{*(n-1)}(y)\,\hat{\pi}_m(x-y) \\
&\leq Cm^{-d} \sum_{y:|y|\geq|x|-2m} \hat{\pi}_m^{*(n-1)}(y) \leq Cm^{-d}\exp\left[-\frac{(|x|-2m)^2}{C(n-1)m^2}\right] \\
&\leq Cm^{-d}\exp\left[-\frac{|x|^2}{Cnm^2}\right].
\end{aligned}$$

■

Let

$$G_m(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \hat{\pi}_m^{*n}(x). \tag{B.7}$$

Corollary B.3
For $|x| \geq m$, we have for some constant $c(d)$

$$G_m(x) = c(d)\frac{1}{\alpha(m)}|x|^{-d+2} + O\left(|x|^{-d}\left(\log\frac{|x|}{m}\right)^{5d}\right).$$

For $|x| \leq m$, we have

$$G_m(x) = \delta_{0,x} + O\left(m^{-d}\right)$$

Proof of Corollary B.3. Assume $|x| \geq m$ and set

$$N(x,m) \stackrel{\text{def}}{=} \frac{|x|^2}{\alpha(m)}\left(\log\frac{|x|^2}{\alpha(m)}\right)^{-10}.$$

Then

$$\begin{aligned}
\sum_{n=N}^{\infty} \hat{\pi}^{*n}(x) &= \sum_{n=N}^{\infty} \frac{1}{(2\pi d\alpha(m)\,n)^{d/2}} \exp\left[-\frac{|x|^2}{2\alpha(m)\,n}\right] \\
&\quad + \sum_{n=N}^{\infty} O\left(\alpha(m)^{-d/2}\,n^{-(d+2)/2}\right).
\end{aligned}$$

$$\sum_{n=N}^{\infty} O\left(\alpha(m)^{-d/2}\,n^{-(d+2)/2}\right) = O\left(|x|^{-d}\left(\log\frac{|x|^2}{\alpha(m)}\right)^{5d}\right)$$

Putting

$$t_n \stackrel{\text{def}}{=} \frac{2\alpha(m)\,n}{|x|^2}$$

we get

$$\begin{aligned}
&\sum_{n=N}^{\infty}\frac{1}{\left(2\pi d\alpha\left(m\right)n\right)^{d/2}}\exp\left[-\frac{\left|x\right|^{2}}{2\alpha\left(m\right)n}\right]\\
&=\frac{\left|x\right|^{-d+2}}{2\left(\pi d\right)^{d/2}\alpha\left(m\right)}\sum_{n=N}^{\infty}\frac{1}{\left(t_{n}\right)^{d/2}}\exp\left[-\frac{1}{t_{n}}\right]\left(t_{n}-t_{n-1}\right)\\
&=\frac{\left|x\right|^{-d+2}}{2\left(\pi d\right)^{d/2}\alpha\left(m\right)}\int_{0}^{\infty}t^{-d/2}\exp\left[-t^{-1}\right]dt+O\left(\left|x\right|^{-d}\right).
\end{aligned}$$

This proves a) for $|x|\geq m$ with

$$c\left(d\right)=\frac{1}{2\left(\pi d\right)^{d/2}}\int_{0}^{\infty}t^{-d/2}\exp\left[-t^{-1}\right]dt.$$

For $|x|\leq m$, the estimate is evident from Proposition B.1. ■

Proof of Lemma 3.10 a). There exists a θ, such that for any $y\in\mathrm{Shell}_L(r(L))$, there exists a unit vector $x\in\mathbb{R}^d$ such that $(y+C_\theta\left(x\right))\cap\partial V_{3r(L)}\left(y\right)\cap V_L=\emptyset$. Using this, we see from (3.12), that our coarse grained Markov chain has after every visit of $\mathrm{Shell}_L\left(r(L)\right)$ a probability of at least $\delta\left(\theta\right)$ to leave V_L in the next step. Therefore, the expected number of visits in this shell is finite, uniformly in the starting point. ■

Proof of Lemma 3.10 b). If $x\in\mathrm{Shell}_L\left(r,2s\right)$, then $\hat{\pi}\left(x,\cdot\right)$ is an averaging over exit distributions from (discrete) balls $V_u\left(x\right)$, the averaging taken over u's with $u\geq\left(\gamma/2\right)d_L\left(x\right)$. Therefore, there exists a $\delta>0$, such that $\hat{\pi}\left(x,\mathrm{Shell}_L\left(d_L\left(x\right)\left(1-\gamma/4\right)\right)\right)\geq\delta$. Therefore, if $x\in\mathrm{Shell}_L\left(a,a+\gamma/8\right)$, $r(L)\leq a\leq 2s(L)$, we have $\hat{\pi}\left(x,\mathrm{Shell}_L\left(a\left(1-\gamma/8\right)\right)\right)\geq\delta$. Therefore, a Markov chain with transition probabilities $\hat{\pi}$ which starts in $\mathrm{Shell}_L\left(a,a+\gamma s(L)/8\right)$ has probability at least δ to reach in one step $\mathrm{Shell}_L\left(a\left(1-\gamma/8\right)\right)$. By Lemma 3.4 c), an nearest neighbor chain starting in $\mathrm{Shell}_L\left(a\left(1-\gamma/8\right)\right)$ has a probability at least $\varepsilon\left(\gamma\right)>0$ of exiting V_L before reentering into $\mathrm{Shell}_L\left(a,a+\gamma/8\right)$. This evidently then applies also to our coarse grained random walk.

We conclude that for the coarse grained chain starting in $x\in\mathrm{Shell}_L\left(a,a+\gamma s/8\right)$, there is a positive probability $\varepsilon>0$, not depending on x,a, that the chain exits from V_L before reentering this shell. It therefore follows that the expected number of visits in $\mathrm{Shell}_L\left(a,a+\gamma s/8\right)$ is bounded, uniformly in the starting point of the chain, and a. From this the conclusion follows by summing over a finite number of such shells. ■

As a preparation for the proof of parts c) and d) of Lemma 3.10, we prove a preliminary result about our coarse grained random walk.

Lemma B.4

$$\sup_{x\in\mathrm{Shell}_L(2s(L))}\sum_{y\in V_{L-2s(L)}}\hat{g}_L\left(x,y\right)\leq C\left(\log L\right)^3.$$

Proof of Lemma B.4. The expression $\sum_{y\in V_{L-2s(L)}}\hat{g}_L\left(x,y\right)$ is the expected total time that the random walk spends in $V_{L-2s}\subset V_L$. When starting in $\mathrm{Shell}_L\left(2s(L)\right)$, the walk has a probability bounded from below, say by $\varepsilon_1 > 0$, of never entering $V_{L-2s(L)}$ before exiting V_L, uniformly in the starting point. If the walk enters $V_{L-2s(L)}$, it has to enter through $\mathrm{Shell}_L(2s, 4s)$. Therefore

$$\sup_{x \in \mathrm{Shell}_L(2s(L))} \sum_{y \in V_{L-2s}} \hat{g}_L(x, y) \leq \varepsilon_1^{-1} \left(1 + \sup_{x \in \mathrm{Shell}_L(2s(L), 4s(L))} E_x \left(T^{\mathrm{CG}}_{\mathrm{Shell}_L(2s(L))} \right) \right),$$

where T_A^{CG} stands for the first entrance time into A by the coarse grained random walk with transition kernel $\hat{\pi}_s$ from $V_{L-2s(L)}$. It therefore suffices to prove

$$\sup_{x \in \mathrm{Shell}_L(2s(L), 4s(L))} E_x \left(T^{\mathrm{CG}}_{\mathrm{Shell}_L(2s(L))} \right) \leq C (\log L)^3,$$

Consider the shells $R_j \stackrel{\mathrm{def}}{=} \mathrm{Shell}_L(js(L), (j+1)s(L))$, $j \geq 2$, and let T_j be the first entrance time of our (coarse grained) random walk into R_j.One then has

$$P_x \left(T^{\mathrm{CG}}_{R_j} < T^{\mathrm{CG}}_{\mathrm{Shell}_L(2s(L))} \right) \leq C P_x \left(T^{\mathrm{RW}}_{R_j} < T^{\mathrm{RW}}_{\mathrm{Shell}_L(2s(L))} \right),$$

and the right hand side we can estimate by Lemma 3.4 c), giving

$$P_x \left(T^{\mathrm{RW}}_{R_j} < T^{\mathrm{RW}}_{\mathrm{Shell}_L(2s(L))} \right) \leq \frac{C}{j},$$

and therefore we get

$$P_x \left(T^{\mathrm{CG}}_{R_j} < T^{\mathrm{CG}}_{\mathrm{Shell}_L(2s(L))} \right) \leq \frac{C}{j}.$$

If $x \in R_j$, we estimate the expected number of visits in R_j by Corollary B.3, which gives

$$\sup_{x \in R_j} \sum_{y \in R_j} G_{\gamma s(L)}(x, y) \leq C (\log L)^3.$$

Combining these estimates completes the proof of Lemma B.4 ■

Let σ be the first entrance time of $\{S_n\}$ into $\mathrm{Shell}_L(2s(L))$. Before time σ, the Markov process $\{S_n\}$ proceeds as a random walk on $\mathbb{Z}^d$ with jump distribution $\hat{\pi}_m$, where $m = \gamma s(L)$.

Proof of Lemma 3.10 c), d), e). From Corollary B.3, we get

$$\sup_{x \in V_L} \sum_{y \in V_{L-2s(L)}} G_{\gamma s(L)}(x, y) \leq C (\log L)^6.$$

Evidently, from Lemma B.4, we get

$$\sup_{x \in V_L} \sum_{y \in V_{L-2s(L)}} \left| G_{\gamma s(L)}(x, y) - \hat{g}_L(x, y) \right| \leq C (\log L)^3,$$

which implies the statement d).

e) follows by the same approximation and

$$\sup_{x, x' \in V : |x - x'| \leq s} \sum_{y \in \mathrm{Bulk}_L} \left| G_{\gamma s(L)}(x, y) - G_{\gamma s(L)}(x', y) \right| \leq C (\log L)^3,$$

which follows again from Corollary B.3.

We turn to the proof of part c). For $x = y$, the result is obvious from the transience of simple random walk. In the sequel, we thus always take $x \neq y$. Write $A_y \overset{\text{def}}{=} \{z : |z-y| \leq s(L)\}$. We first prove the result for $x \in A_y$ and $d_L(y) \geq 5s(L)$. In that case,

$$\sup_{x\in A_y : x\neq y} \hat{g}_L(x,y) \leq G_{\gamma s(L)}(x,y) + \max_{z\in \mathrm{Shell}_L(2s(L))} P_z^{\mathrm{RW}}(T_{A_y} < T_{V_L}) \sup_{x\in A_y : x\neq y} \hat{g}_L(x,y)\,.$$

Since

$$\max_{z\in \mathrm{Shell}_L(2s(L))} P_z^{\mathrm{RW}}(T_{A_y} < T_{V_L}) < 1$$

uniformly in L by Donsker's invariance principle, we conclude that

$$\sup_{x\in A_y : x\neq y} \hat{g}_L(x,y) \leq C G_{\gamma s(L)}(x,y)\,.$$

Corollary B.3 then completes the proof in this case.

Consider next $x \in A_y$ but $s(L) \leq d_L(y) \leq 5s(L)$, and set $B_y \overset{\text{def}}{=} \{z : |z-y| \leq s(L)/2\}$ and $C_y \overset{\text{def}}{=} \{z : |z-y| \leq 5s(L)\}$. We note that

$$\sup_{x\in A_y : x\neq y} \hat{g}_L(x,y) \leq \frac{C}{s(L)^d} + \sup_{x\notin A_y} \hat{g}_L(x,y) \leq \frac{C}{s(L)^d} + \sup_{z\notin A_y} P_z^{\mathrm{RW}}(T_{B_y} < T_{V_L}) \sup_{x\in A_y : x\neq y} \hat{g}_L(x,y)\,.$$

Since $\sup_{z\notin A_y} P_z^{\mathrm{RW}}(T_{B_y} < T_{V_L}) < 1$ uniformly in L, again by Donsker's invariance principle, we conclude that

$$\sup_{x\in A_y : x\neq y} \hat{g}_L(x,y) \leq \frac{C}{s(L)^d}\,,$$

which proves the claim in this case.

We next consider $x \notin A_y$. Let σ' denote the first entrance time of the simple random walk into $\mathrm{Shell}_L(2s(L))$. Clearly, $\sigma' \leq \sigma$. We then have

$$\begin{aligned}
\hat{g}_L(x,y) &\leq G_{\gamma s(L)}(x,y) + C \sum_{z\in \mathrm{Shell}_L(2s(L))} P_x^{\mathrm{RW}}(S_{\sigma'} = z) P_x^{\mathrm{RW}}(T_{A_y} < T_{V_L}) \sup_{w\in A_y : w\neq y} \hat{g}_L(w,y) \\
&\leq \frac{C}{s(L)^2 |x-y|^{d-2}} + \frac{C d_L(x) d_L(y)}{s(L)^2} \sum_{z\in \mathrm{Shell}_L(2s(L))} \frac{1}{(|x-z|\vee 1)^d (|y-z|\vee 1)^d} \\
&\leq \frac{C}{s(L)^2 |x-y|^{d-2}} + \frac{C}{s(L)^2} \sum_{z\in \mathrm{Shell}_L(2s(L))} \frac{1}{(|x-z|\vee 1)^{d-1} (|y-z|\vee 1)^{d-1}} \\
&\leq \frac{C}{s(L)^2 |x-y|^{d-2}}\,,
\end{aligned} \tag{B.8}$$

where the second inequality uses Corollary B.3, the estimate on $\hat{g}_L(x,y)$ for $x \in A_y$ that was already proved, and Lemma 3.5. This completes the proof. ■

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