SPLIT CHERN-SIMONS THEORY IN THE BV-BFV FORMALISM

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Abstract. The goal of this note is to give a brief overview of the BV-BFV formalism developed by the first two authors and Reshetikhin in [CMR14], [CMR15] in order to perform perturbative quantisation of Lagrangian field theories on manifolds with boundary, and present a special case of Chern-Simons theory as a new example.

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Contents

1. Introduction

Since the proposal of functorial quantum field theory by Atiyah and Segal ([Ati88],[Seg88]) mathematical research in this topic has progressed far and in many directions (see e.g. the books [Tur10] and [Oht02], or the review article [Sch07]). Recently, the first two authors together with Reshetikhin introduced the BV-BFV formalism, which can be seen either as an extension of functorial QFT to perturbative quantisation or, from another viewpoint, as a method to perturbatively quantise gauge theory in the presence of a boundary. The main idea is to unify the Lagrangian Batalin-Vilkovisky (BV) formalism [BV81, BV83] in the bulk and the Hamiltonian Batalin-Fradkin-Vilkovisky (BFV) formalism [BF83] on the boundary.
One possible application is to shed new light on the relation between perturbative techniques and mathematical ideas that are concepts of non-perturbative quantisation, like the Reshitikhin-Turaev invariants ([RT91], see also [KM91]), and thus ultimately about non-perturbative results to the path integral itself. In this note, a very first step on this road is taken by applying the formalism to a special form of Chern-Simons theory.
The note is structured as follows: Section 2 delivers a short overview of the relevant formal concepts via the example of abelian BF theory. Section 3 discusses a variant of Chern-Simons theory known as split Chern-Simons theory, in its BV-BFV formulation. Section 4 computes the state of this theory explicitly in lowest orders on the solid torus, which is a first step towards constructing the Chern-Simons invariant for lens spaces.

2. Overview of the BV and BV-BFV formalisms

The goal of this section is to give a very brief introduction to the BV-formalism on manifolds without boundary, see also [CN16] in the present volume, and the BV-BFV formalism on manifolds with boundary, for two special examples. For the technical details we will refer to the papers [CMR14] and [CMR15] where the Classical and Quantum BV-BFV formalisms were discussed in depth.

2.1. **Perturbative Quantisation of Lagrangian field theories.** Fix a dimension d. A Lagrangian field theory assigns to every closed d-dimensional manifold a *space of fields* F_M and an *action functional* $S_M\colon F_M \to \mathbb{R}$. This action functional is required to be local, i.e. of the form

$$S_M[\phi] = \int_M \mathcal{L}[\phi(x), \partial\phi(x), \ldots],$$

where $\mathcal{L}$, the so-called *Lagrangian density*, should depend only on the fields ϕ and finitely many of their derivatives. The critical points of the action functional are called the classical solutions of the theory, and are obtained by solving the Euler-Lagrange equations, also called equations of motion.

One way of quantising such a theory, suggested by the path integral from quantum mechanics, is to compute "integrals" of the form

$$\int_{F_M} \mathcal{O}[\phi] e^{\frac{i}{\hbar} S_M[\phi]} \mathcal{D}\phi,$$

where $\mathcal{O}$ is an "observable", over the space of fields F_M (these integrals are usually also called path integrals, even though they do not involve any paths). In this note we are only interested in the so-called *vaccuum state* or *partition function*

$$\psi = \int_{F_M} e^{\frac{i}{\hbar} S_M[\phi]} \mathcal{D}\phi. \tag{1}$$

However, in almost all relevant examples the spaces of fields have infinite dimension, and there is no sensible integration theory at hand. One way to still make sense of such expressions in the limit $\hbar \to 0$ is to use (formally) the principle of stationary phase. This produces an expansion in powers of $\hbar$ around critical points of the action. The terms in such an expansion can conveniently be labelled by diagrams, which after their inventor are called Feynman diagrams. A concise introduction can be found in [Pol05].

Remark 2.1 (Perturbative expansion). We will only consider actions of the form $S = S_0 + S_{\text{int}}$ where S_0 is the quadratic part (also called "free" or "kinetic" part). In this case one usually considers the interaction or perturbation term to be small ("weak coupling") so we can expand the action around critical points of S_0 in powers of the interaction ("coupling constant"), and the integral then can be formally computed from the theory of Gaussian moments[1], usually referred to as *Wick's theorem* in quantum field theory. Details can be found e.g. in the Book by Peskin and Schroeder [PS95] or lecture notes such as [Ton06, Bei14].

2.2. **Perturbative quantisation of gauge theories.** In many cases important for physics and mathematics, the Lagrangian is actually *degenerate*, i.e. its critical points are not isolated, and we cannot apply the stationary phase expansion, see e.g. [Res10]. This is usually due to the presence of symmetries on the space of fields that leave the action invariant.
This problem can often be solved by so-called *gauge-fixing* procedures (a thorough introduction to gauge theories from a physical viewpoint can be found in [HT94], a concise introduction to the mathematical formalisms in [Mne08]). The common idea is to add more fields, corresponding to the generators of those symmetries, to remove the degeneracies in the Lagrangian. The most powerful gauge-fixing procedure (in the sense that it deals with the most general situation) is the Batalin-Vilkovsky formalism([BV81, BV83], for a short introduction to the mathematics see [Fio03]). We will not discuss it in full generality, but rather explain the idea using the example of abelian BF theory, which will be important later in this note.

2.2.1. *Abelian BF theory.* Let M be a closed manifold, i.e. a compact manifold without boundary. Abelian BF theory has the space of fields

$$F_M = \Omega^1(M, \mathbb{R}) \oplus \Omega^{d-2}(M, \mathbb{R}) \ni (A, B).$$

Here $\Omega^p(M, \mathbb{R})$ denotes the vector space of real-valued differential p-forms on M. The action functional is

$$S_M[A, B] = \int_M B \wedge \mathrm{d}A$$

and the critical points are simply closed forms $\mathrm{d}A = 0, \mathrm{d}B = 0$. Clearly, the critical points are not isolated. In fact, adding any exact form to either A or B will leave the action invariant by Stokes' theorem. Therefore, the symmetries of the theory are generated by $\mathcal{A} := C^\infty(M) \oplus \Omega^{d-3}(M)$. An element $(c, \tau) \in \mathcal{A}$ acts on F_M by $(A, B) \mapsto (A + \mathrm{d}c, B + \mathrm{d}\tau)$. Since both the space of fields and the space of symmetries are linear here, the space of symmetries can be identified with the space of generators of the symmetries. We then declare the new space of fields to be

$$F^1_M := F_M \oplus \mathcal{A}[1].$$

Here $\mathcal{A}[1]$ means that we give the fields in $\mathcal{A}$ ghost number 1.

[1]Slight abuse of language as we are actually considering *Fresnel* integrals, i.e. with complex exponent.

Remark 2.2 (Reducible symmetries). In this note we will only be concerned with dimension $d = 3$, which we fix from now. However, in dimension $D \geq 4$, the symmetries of BF theory are *reducible*, that is, "the symmetries have some symmetries themselves": We do not change the symmetry of the action given by (c, τ) if we add to τ the differential of a $D-4$-form τ_2. In this case one has to introduce the so-called "ghosts-for-ghosts" of ghost number 2, which amounts to adding to the space of fields $\Omega^{D-4}(M)[2]$, and continue all the way until we reach $\Omega^{D-D}(M)[D-2]$.

Remark 2.3 (Total degree). Forms commute or anticommute according to their form degree, i.e. if ω is a p-form and τ is a q-form we have $\omega \wedge \tau = (-1)^{pq}\tau \wedge \omega$. If we introduce ghost fields, fields commute or anticommute according to their *total degree*, which is defined to be the form degree plus the ghost number. In BF theory in 3 dimensions, all fields have total degree 1, so all fields anticommute.

These new fields are not enough to make the action nondegenerate. One way to resolve the situation is to pass to the *BV space of fields*

$$\mathcal{F}_M := T^*[-1]F^1_M = F^1_M \oplus (F^1_M)^*[-1] = F_M \oplus \mathcal{A}[1] \oplus F^*_M[-1] \oplus \mathcal{A}^*[-2].$$

The prescription to use cotangent bundle comes from finite dimensions where the dual of a vector space is always unique. Here we will not use the real dual spaces of differential forms (i.e. currents[2]), but use the Poincaré pairing

$$(\cdot,\cdot)\colon \Omega^p(M,\mathbb{R}) \times \Omega^{D-p}(M,\mathbb{R}) \to R$$
$$(\alpha,\beta) \mapsto \int_M \alpha \wedge \beta$$

to set $F(M)^* = (\Omega^1(M) \oplus \Omega^{d-2}(M))^* = \Omega^{d-1}(M) \oplus \Omega^2(M)$ and $\mathcal{A}^* = (\Omega^0(M) \oplus \Omega^{d-3}(M))^* = \Omega^d(M) \oplus \Omega^3(M)$. Denoting the dual fields with a $^+$, we summarise the fields and their degrees:

Field	Form degree	Ghost number	total degree = ghost number + form degree
A	1	0	1
B	d-2 = 1	0	d-2 = 1
c	0	1	1
τ	d-3 = 0	1	d-2=1
A^+	d-1=2	-1	1
B^+	2	-1	1
c^+	d=3	-2	1
τ^+	3	-2	1

Table 1. The fields involved in BV veriosn of abelian BF theory in dimension 3, with their form degree, ghost number and total degree

The new (BV) action is then

$$\mathcal{S}_M = \int_M B \wedge \mathrm{d}A + A^+ \wedge \mathrm{d}c + B^+ \wedge \mathrm{d}\tau$$

which leads to Euler-Lagrange equations

$$\mathrm{d}A = \mathrm{d}B = \mathrm{d}c = \mathrm{d}A^+ = \mathrm{d}B^+ = \mathrm{d}c^+ = 0.$$

Of course, right now it seems we only introduced more degeneracy, but this is where the gauge fixing comes into play. First, however, we shall need a couple of remarks.

Remark 2.4 (Superfields). At this point it is very convenient to introduce the "superfields"

$$\mathsf{A} = c + A + B^+ + \tau^+ \in \Omega^\bullet(M),$$
$$\mathsf{B} = \tau + B + A^+ + c^+ \in \Omega^\bullet(M).$$

The action now simply reads

$$\mathcal{S}_M = \int_M \mathsf{B} \wedge \mathrm{d}\mathsf{A},$$

[2]This leads to another theory with a larger space of fiels called *canonical BF theory*, see [Mne08].

where only the integral of the top-degree part is non-zero, and the Euler-Lagrange equations can be summarised as

$$\mathsf{dB} = \mathsf{dA} = 0.$$

Remark 2.5 (Structure of the space of fields). The grading by ghost number endows $\mathcal{F}_M$ with the structure of a *graded vector space*. The pairing of fields and anti-fields endows $\mathcal{F}_M$ with a so-called *odd symplectic structure* (odd because it pairs fields whose degrees add up to -1, rather than to 0). If δ denotes the de Rham differential on $\mathcal{F}_M$, it is given by

$$\omega_M = \int_M \delta\mathsf{A} \wedge \delta\mathsf{B}. \tag{2}$$

As every odd symplectic structure it induces an odd Poisson bracket on $\mathrm{Fun}(\mathcal{F}_M)$, which in this case is called the *BV bracket*. It is well defined on the subspace of local functionals (see the discussion of BV formalism in [CF00]). Also, one has the *BV Laplacian*

$$\Delta = \sum_{k=0}^{3} (-1)^{k+1} \int_M \frac{\delta^2}{\delta\mathsf{A}^{(k)}(x)\delta\mathsf{B}^{(k)}(x)},$$

where $\mathsf{A}^{(k)}$ denotes the k-form part of A. Together with the BV bracket, it gives $\mathrm{Fun}(\mathcal{F}_M)$ the structure of a so-called *BV algebra*. However, in the infinite-dimensional setting this expression for the BV Laplacian is very singular and needs to be regularised carefully.

The BV formalism to compute integral (1) now proceeds as follows: one picks a Lagrangian subspace $\mathcal{L}$ of $\mathcal{F}_M$ such that the BV action has isolated critical points on $\mathcal{L}$. This is the gauge fixing in the BV formalism. The integral

$$\psi = \int_{\mathcal{L}} e^{\frac{i}{\hbar}\mathcal{S}[\phi]}\mathcal{D}\phi$$

can be computed by methods of Feynman diagrams. If the BV action satisfies the *Quantum Master Equation* $\Delta(e^{\frac{i}{\hbar}S}) = 0$, then under deformations of $\mathcal{L}$, the result changes by a Δ-exact term.

Remark 2.6. (Quantum and Classical Master Equations) The Quantum Master equation $\Delta(e^{\frac{i}{\hbar}\mathcal{S}}) = 0$ is equivalent to $(\mathcal{S}, \mathcal{S}) - 2i\hbar\Delta\mathcal{S} = 0$, where $(\cdot, \cdot)$ is the BV bracket. Expanding $\mathcal{S}$ as a power series in $\hbar$, the degree 0 part S_0 has to satisfy $(S_0, S_0) = 0$. This is called the *Classical Master Equation*.

Remark 2.7. The statements above can be made entirely precise and rigorously proven for finite-dimensional spaces of fields (see e.g. [CN16] in this volume, [Fio03] or [CMR15], Chapter 2). In the infinite-dimensional setting, the BV formalism produces a number of postulates that one has to prove *a posteriori*. An example (for the extension to manifolds with boundary, the BV-BFV formalism) in this note is the mQME (4) which is proven for abelian BF theory. In [CMR15] a general procedure to prove the mQME is described.

Remark 2.8 (Perturbative expansion of interacting gauge theories). Abelian BF theory is an example for a free theory (i.e. $S_{\text{int}} \equiv 0$). For theories that are perturbations of free theories, the gauge-fixing for the free part of the theory can be used to compute the expansion in powers of the coupling constant. We will call theories that are perturbations of abelian BF theory "BF-like". Examples are the Poisson Sigma model and non-abelian BF theory, and, most importantly for this note, split Chern-Simons theory.

2.2.2. *Residual fields.* It can happen that the degeneracy in the quadratic part of the action does not stem from the gauge symmetries alone. This is the case when the operator in the quadratic part of the action has non-trivial "zero modes" i.e. it has zeros that are not related under gauge symmetries. In the case of abelian BF theory, the operator in question is the de Rham differential, while the gauge symmetries are given by shifting the fields by exact forms. It follows that the space of inequivalent zero modes is precisely the de Rham cohomology of M.
In this case the procedure is as follows. One splits the space of fields $\mathcal{F}_M = \mathcal{Y}' \times \mathcal{Y}''$ into a space of *residual fields*[3] $\mathcal{Y}'$, consisting of representatives of the zero modes, and a complement $\mathcal{Y}''$ that we will call *fluctuations*[4]. Then one only integrates over a Lagrangian subspace $\mathcal{L}$ of $\mathcal{Y}''$, so that the result depends on the residual fields. This yields the definition of the *effective action*:

$$e^{\frac{i}{\hbar}\mathcal{S}^{\text{eff}}(\phi')} = \int_{\phi'' \in \mathcal{L} \subset \mathcal{Y}''} e^{\frac{i}{\hbar}S(\phi', \phi'')}\mathcal{D}\phi''.$$

[3]Also known as background fields, slow fields, infrared fields.
[4]Otherwise known as fast fields or ultraviolet fields.

To be compatible with the BV formalism, $\mathcal{Y}'$ and $\mathcal{Y}''$ should be odd symplectic themselves, such that $\mathcal{F}_M$ has the product structure. In this case, one can prove that in the finite-dimensional case, the QME for the action on $\mathcal{F}$ induces the QME for the effective action. In the case at hand of abelian BF theory, we choose a finite-dimensional space of residual fields, the de Rham cohomology, and one can prove explicitly that the effective action satisfies the QME. Therefore $\mathcal{Y}'$ should be given by representatives of the de Rham cohomology of M. Such a splitting (and a suitable choice of Lagrangian) can then be found e.g. by Hodge decomposition. Choosing a Riemannian metric g, the space $\mathcal{Y}'$ is given by g-harmonic forms and the Lagrangian $\mathcal{L}$ by the Lorentz gauge condition $d^*\phi = 0$. On this space d has no kernel and therefore the restriction of the BV extension of the abelian BF action to this Lagrangian subspace is non-degenerate.

2.3. **On manifolds with boundary.** We will now consider the case of manifolds with boundary. The strategy that is compatible with the mathematical idea of gluing of manifolds along boundary components is not to fix boundary conditions, but instead to think of the state as a functional on the possible boundary fields.

Consider first the case of a theory without gauge symmetries. Under some assumptions, one can show that a d-dimensional field theory induces a space of fields F^∂_Σ on $(d-1)$-dimensional manifolds Σ that has a natural even symplectic structure. The space of states should be a quantisation of this symplectic manifold. In many examples, F^∂_Σ is actually an affine space, and one can define a quantisation from a Lagrangian polarisation[5] with a smooth leaf space (examples of this are the position or momentum space) B_Σ. In this case, the space of states is the space of functionals on B_Σ. If $\Sigma = \partial M$, there is a surjective submersion $F_M \to F^\partial_{\partial M}$ given by restriction of fields to the boundary. If we denote by p the composition of this map with the projection $F^\partial_{\partial M} \to B_{\partial M}$, we can define the state by the "integral"

$$\widehat{\psi}_M(\beta) = \int_{p^{-1}(\beta)} e^{\frac{i}{\hbar}S[\phi]}\mathcal{D}\phi$$

for $\beta \in B_{\partial M}$.

2.4. **The BV-BFV formalism.** Now we want to combine this with the method used to deal with gauge theories discussed above. Given a space of BV fields $\mathcal{F}_M$ for every d-dimensional manifold M, there is again an induced space of fields $\mathcal{F}^\partial_\Sigma$ on $d-1$-dimensional manifolds endowed with what is called a BFV structure (see [Sch10] for a mathematical discussion of BFV structure). The result is what is called a *BV-BFV manifold*, whose definition we will now recall.

Definition 2.1 (BFV manifold)**.** A *BFV manifold* is a triple $(\mathcal{F}, \omega, Q)$, where

- $\mathcal{F}$ is a $\mathbb{Z}$-graded manifold,
- $\omega = \delta\alpha$ is an exact degree 0 symplectic form on $\mathcal{F}$,
- Q is a degree $+1$ vector field on $\mathcal{F}$,

such that

- Q is symplectic for ω, i.e. $L_Q\omega = 0$,
- Q is cohomological, i.e. $Q^2 = 0$ or equivalently $[Q, Q] = 0$.

For degree reasons this implies the existence of a degree 1 Hamiltonian function S for Q, i.e. $\iota_Q\omega = \delta S$ (and the datum of such function specifies a cohomological symplectic vector field) and this function S automatically satisfies the Classical Master Equation $(S, S) = 2\iota_Q\iota_Q\omega = 0$. The $\mathbb{Z}$-grading of the manifold is the ghost number we briefly explained above.

Definition 2.2 (BV-BFV manifold)**.** A *BV-BFV manifold* over a given BFV manifold $(\mathcal{F}^\partial, \omega^\partial = \delta\alpha^\partial, Q^\partial$ is a quintuple $(\mathcal{F}, \omega, Q, \mathcal{S}, \pi)$ where

- $\mathcal{F}$ is a $\mathbb{Z}$-graded manifold,
- ω is a degree -1 symplectic form,
- Q is a degree $+1$ cohomological vector field,
- $\mathcal{S}$ is a degree 0 function on $\mathcal{F}$,
- π is a surjective submersion $\mathcal{F} \to \mathcal{F}^\partial$,

such that[6]

- $\delta\pi(Q) = Q^\partial$,

[5]This is basically a choice of coordinates and canonically conjugate momenta, similar to the p and q variables in quantum mechanics.

[6]This definition differs from the one in [CMR14] by a purely conventional sign $(-1)^n$ in front of δS.

- $\iota_Q\omega = \delta\mathcal{S} + \pi^*\alpha^\partial$.

The axioms imply the *modified Classical Master Equation* (mCME)

$$\frac{1}{2}\iota_Q\iota_Q\omega - \pi^*\mathcal{S}^\partial = 0. \tag{3}$$

Remark 2.9 (Shifting α). Given a BV-BFV theory and a functional f on the space of boundary fields, we can define a new BV-BFV theory by $\alpha^\delta \mapsto \alpha^\partial + \delta f, \mathcal{S} \mapsto \mathcal{S} - \pi^* f$. It will coincide with the previous theory on closed manifolds.

In many cases, the BV structure on the bulk and the BFV structure on the boundary look very similar in the superfield formalism.
Let us look at the example of abelian BF theory on a 3-manifold M with boundary ∂M that is included via $\iota\colon \partial M \to M$. Let $\mathcal{F}_M$ be the space of BV fields $\Omega^\bullet(M)[1] \oplus \Omega^\bullet(M)[1] \ni (\mathsf{A}, \mathsf{B})$. Denote by $\mathsf{A}^\partial := \iota^*\mathsf{A}, \mathsf{B}^\partial := \iota^*\mathsf{B}$ the restrictions of these fields to the boundary. Then the space of boundary BFV fields is $\mathcal{F}^\partial_{\partial M} = \Omega^\bullet(\partial M)[1] \oplus \Omega^\bullet(\partial M)[1] \ni (\mathsf{A}^\partial, \mathsf{B}^\partial)$. The symplectic form and action have the same form as before

$$\begin{aligned}\omega^\partial_{\partial M} &= \int_{\partial M} \delta\mathsf{A}^\partial \wedge \delta\mathsf{B}^\partial,\\ S^\partial_{\partial M} &= \int_{\partial M} \mathsf{B}^\partial \wedge \mathrm{d}\mathsf{A}^\partial,\end{aligned}$$

and the corresponding Hamiltonian vector field on $\mathcal{F}^\partial_{\partial M}$ is

$$Q^\partial_{\partial M} = \int_{\partial M} \mathrm{d}\mathsf{A}^\partial \frac{\delta}{\delta\mathsf{A}^\partial} + \mathrm{d}\mathsf{B}^\partial \frac{\delta}{\delta\mathsf{B}^\partial}.$$

However, considering table 1 and that the dimension of ∂M is 2, notice that $\omega^\partial_{\partial M}$ pairs fields of *opposite* ghost number, and thus has degree 0. I.e., $(\mathcal{F}^\partial_{\partial M}, \omega^\partial_{\partial M}, Q^\partial_{\partial M})$ is a BFV manifold.

Claim 2.10. *If we denote*

$$Q_M = \int_M \mathrm{d}\mathsf{A}\frac{\delta}{\delta\mathsf{A}} + \mathrm{d}\mathsf{B}\frac{\delta}{\delta\mathsf{B}}$$

and $\pi_M = \iota^*\colon \mathcal{F}_M \to \mathcal{F}^\partial_{\partial M}$ the restriction of fields to the boundary, then in abelian BF theory the quintuple $(\mathcal{F}_M, \omega_M, Q_M, S_M, \pi_M)$ is a BV-BFV manifold over the BFV manifold $(\mathcal{F}^\partial_{\partial M}, \omega^\partial_{\partial M}, Q^\partial_{\partial M})$.

Proof. We will just prove the central BV-BFV identity $\iota_{Q_M}\omega_M = \delta S_M + \pi^*\alpha^\partial_{\partial M}$. Notice that the de Rham differential on $\mathcal{F}_M$ is given by

$$\delta = \int_M \delta\mathsf{A}\frac{\delta}{\delta\mathsf{A}} + \delta\mathsf{B}\frac{\delta}{\delta\mathsf{B}}$$

and one choice of $\alpha^\partial_{\partial M}$ is

$$\alpha^\partial_{\partial M} = \int_{\partial M} \mathsf{B} \wedge \delta\mathsf{A}.$$

On the one hand,

$$\iota_{Q_M}\omega_M = \int_M \mathrm{d}\mathsf{A} \wedge \delta\mathsf{B} + \delta\mathsf{A} \wedge \mathrm{d}\mathsf{B}.$$

On the other hand, integrating by parts yields

$$\begin{aligned}\delta\mathcal{S}_M &= \delta\int_M \mathsf{B} \wedge \mathrm{d}\mathsf{A} = \int_M \mathsf{B} \wedge \mathrm{d}\delta A + \int_M \delta B \wedge \mathrm{d}\mathsf{A}\\ &= \int_M \mathrm{d}\mathsf{B} \wedge \delta\mathsf{A} + \int_M \delta B \wedge \mathrm{d}\mathsf{A} - \int_{\partial M} \mathsf{B} \wedge \delta\mathsf{A} = \iota_{Q_M}\omega_M - \pi^*_M\alpha^\partial_{\partial M}.\end{aligned}$$

□

2.5. **The quantum BV-BFV formalism.** We now explain the data of a quantum BV-BFV theory and show how to quantise in the example of abelian BF theory, before turning to the example of Chern-Simons theory. The perturbative quantisation of a BV-BFV theory consists of the following data:

(1) A cochain complex $(\mathcal{H}^{\mathcal{P}}_{\Sigma}, \Omega^{\mathcal{P}}_{\Sigma})$ for every $(d-1)$-manifold Σ with a choice of polarisation in $\mathcal{F}^{\partial}_{\Sigma}$.
(2) A finite-dimensional BV manifold $(\mathcal{V}_M, \Delta_{\mathcal{V}_M})$ - called the space of *residual fields* - associated to every d-manifold M and polarisation $\mathcal{P}$ on $\mathcal{F}^{\partial}_{\partial M}$.
(3) Let $\widehat{\mathcal{H}}^{\mathcal{P}}_M := \mathcal{H}^{\mathcal{P}}_{\partial M}\hat{\otimes} C^\infty(\mathcal{V}_M)$ and endow it with the two commuting coboundary operators $\widehat{\Omega}^{\mathcal{P}}_M := \Omega^{\mathcal{P}}_{\partial M}\otimes \mathrm{id}$ and $\widehat{\Delta}^{\mathcal{P}}_M = \mathrm{id}\otimes\Delta_{\mathcal{V}_M}$. Then we require the existence of a *state* $\widehat{\psi}_M$ satisfying the *modified Quantum Master Equation* (mQME)

$$(\hbar^2\widehat{\Delta}^{\mathcal{P}}_M + \widehat{\Omega}^{\mathcal{P}}_M)\widehat{\psi}_M = 0, \tag{4}$$

the quantum counterpart of the mCME (3).

Some comments are in order. The cochain complex $(\mathcal{H}^{\mathcal{P}}_{\Sigma}, \Omega^{\mathcal{P}}_{\Sigma})$ is to be constructed as a sort of geometric quantisation of the symplectic manifold $\mathcal{F}^{\partial}_{\partial M}$ with the polarisation $\mathcal{P}$ and the action $\mathcal{S}^{\partial}_{\partial M}$. The general construction of the boundary quantisation is not important in this note. More important is the idea of residual fields that was explained in section 2.2.2. The state is then computed by combining the methods of sections 2.2 and 2.3. Again, assume we have a polarisation $\mathcal{P}$ of $\mathcal{F}^{\partial}_{\partial M}$ with smooth leaf space $\mathcal{B}^{\mathcal{P}}_{\partial M}$. In this case $\mathcal{H}^{\mathcal{P}}_{\Sigma}\subset \mathrm{Fun}(\mathcal{B}^{\mathcal{P}}_{\partial M})$ is a certain subspace of functionals on boundary conditions defined in detail in sections 3.5.1 and 4.1.1 to 4.1.3 in [CMR15][7]. We will further assume that actually $\mathcal{F}_M = \mathcal{B}^{\mathcal{P}}_{\partial M}\times\mathcal{Y}$ so that the fibers of the projection $p\colon \mathcal{F}_M \to \mathcal{B}^{\mathcal{P}}_{\partial M}$ are just $\{b\}\times\mathcal{Y}$. Moreover, we assume there is a functional $f^{\mathcal{P}}_{\partial M}$ such that $\alpha_{\partial M} - \delta f^{P}_{\partial M}$ vanishes when restricted to the fibers, i.e. on $\mathcal{Y}$, and then adapt the bulk action as in remark 2.9. We then split $\mathcal{Y} = \mathcal{V}_M\times\mathcal{Y}''$ into a space of residual fields and fluctuations $\mathcal{Y}''$. Then we can finally define the state $\widehat{\psi}_M$ by

$$\widehat{\psi}_M(b,\phi) = \int_{\mathcal{L}\subset\mathcal{Y}''} e^{\frac{i}{\hbar}S_M(b,\phi,\phi'')}\mathcal{D}\phi'' \in \widehat{\mathcal{H}}^{\mathcal{P}}_M = \mathcal{H}^{\mathcal{P}}_{\partial M}\hat{\otimes}C^\infty(\mathcal{V}_M).$$

Again, we define the BV effective action by

$$\widehat{\psi}_M(b,\phi) = e^{\frac{i}{\hbar}S^{\mathrm{eff}}(b,\phi)}.$$

Instead of entering a general discussion of the above, let us continue the example of abelian BF theory.

2.6. **Abelian BF theory in the quantum BV-BFV formalism.**

2.6.1. *Polarisations.* Here there are two easy polarisations on $\mathcal{F}^{\partial}_{\partial M} = \Omega^\bullet(\partial M)[1]\oplus\Omega^\bullet(\partial M)[1]$, namely the ones given by $\frac{\delta}{\delta \mathsf{A}^\partial}$ (whose leaf space can be identified with the B^∂ fields) and $\frac{\delta}{\delta\mathsf{B}^\partial}$ (whose leaf space can be identified with the A^∂ fields). Let now M be a manifold with boundary $\partial M = \partial_1 M\sqcup\partial_2 M$. We then define the polarisation $\mathcal{P}$ to be the $\frac{\delta}{\delta\mathsf{B}^\partial}$-polarisation on $\partial_1 M$ and the $\frac{\delta}{\delta\mathsf{A}^\partial}$-polarisation on $\partial_2 M$, so that we have the leaf space $\mathcal{B}^{\mathcal{P}}_{\partial M} = \Omega^\bullet(\partial_1 M)[1]\oplus\Omega^\bullet(\partial_2 M)[1]$, we denote the coordinates on it by $(\mathbb{A},\mathbb{B})$. The correct way to adapt the boundary 1-form is to subtract the differential $f^{\mathcal{P}}_{\partial M} = \int_{\partial_2 M}\mathsf{B}^\partial\wedge\mathsf{A}^\partial$ from it.

2.6.2. *Choosing a splitting.* We now split the space of fields $\mathcal{F}_M$ by choosing extensions $\widetilde{\mathbb{A}},\widetilde{\mathbb{B}}$ of $\mathbb{A}$ and $\mathbb{B}$ from the boundary to the bulk of the manifold and splitting $\mathsf{A} = \widetilde{\mathbb{A}} + \widehat{\mathsf{A}}$, $\mathsf{B} = \widetilde{\mathbb{B}} + \widehat{\mathsf{B}}$ where $\widehat{\mathsf{A}}$ and $\widehat{\mathsf{B}}$ restrict to 0 on $\partial_1 M$ resp. $\partial_2 M$. As discussed in [CMR15], one needs to require the extensions to be discontinuous extensions by 0 outside of the boundaries. One way to make this more precise is to work with a family of regular decompositions approximating this singular one, resulting a family of states that only in the limit will satisfy the mQME. We will therefore choose these extensions and identify $\widetilde{\mathbb{A}} = \mathbb{A}, \widetilde{\mathbb{B}} = \mathbb{B}$. This is our splitting $\mathcal{F}_M = \mathcal{B}^{\mathcal{P}}_{\partial M}\times\mathcal{Y}$.

[7]There are some subtleties arising from the regularisation of higher functional derivatives that would be too much for the purpose of this note.

2.6.3. *Residual fields and fluctuations, gauge fixing.* We now want to split $\mathcal{Y}$ into residual fields and fluctuations. As discussed above, in abelian BF theory the residual fields should contain the de Rham cohomology of M. In the case with boundary, for our polarisation, the space of residual fields is

$$\mathcal{V}_M = H^\bullet(M, \partial_1 M)[1] \oplus H^\bullet(M, \partial_2 M)[1].$$

We choose representatives $\chi_i \in \Omega^\bullet_{closed}(M, \partial_1 M)$ and $\chi^j \in \Omega^\bullet_{closed}(M, \partial_2 M)$ such that their cohomology classes form a basis of $H^\bullet(M, \partial_1 M)$ resp. $H^\bullet(M, \partial_2 M)$ and $\int_M \chi_i \wedge \chi^j = \delta_i^j$. Then, we write $\mathsf{a} = \sum_i z^i \chi_i, \mathsf{b} = \sum_i z_i^+ \chi^i$ for elements of $\mathcal{V}_M \subset \mathcal{F}_M$. The BV Laplacian $\Delta_{\mathcal{V}_M}$ is then

$$\Delta_{\mathcal{V}_M} = \sum_i -\frac{\partial}{\partial z^i}\frac{\partial}{\partial z_i^+}.$$

A possible way to choose such a basis, a complement $\mathcal{Y}''$ and a Lagrangian $\mathcal{L} \subset \mathcal{Y}''$ is to pick a Riemannian metric and use Hodge decomposition on manifolds with boundary (see [CDGM07]). This is the choice of gauge fixing (it is a variant of the Lorentz Gauge Fixing mentioned earlier). Its most important feature is that the gauge-fixing Lagrangian does not depend the boundary and background fields. We will avoid the details of this lengthy discussion, referring the interested reader again to [CMR15](section 3.3 and Appendix A), and simply assume we can decompose the fields $\widehat{\mathsf{A}} = \mathsf{a} + \alpha, \widehat{\mathsf{B}} = \mathsf{b} + \beta$ into residual fields and fluctuations.

Remark 2.11 (Decomposition of the action). The decomposition of the fields also induces a decomposition of the adapted action

$$\mathcal{S}^{\mathcal{P}}_{\mathcal{M}} = \widehat{\mathcal{S}}_{M,0} + \mathcal{S}^{\text{back}}_M + \mathcal{S}^{\text{source}}_M, \tag{5}$$

where

$$\begin{aligned}
\widehat{\mathcal{S}}_{M,0} &= \int_M \beta \wedge \mathrm{d}\alpha, \\
\mathcal{S}^{\text{back}}_M &= -\left(\int_{\partial_2 M} \mathbb{B} \wedge \mathsf{a} + \int_{\partial_1 M} \mathsf{b} \wedge \mathbb{A}\right), \\
\mathcal{S}^{\text{source}}_M &= -\left(\int_{\partial_2 M} \mathbb{B} \wedge \alpha + \int_{\partial_1 M} \beta \wedge \mathbb{A}\right).
\end{aligned}$$

Proof. Assume we have chosen non-singular extensions $\widetilde{\mathbb{A}}, \widetilde{\mathbb{B}}$ and split $\mathsf{A} = \widetilde{\mathbb{A}} + \mathsf{a} + \alpha, \mathsf{B} = \widetilde{\mathbb{B}} + \mathsf{b} + \beta$. The action then reads

$$\mathcal{S}^{\mathcal{P}}_{\mathcal{M}} = \int_M (\widetilde{\mathbb{B}} + \mathsf{b} + \beta) \wedge \mathrm{d}(\widetilde{\mathbb{A}} + \mathsf{a} + \alpha) - \int_{\partial_2 M} \iota_2^*((\widetilde{\mathbb{B}} + \mathsf{b} + \beta) \wedge (\widetilde{\mathbb{A}} + \mathsf{a} + \alpha))$$

where ι_2 denotes the inclusion $\partial_2 M \hookrightarrow M$. We can assume the supports of $\widetilde{\mathbb{B}}$ and $\widetilde{\mathbb{A}}$ are disjoint. Furthermore, we have that $\iota_2^*\mathsf{b} = \iota_2^*\beta = \iota_2^*\widetilde{\mathbb{A}} = 0$ and $\mathrm{d}\mathsf{a} = \mathrm{d}\mathsf{b} = 0$. We then get

$$\mathcal{S}^{\mathcal{P}}_{\mathcal{M}} = \int \mathsf{b} \wedge \mathrm{d}\widetilde{\mathbb{A}} + \beta \wedge \mathrm{d}\widetilde{\mathbb{A}} + \widetilde{\mathbb{B}} \wedge \mathrm{d}\alpha + \mathsf{b} \wedge \mathrm{d}\alpha + \beta \wedge \mathrm{d}\alpha - \left(\int_{\partial_2 M} \widetilde{\mathbb{B}} \wedge \mathsf{a} + \widetilde{\mathbb{B}} \wedge \alpha\right)$$

The integral of $\mathsf{b} \wedge \mathrm{d}\alpha$ vanishes by integration by parts since b is closed and $\mathsf{b} \wedge \mathrm{d}\alpha$ is zero restricted to ∂M. Now integrate the $\widetilde{\mathbb{A}}$ terms by parts, resulting in

$$\mathcal{S}^{\mathcal{P}}_{\mathcal{M}} = \int_M \mathrm{d}\beta \wedge \widetilde{\mathbb{A}} + \widetilde{\mathbb{B}} \wedge \mathrm{d}\alpha + \beta \wedge \mathrm{d}\alpha - \left(\int_{\partial_1 M} \mathsf{b} \wedge \widetilde{\mathbb{A}} + \beta\widetilde{\mathbb{A}}\right) - \left(\int_{\partial_2 M} \widetilde{\mathbb{B}} \wedge \mathsf{a} + \widetilde{\mathbb{B}} \wedge \alpha\right)$$

Sending $\widetilde{\mathbb{A}}, \widetilde{\mathbb{B}}$ to singular extensions proves the claim as the first two terms will vanish. □

2.6.4. *The state.* We now would like to compute the state

$$\widehat{\psi}_M(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = \int_{(\alpha,\beta)\in\mathcal{L}} e^{\frac{i}{\hbar}\mathcal{S}^{\mathcal{P}}_M(\mathbb{A}+\mathsf{a}+\alpha, \mathbb{B}+\mathsf{b}+\beta)} \mathcal{D}\alpha\mathcal{D}\beta \quad \in \widehat{\mathcal{H}}^{\mathcal{P}}_M \subset \mathrm{Fun}(\mathcal{B}^{\mathcal{P}}_{\partial M})\hat{\otimes} C^\infty(\mathcal{V}_M). \tag{6}$$

as a formal Gaussian integral. Applying decomposition (5) of the action, and the general theory of performing such Gaussian integrals in quantum field theory (see [Pol05, Res10]), we need to understand the integral

$$T_M := \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widehat{\mathcal{S}}_M} \mathcal{D}\alpha\mathcal{D}\beta. \tag{7}$$

as a regularised determinant of the inverse of the operator d in the quadratic part of the action. This is not an easy task (see [Sch78, Mne14]), but for our purposes it is enough to say that T_M is a number independent of the choice of $\mathcal{L}$ (but that can depend on our choice of representatives of cohomology). The integral (6) can then be expressed in terms of the so-called *propagator*[8]

$$\eta(x_1, x_2) = \frac{-1}{T_M}\frac{1}{i\hbar}\int_{\mathcal{L}} e^{\frac{i}{\hbar}\widehat{\mathcal{S}}_M}\alpha(x_1)\beta(x_2)\mathcal{D}\alpha\mathcal{D}\beta. \tag{8}$$

Namely,

$$\widehat{\psi}_M(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = T_M e^{\frac{i}{\hbar}S^{\mathrm{eff}}(\mathbb{A},\mathbb{B},\mathsf{a},\mathsf{b})}, \tag{9}$$

with

$$S^{\mathrm{eff}}(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = -\left(\int_{\partial_2 M}\mathbb{B}\wedge\mathsf{a} - \int_{\partial_1 M}\mathsf{b}\wedge\mathbb{A}\right) - \int_{\partial_2 M\times\partial_1 M}\pi_1^*\mathbb{A}\wedge\eta\wedge\pi_2^*\mathbb{B}. \tag{10}$$

2.6.5. *The propagator.* The propagator η is a $(d-1)$-form on the configuration space $C_2^0(M) = \{(x_1, x_2) \in M\times M : x_1 \neq x_2\}$ that vanishes for $x_2 \in \partial_1 M$ or $x_1 \in \partial_2 M$. It is determined by our choice of gauge fixing Lagrangian. It has two important properties:

- Its differential satisfies
$$\mathrm{d}\eta = \sum_i (-1)^{\deg \chi_i}\pi_1^*\chi_i\pi_2^*\chi^i. \tag{11}$$
- For any $x \in M$, if we fix a chart $\phi\colon U \to \mathbb{R}^3$ satisfying $\phi(x) = 0$, then
$$\lim_{\varepsilon\to 0}\int_{y\in\partial B_\varepsilon(0)}\eta(\phi^{-1}(y), x) = 1 = -\lim_{\varepsilon\to 0}\int_{y\in\partial B_\varepsilon(0)}\eta(x, \phi^{-1}(y)). \tag{12}$$

A choice of such a propagator (and representatives of cohomology) also leads to the definition of a gauge-fixing Lagrangian. For computations with Feynman diagrams it is often desirable to have a propagator satisfying also

- $$\int_{y\in M}\eta(x, y)\chi_i(y) = \int_{x\in M}\chi^i(x)\eta(x, y) = 0, \tag{13}$$
- $$\int_{y\in M}\eta(x, y)\eta(y, z) = \int_{x\in M}\eta(z, x)\eta(x, y) = 0. \tag{14}$$

These properties do not automatically follow from the definition but they can always be satisfied by picking a suitable $\mathcal{L}$ (see section 4 in [CM08] for a discussion on manifolds without boundary, arguments there can be adapted to the case with boundary using machinery in [CMR15]).

2.6.6. *mQME.* In the case of abelian BF theory, the quantisation of the boundary is simply the "standard" or "canonical" quantisation. It is obtained by the following recipe: In the boundary action, on $\partial_1 M$ we have to replace every occurence of $\widehat{\mathsf{B}}$ by $(-i\hbar\frac{\delta}{\delta\mathbb{A}})$, on $\partial_2 M$, $\widehat{\mathsf{A}}$ has to replaced by $(-i\hbar\frac{\delta}{\delta\mathbb{B}})$. Here we have to integrate by parts to do so. The result is

$$\Omega^{\mathcal{P}}_{\partial M} = (-i\hbar)\left(\int_{\partial_1 M}\mathrm{d}\mathbb{A}\frac{\delta}{\delta\mathbb{A}} + \int_{\partial_2 M}\mathrm{d}\mathbb{B}\frac{\delta}{\delta\mathbb{B}}\right). \tag{15}$$

Claim 2.12. *The state $\widehat{\psi}_M$ defined by* (9) *satisfies the mQME* (4)

$$(\hbar^2\widehat{\Delta}^{\mathcal{P}}_M + \widehat{\Omega}^{\mathcal{P}}_M)\widehat{\psi}_M = 0. \tag{16}$$

Proof. Since the effective action S^{eff} given in (10) is only linear in coordinates on $\mathcal{V}_M$, it is immediate that $\Delta S^{\mathrm{eff}} = 0$. In this case $(\hbar^2\Delta + \Omega)e^{\frac{i}{\hbar}S^{\mathrm{eff}}} = -\frac{1}{2}(S^{\mathrm{eff}}, S^{\mathrm{eff}})e^{\frac{i}{\hbar}S^{\mathrm{eff}}} + \Omega e^{\frac{i}{\hbar}S^{\mathrm{eff}}}$. Only the first two terms in the action depend on the residual fields and hence contribute to the BV bracket. Also, only the bracket of b with a is nontrivial, so we have

$$\begin{aligned}\frac{1}{2}(\mathcal{S}^{\mathrm{eff}}, \mathcal{S}^{\mathrm{eff}}) = \left(\int_{\partial_2 M}\mathbb{B}\wedge\mathsf{a}, \int_{\partial_1 M}\mathsf{b}\wedge\mathbb{A}\right) &= \sum_{i,j}\left(\int_{\partial_2 M}\mathbb{B}\wedge z^i\chi_i, \int_{\partial_1 M}z_j^+\chi^j\wedge\mathbb{A}\right)\\ &= \sum_i(-1)^{\deg z^i}\int_{\partial_2 M}\mathbb{B}\wedge\chi_i\int_{\partial_1 M}\chi^j\wedge\mathbb{A},\end{aligned}$$

[8]Also known to physicists as 2-point function or - slightly abusing language - Green's function.

since $(z^i, z_j^+) = (-1)^{\deg z^i}\Delta(z^i z_j^+) = (-1)^{\deg z^i}$. On the other hand,

$$\begin{aligned}\Omega e^{\frac{\mathrm{i}}{\hbar}S^{\text{eff}}} &= \left(\left(\int_{\partial_1 M} \mathrm{d}\mathbb{A}\frac{\delta}{\delta\mathbb{A}} + \int_{\partial_2 M} \mathrm{d}\mathbb{B}\frac{\delta}{\delta\mathbb{B}}\right)\mathcal{S}^{\text{eff}}\right) e^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}} = \left(\int_{\partial_2 M\times\partial_1 M} \pi_1^*\mathbb{A}\wedge \mathrm{d}\eta\wedge\pi_2^*\mathbb{B}\right) e^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}} \\ &= \sum_i (-1)^{\deg\chi^i+1}\int_{\partial_2 M}\mathbb{B}\wedge\chi_i\int_{\partial_1 M}\chi^j\wedge\mathbb{A},\end{aligned}$$

where we integrated by parts and used property (11). Now the claim follows from the fact that $\deg z^i = 1 - \deg\chi^i$. □

2.6.7. *Dependence of the state on the gauge-fixing.* Clearly, the state defined in (9) depends on the choice of the gauge-fixing. However, one can show (and, by finite-dimensional arguments, this is supposed to hold in any quantum BV-BFV theory) that, upon deformations of the gauge fixing, the state changes as

$$\frac{d}{dt}\widehat{\psi} = (\hbar^2\widehat{\Delta}_M + \widehat{\Omega}_M^{\mathcal{P}})\widehat{\zeta} \tag{17}$$

for some $\widehat{\zeta}\in\widehat{\mathcal{H}}_M^{\mathcal{P}}$.

2.6.8. *Gluing.* Suppose we have two manifolds M_1 and M_2 that share a boundary component Σ. Then we can glue them together along Σ to obtain a new manifold $M = M_1\cup_\Sigma M_2$. The state $\widehat{\psi}_M$ can now be computed from the states $\widehat{\psi}_{M_1}$ and $\widehat{\psi}_{M_2}$ in the following way: Fix polarisations such that $\Sigma\subseteq\partial_1 M_1$ on M_1 and $\Sigma\subseteq\partial_2 M_2$ on M_2. Denote by $\mathbb{A}^\Sigma$ coordinates on $\Omega^\bullet(\Sigma)[1]\subseteq\mathcal{B}^{\mathcal{P}}_{\partial M_1}$ and by $\mathbb{B}^\Sigma$ coordinates on $\Omega^\bullet(\Sigma)[1]\subseteq\mathcal{B}^{\mathcal{P}}_{\partial M_2}$. Then we define $\widetilde{\psi}_M$ by

$$\widetilde{\psi}_M = \int_{\mathbb{A}^\Sigma,\mathbb{B}^\Sigma} e^{\frac{i}{\hbar}\int_\Sigma \mathbb{B}^\Sigma\mathbb{A}^\Sigma}\widehat{\psi}_{M_1}\widehat{\psi}_{M_2}.$$

Again, this integration is defined by a variant of Wick's theorem[9] : The integral of a term in the product of the states is nonzero if we can contract every $\mathbb{A}^\Sigma$ with to a $\mathbb{B}^\Sigma$. In this case, we sum over all possibilities to do so, and every contraction of a $\mathbb{A}^\Sigma(x)$ with a $\mathbb{B}^\Sigma(y)$ yields a $\delta^{(2)}_{\partial M}(x,y)$.
One also has to take care of the residual fields: This glued state will usually depend on a non-minimal amount of residual fields, and one can pass to the minimal amount of residual fields by a BV pushforward, yielding the "correct" state $\widehat{\psi}_M$.

2.6.9. *BF-like theories.* As above, we call "BF-like" those theories whose action can be decomposed as $\mathcal{S}_{BF}+\mathcal{S}_{\text{int}}$. It is useful to also allow for the free part to consist of several copies of abelian BF theories. One way to do this is to change the space of fields to $\mathcal{F}_M = (\Omega^\bullet(M)\otimes V[1])\oplus(\Omega^\bullet(M)\otimes V^*[1])$ with action

$$\mathcal{S}_{M,0} = \int_M\langle\mathsf{B},\mathsf{dA}\rangle$$

where V is a finite-dimensional vector space and $\langle\cdot,\cdot\rangle$ denotes the pairing between V and V^*. The above discussion goes through. The only thing that changes in the gauge fixing is that we should replace η by $\tilde{\eta} = \eta\otimes\mathrm{id}_V\in\Omega(C_2^0(M))\otimes(V\otimes V^*)$, so that in any basis ξ_i of V with dual basis ξ^i it is given by

$$\tilde{\eta}(x_1,x_2) = \sum_{i,j}\eta(x_1,x_2)\delta^i_j\xi_i\otimes\xi^j.$$

3. Chern-Simons theory as a BF-like theory

3.1. **Split BV Chern-Simons theory.** Let $\mathfrak{g}$ be a Lie algebra with an non-degenerate ad-invariant pairing $\langle\cdot,\cdot\rangle\colon\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$, i.e we have for all $x,y,z\in\mathfrak{g}$ that $\langle x,[y,z]\rangle = \langle[x,y],z\rangle$. Let M be a 3-manifold, and $\mathsf{C}\in\Omega^\bullet(M)\otimes\mathfrak{g}[1]$. Then the BV Chern-Simons action is [CMR14]

$$S[\mathsf{C}] = \int_M\frac{1}{2}\langle\mathsf{C},\mathsf{dC}\rangle+\frac{1}{6}\langle\mathsf{C},[\mathsf{C},\mathsf{C}]\rangle,$$

where for homogeneous elements $A\otimes v, B\otimes w\in\Omega^\bullet(M)\otimes\mathfrak{g}$ the bracket and the pairing are defined by

$$[A\otimes v, B\otimes w] = A\wedge B\otimes[v,w]$$

and

$$\langle A\otimes v, B\otimes w\rangle = \langle v,w\rangle A\wedge B$$

[9] In the sense that we compute it formally as a Gaussian (or rather, Fresnel) integral.

respectively. Now assume that the Lie Algebra $\mathfrak{g}$ admits a splitting $g = V \oplus W$ into maximally isotropic subspaces, i.e. the pairing restricts to 0 on V and W and $\dim V = \dim W = \frac{\dim \mathfrak{g}}{2}$. Then we can identify $W \cong V^*$ via the pairing and decompose $\mathsf{C} = \mathsf{A} + \mathsf{B}$, where $\mathsf{A} \in \Omega^\bullet(M) \otimes V[1]$and $\mathsf{B} \in \Omega^\bullet(M) \otimes W[1]$. The action decomposes into a "free" or "kinetic" part

$$S_{free} = \int_M \frac{1}{2}\langle \mathsf{C}, \mathrm{d}\mathsf{C}\rangle = \int_M \frac{1}{2}\langle \mathsf{A}+\mathsf{B}, \mathrm{d}\mathsf{A} + \mathrm{d}\mathsf{B}\rangle = \int_M \frac{1}{2}\langle \mathsf{A}, \mathrm{d}\mathsf{B}\rangle + \frac{1}{2}\langle \mathsf{B}, \mathrm{d}\mathsf{A}\rangle = \int_M \langle \mathsf{B}, \mathrm{d}\mathsf{A}\rangle$$

(where $\langle \mathsf{A}, \mathrm{d}\mathsf{A}\rangle = 0 = \langle \mathsf{B}, \mathrm{d}\mathsf{B}\rangle$ by isotropy and we integrate by parts) and an "interaction" term

$$\mathcal{V}\langle \mathsf{A}, \mathsf{B}\rangle = \frac{1}{6}\langle \mathsf{A}+\mathsf{B}, [\mathsf{A}+\mathsf{B}, \mathsf{A}+\mathsf{B}]\rangle.$$

Hence, the theory is "BF-like".

3.2. **Perturbative Expansion.** Let M be a 3-manifold, possibly with boundary. We want to compute the state $\widehat{\psi}_M$. As described above for the BF example, we choose a decomposition of the boundary $\partial M = \partial_1 M \sqcup \partial_2 M$ and get a polarisation on the space of boundary fields such that $\mathcal{B}^{\mathcal{P}}_{\partial M} = \mathcal{B}_1 \times \mathcal{B}_2 \ni (\mathbb{A}, \mathbb{B})$. Decomposing $\mathsf{A} = \mathbb{A} + \mathsf{a} + \alpha$, $\mathsf{B} = \mathbb{B} + \mathsf{b} + \beta$, we can decompose the action as explained in remark 2.11:

$$\mathcal{S}^{\mathcal{P}}_M = \widehat{\mathcal{S}}_{M,0} + \widehat{\mathcal{S}}_{M,\mathrm{pert}} + \mathcal{S}^{\mathrm{back}}_M + \mathcal{S}^{\mathrm{source}}_M,$$

where

$$\begin{aligned}
\widehat{\mathcal{S}}_{M,0} &= \int_M \langle \beta, \mathrm{d}\alpha\rangle, \\
\widehat{\mathcal{S}}_{M,\mathrm{pert}} &= \int_M \mathcal{V}(\widehat{A}, \widehat{B}), \\
\mathcal{S}^{\mathrm{back}}_M &= -\left(\int_{\partial_2 M} \langle \mathbb{B}, \mathsf{a}\rangle + \int_{\partial_1 M} \langle \mathsf{b}, \mathbb{A}\rangle\right), \\
\mathcal{S}^{\mathrm{source}}_M &= -\left(\int_{\partial_2 M} \langle \mathbb{B}, \alpha\rangle + \int_{\partial_1 M} \langle \beta, \mathbb{A}\rangle\right).
\end{aligned}$$

The state is given by

$$\widehat{\psi}_M = \widehat{\psi}_M(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = \int_{\mathcal{L}} e^{\frac{i}{\hbar}\mathcal{S}^{\mathcal{P}}_M},$$

where $\mathcal{L} \ni (\alpha, \beta)$, the gauge-fixing Lagrangian, is the same as for abelian BF theory (cf. remark 2.8). Therefore it does not depend on the boundary and background fields. By virtue of the above decomposition, we can rewrite this as

$$\widehat{\psi}_M(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = e^{\frac{i}{\hbar}\mathcal{S}^{\mathrm{back}}_M} \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widehat{\mathcal{S}}_{M,0}} e^{\frac{i}{\hbar}\widehat{\mathcal{S}}_{M,\mathrm{pert}}} e^{\frac{i}{\hbar}\mathcal{S}^{\mathrm{source}}_M}.$$

To do a perturbative (power series) expansion[10], expand the exponentials

$$\begin{aligned}
\widehat{\psi}_M(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = &\sum_k \frac{1}{k!}\left(-\frac{i}{\hbar}\right)^k \left(\int_{\partial_2 M} \langle \mathbb{B}, \mathsf{a}\rangle + \int_{\partial_1 M} \langle \mathsf{b}\mathbb{A}\rangle\right)^k \int_{\mathcal{L}} e^{i\widehat{\mathcal{S}}_{M,0}} \sum_l \frac{1}{l!}\left(\frac{i}{\hbar}\right)^l \left(\int_M \mathcal{V}(\widehat{A}, \widehat{B})\right)^l \\
&\sum_m \frac{1}{m!}\left(-\frac{i}{\hbar}\right)^m \left(\int_{\partial_2 M} \langle \mathbb{B}, \alpha\rangle + \int_{\partial_1 M} \langle \beta, \mathbb{A}\rangle\right)^m \\
= &\sum_{k,l,m} \frac{1}{k!l!m!}(-1)^{k+m}\left(\frac{i}{\hbar}\right)^{k+l+m} \left(\int_{\partial_2 M} \langle \mathbb{B}, \mathsf{a}\rangle + \int_{\partial_1 M} \langle \mathsf{b}, \mathbb{A}\rangle\right)^k \\
&\int_{\mathcal{L}} e^{i\widehat{\mathcal{S}}_{M,0}} \left(\int_M \mathcal{V}(\widehat{\mathsf{A}}, \widehat{\mathsf{B}})\right)^l \left(\int_{\partial_2 M} \langle \mathbb{B}, \alpha\rangle + \int_{\partial_1 M} \langle \beta, \mathbb{A}\rangle\right)^m \\
= &\sum_{l,k,m} \frac{1}{k!l!m!}(-1)^{k+m}\left(\frac{i}{\hbar}\right)^{k+l+m} \left(\int_{\partial_2 M} \langle \mathbb{B}, \mathsf{a}\rangle + \int_{\partial_1 M} \langle \mathsf{b}, \mathbb{A}\rangle\right)^k \\
&\int_{\mathcal{L}} e^{i\widehat{\mathcal{S}}_{M,0}} \left(\int_M \frac{1}{6}\left\langle \widehat{\mathsf{A}} + \widehat{\mathsf{B}}, \left[\widehat{\mathsf{A}} + \widehat{\mathsf{B}}, \widehat{\mathsf{A}} + \widehat{\mathsf{B}}\right]\right\rangle\right)^l \left(\int_{\partial_2 M} \langle \mathbb{B}, \alpha\rangle + \int_{\partial_1 M} \langle \beta, \mathbb{A}\rangle\right)^m.
\end{aligned}$$

[10]Actually, a semiclassical expansion around the classical solution given by the trivial connection

Now we choose a basis ξ_i of V and let ξ^i be the corresponding dual basis of W. We expand our fields[11] $\mathsf{A} = \mathsf{A}^i\xi_i, \mathsf{B} = \mathsf{B}_i\xi^i$ and also their decompositions accordingly, i.e. $\alpha = \alpha^i\xi_i$, and so on. We then get e.g. $\langle \mathsf{B}, \mathrm{d}\mathsf{A}\rangle = \mathsf{B}_i\mathrm{d}\mathsf{A}^i$. We now want to expand the perturbation term in this basis. For this purpose we make use of the fact that $\langle X, [Y, Z]\rangle = \langle Z, [X, Y]\rangle = \langle Y, [Z, X]\rangle$ for any $X, Y, Z \in \Omega^\bullet(M) \otimes \mathfrak{g}[1]$, so we can decompose the interaction term as

$$\mathcal{V}(\widehat{\mathsf{A}}, \widehat{\mathsf{B}}) = \frac{1}{6}\langle \widehat{\mathsf{A}}, [\widehat{\mathsf{A}}, \widehat{\mathsf{A}}]\rangle + \frac{1}{2}\langle \widehat{\mathsf{B}}, [\widehat{\mathsf{A}}, \widehat{\mathsf{A}}]\rangle + \frac{1}{2}\langle \widehat{\mathsf{A}}, [\widehat{\mathsf{B}}, \widehat{\mathsf{B}}]\rangle + \frac{1}{6}\langle \widehat{\mathsf{B}}, [\widehat{\mathsf{B}}, \widehat{\mathsf{B}}]\rangle.$$

Now we make the following simplifying assumption on $\mathfrak{g}$.

Assumption 3.1. The splitting $\mathfrak{g} = V \oplus W$ is actually a splitting into *Lie subalgebras*, i.e. $(\mathfrak{g}, V, W)$ is a Manin triple.

By isotropy of the subspaces, this implies that the terms $\langle \widehat{\mathsf{A}}, [\widehat{\mathsf{A}}, \widehat{\mathsf{A}}]\rangle$ and $\langle \widehat{\mathsf{B}}, [\widehat{\mathsf{B}}, \widehat{\mathsf{B}}]\rangle$ vanish. Splitting $\widehat{\mathsf{A}} = \mathsf{a} + \alpha, \widehat{\mathsf{B}} = \mathsf{b} + \beta$, we expand the perturbation term in terms of the type $\langle \gamma_1, [\gamma_2, \gamma_3]\rangle$, where $\gamma_i \in \{\mathsf{a}, \alpha, \mathsf{b}, \beta\}$. These we can express as $\sum_{i,j,k} f_{ijk}\gamma_1^i\gamma_2^j\gamma_3^k$, where f_{ijk} are the structure constans of $\mathfrak{g}$ in the basis $\xi_1, \ldots \xi_n, \xi^1, \ldots \xi^n$. Integration over $\mathcal{L}$ can then be performed using Wick's theorem. Let η be an abelian BF propagator on M as discussed above. We exchange integrals over $M, \partial_i M$ and $\mathcal{L}$ and get an integrand which is a sum of products of forms γ. By the Wick theorem, the integral vanishes except for the case where there are precisely as many α's as β's, in which case

$$\int_{\mathcal{L}} e^{i\widehat{\mathcal{S}}_{M,0}}\alpha^{j_1}(x_1)\cdots\alpha^{j_n}(x_n)\beta^{k_1}(y_1)\cdots\beta^{k_n}(y_n) = T_M(-i\hbar)^n \sum_{\sigma\in S_n} \delta^{j_1 k_{\sigma(1)}}\eta(x_1, y_{\sigma(1)})\cdots\delta^{j_n k_{\sigma(n)}}\eta(x_n, y_{\sigma(n)}),$$

where $T_M = \int_{\mathcal{L}} e^{i\widehat{\mathcal{S}}_{M,0}}$.

3.3. **Feynman graphs and rules.** After integration over $\mathcal{L}$, we can label the terms in the perturbative expansion by graphs as follows. Fix $k, l, m \in \mathbb{N}_0$. We consider graphs Γ with three types of vertices:

- *Boundary background vertices:* There are k of these distributed on ∂M. They are labelled by $\mathbb{B}\mathsf{a}$ if they lie on $\partial_2 M$ and $\mathsf{b}\mathbb{A}$ if they lie on $\partial_1 M$.
- *Boundary source vertices:* There are m boundary source vertices distributed on ∂M. They are labelled by $\mathbb{B}\alpha$ on $\partial_2 M$ and $\mathbb{A}\beta$ on $\partial_1 M$. Vertices on $\partial_2 M$ have an arrow tail originating from them, whereas vertices on $\partial_1 M$ have an arrowhead pointing towards them.
- *Internal interaction vertices:* There are l internal vertices. They come with three half-edges which are labelled by γ_i's in $\{\mathsf{a}, \alpha, \mathsf{b}, \beta\}$. These half-edges are either marked as leaves if they are labelled by a background, as an arrow tail if they are labelled by α, or an arrowhead if they are labelled by β

If it is possible to connect every arrow tail α to an arrowhead β (possibly at the same vertex), then the graph resulting from this procedure is called an *admissible graph.* To such a graph we can associate a functional on the space of boundary fields as follows:

- For every background boundary vertex, multiply by $(-i/\hbar)$ times the label and integrate over the corresponding boundary point.
- For every internal vertex multiply by $(-i/\hbar)$ times the correct structure constants (specified by the half-edge labels) and integrate over M.
- For every leaf, multiply by the corresponding background field evaluated at the point.
- For every arrow between vertices in different positions $i \neq j$, with tail labelled by α^k and head β_l, multiply by a propagator $(-i\hbar)\delta_l^k\eta(x_i, y_j)$.
- For every short loop (also called tadpole), i.e. an arrow issueing and ending at the same vertex i ,with tail labelled by α^k and head β_l, multiply by $(-i\hbar)\delta_l^k\alpha(x_i)$, where $\alpha \in \Omega^2(M)$ is a so-called "tadpole form".[12]
- For every source boundary vertex, we multiply by $(-i/\hbar)$ times the corresponding boundary field and integrate over the corresponding boundary point.

We denote the result by $\widehat{\psi}_\Gamma$. Denoting the set of all admissible graphs for k, l, m by $\Lambda_{k,l,m}$, we get

$$\widehat{\psi}_M(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = T_M \sum_{k,l,m} \sum_{\Gamma\in\Lambda_{k,l,m}} \widehat{\psi}_\Gamma.$$

[11]From now on, we will make use of Einstein summation (sums over repeated indices are implied).

[12]These contributions can be ignored if the Lie algebra is unimodular (i.e. the structure constants satisfy $f_{ik}^i = 0$) or the Euler characteristic of M is 0. We will restrict ourselves to these cases.

Remark 3.2. We can factor out the non-interacting diagram parts (background boundary vertices and source boundary vertices connecting to other source boundary vertices). This will yield a prefactor of $e^{\frac{i}{\hbar}\mathcal{S}_0^{\text{eff}}}$ where $\mathcal{S}_0^{\text{eff}}$ is the free effective action

$$\mathcal{S}_0^{\text{eff}} = -\left(\int_{\partial_2 M} \langle \mathbb{B}, \mathsf{a}\rangle + \int_{\partial_1 M} \langle \mathsf{b}, \mathbb{A}\rangle\right) - \int_{\partial_2 M \times \partial_1 M} \pi_1^* \mathbb{B}_i \eta \mathbb{A}^i \tag{18}$$

i.e. the effective action of the unperturbed theory.

The remaining interaction diagrams have $l \geq 1$ internal vertices and $m \leq 3l$ boundary vertices. Denoting the set of admissible interaction diagrams by $\Lambda_{l,m}^{int}$, the above expression becomes

$$\widehat{\psi}_M(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = T_M e^{\frac{i}{\hbar}\mathcal{S}_0^{\text{eff}}} \left(1 + \sum_{l=1}^{\infty} \sum_{m=0}^{3l} \sum_{\Gamma \in \Lambda_{l,m}^{int}} \widehat{\psi}_\Gamma\right).$$

Our goal is now to give an asymptotic expansion of the state of the form

$$\widehat{\psi}_M(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = T_M e^{\frac{i}{\hbar}\mathcal{S}_M^{\text{eff}}} \sum_{j \geq 1} \hbar^j R_j,$$

where $\mathcal{S}_M^{\text{eff}}$ is the so-called *tree effective action*, i.e the sum of all diagrams whose underlying graphs are trees, and R_j denotes the sum of all diagrams that contain at least one loop.

4. Split Chern-Simons theory on the solid torus

In this section we compute a first approximation for the state on the solid torus $K := D \times S^1$ with boundary $\partial M = S^1 \times S^1 =: \mathbb{T}^2$. Here we think of $D = \{z \in \mathbb{C}, |z| \leq 1\}$ as the closed unit disk in the complex plane. This is not just a simple exercise: Note that since the quantum BV-BFV formalism allows also for the gluing of states, given a state on the solid torus one can compute it also for any manifold that can be glued together from tori (namely, all lens spaces).
Since the boundary $\mathbb{T}^2$ is connected, there are only two possible choices for $\partial_1 M$ and $\partial_2 M$, we choose $\partial_1 M := \partial M$ and $\partial_2 M := \emptyset$. In a future paper we plan to do a similar computation for handlebodies, and due to Heegard decomposition this would lead to state for general 3-folds. This leads to the following space of backgrounds:

$$\begin{aligned}\mathcal{V}_M &= H^\bullet_{D1}(M)[1] \otimes V \oplus H^\bullet_{D2}(M)[1] \otimes W = H^\bullet(M, \partial M)[1] \otimes V \oplus H^\bullet(M) \otimes W \\ &\cong (H^\bullet(D, \partial D) \otimes H^\bullet(S^1)) \otimes V \oplus H^\bullet(S^1))[1] \otimes W.\end{aligned}$$

Let μ be a normalised generator of $H^\bullet(D, \partial D)$, i.e $\int_D \mu = 1$. Denoting t the coordinate on S^1, we get that $\chi_1 = \mu dt, \chi_2 = \mu$ is a basis of $H^\bullet_{D1}(M)[1]$, with dual basis $\chi^1 = 1, \chi^2 = dt$ of $H^\bullet_{D2}(M)[1]$. We can then expand

$$\begin{aligned}\mathsf{a}^i &= z^{1i}\mu dt + z^{2i}\mu, \\ \mathsf{b}_i &= z_{1i}^+ 1 + z_{2i}^+ dt.\end{aligned}$$

The canonical BV Laplacian on $\mathcal{V}_M$ is then given by

$$\Delta_{\mathcal{V}_M} = -\left(\frac{\partial}{\partial z^1}\frac{\partial}{\partial z_1^+} + \frac{\partial}{\partial z^2}\frac{\partial}{\partial z_2^+}\right).$$

4.1. **Effective Action on the solid torus.**

Assume as above that $\mathfrak{g} = V \oplus W$ is a Manin triple, i.e

- $V \cong W^*$ as vector spaces
- V, W Lie algebras.

Let us introduce bases $\xi_1, \ldots, \xi_n$ of V, $\xi^1, \ldots, \xi^n$ of W such that $\langle \xi_i, \xi^j \rangle = \delta_i^j$ and structure constants in these bases: $[\xi_i, \xi_j]_V = f_{ij}^k \xi_k, [\xi^i, \xi^j]_W = g_k^{ij}\xi^k$. We can then also decompose the fields

$$\begin{aligned}\mathsf{B} &= \mathsf{B}_i \xi^i = \mathsf{b}_i \xi^i + \beta_i \xi^i + \mathbb{B}_i \xi^i, \\ \mathsf{A} &= \mathsf{A}^i \xi_i = \mathsf{a}^i \xi_i + \alpha^i \xi_i + \mathbb{A}^i \xi_i.\end{aligned}$$

The fact we have a Manin triple means that in terms of the structure constants we have

$$f_{ij}^k g_k^{lm} = f_{ik}^l g_j^{km} - f_{jk}^l g_i^{km} + f_{ik}^m g_j^{lk} - f_{jk}^m g_i^{lk}. \tag{19}$$

We now want to compute an approximation to the tree effective action by considering tree diagrams that have at most two interaction vertices and at most two boundary vertices.
We will proceed by the number of interaction vertices. There is only a single connected diagram with no interaction vertices, consisting of a single point on the boundary. It yields the free effective action (18) for $\partial_2 M = \emptyset$, namely

$$S_0^{\text{eff}} = -\int_{\partial_1 M} \mathsf{b}_k \mathbb{A}^k.$$

4.1.1. *1-point contribution.* Let us continue with diagrams containing a single interaction vertex. It is now important that the solid torus has zero Euler characteristic, so we do not need to consider tadpoles. Since there can be no arrows issuing from $\partial_1 M$, diagrams with a half-edge labelled by β at the interaction point are not admissible. Also notice that $\mathsf{a} \wedge \mathsf{a} = 0$ (it is a 4-form on a 3-manifold). In the end, there are only three contributing diagrams:

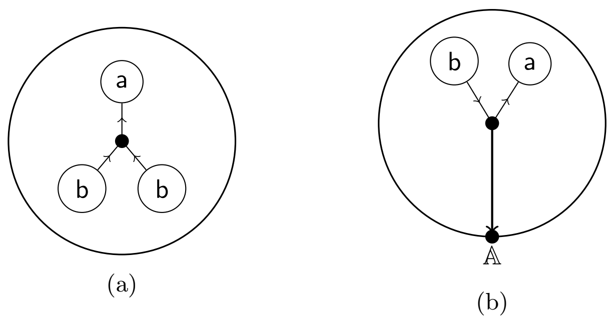

FIGURE 1. Graphs in the solid torus (depicted in a cross-section) with 1 interaction vertex. A bullet denotes a point we integrate over, a long arrow denotes a propagator.

a) The single interaction vertex with three leaves labelled by a, b and b, corresponding to

$$S_1^{\text{eff}} := \frac{1}{2}\int_M \langle \mathsf{a}, [\mathsf{b}, \mathsf{b}] \rangle.$$

We should explain some notation. We denote by $C_{m,n}(M, \partial M)$ (a suitable compactification of) the configuration space of m points in the bulk and n in the boundary. It comes with natural projections

$$\pi_i \colon C_{m,n}(M, \partial M) \to \begin{cases} M & i \leq m \\ \partial M & i \geq m \end{cases}$$

and

$$\pi_{ij} \colon C_{i,j}(M, \partial M) \to \begin{cases} C_2(M) & i, j \leq m \\ C_{1,1}(M, \partial M) & i \leq m, j \leq n \\ C_2(\partial M) & i, j \geq m \end{cases}.$$

By writing γ_i resp. γ_{ij} we mean the pullback of γ under the corresponding projection.

b) The single interaction vertex with two leaves labelled b and a and an arrow connecting to a boundary source vertex $\beta\mathbb{A}$. It evaluates to

$$S_2^{\text{eff}} := -\int_{C_{1,1}(M, \partial_1 M)} f^i_{jk} \mathsf{b}_{1,i} \mathsf{a}^j_1 \eta_{12} \mathbb{A}^k_2.$$

c) The single interaction vertex with a leaf labelled by b and two arrows connecting to two different boundary source vertices. This evaluates to

$$S_{\text{eff},3} := \frac{1}{2}\int_{C_{1,2}(M, \partial_1 M)} f^i_{jk} b_{1,i} \eta_{12} \eta_{13} \mathbb{A}^j_2 \mathbb{A}^k_3.$$

4.1.2. *2-point contribution.* Now we consider tree diagrams with two interaction vertices. Since the diagrams have to be connected, there has to be at least one arrow between the vertices. Since we are only considering trees, there is exactly one arrow between them. Also, we are considering only diagrams that have at most two boundary vertices. The diagrams in figure 2 below show the admissible graphs in the relevant degrees (admissible graphs with no boundary vertices all evaluate to 0 because of property 13) We will discuss the results below.

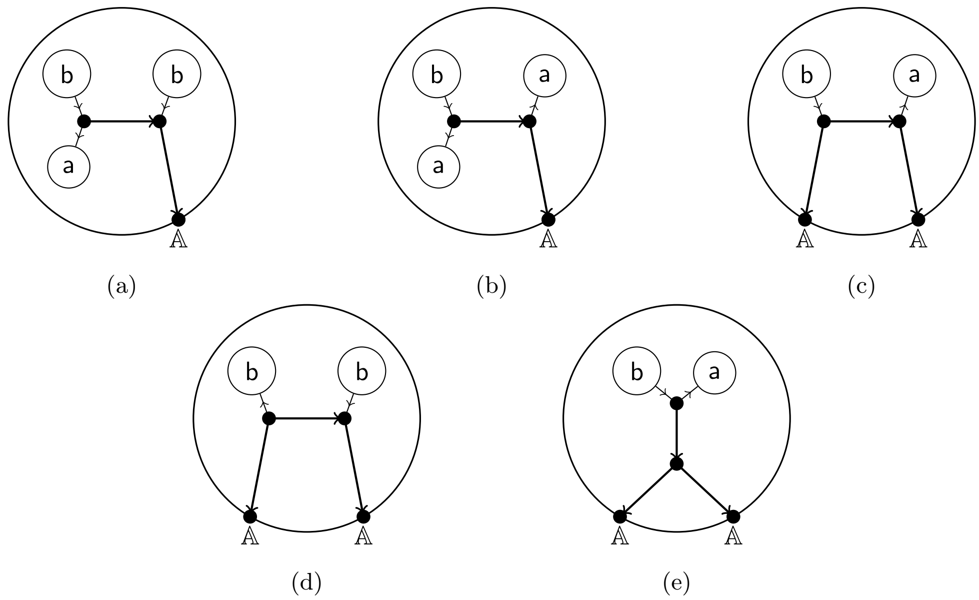

FIGURE 2. Graphs with 2 interaction vertices. A bullet denotes a point we integrate over, long arrow denotes a propagator.

4.1.3. *Performing integration over M.* We now want to perform the integration over the bulk points. There are two possibilities to proceed:

(1) One constructs an explicit propagator on M and computes the integrals analytically.
(2) One analyses how the resulting form on the boundary behaves under de Rham differential and integration of points, and picks a form which is a product of propagators and representatives of cohomology on the boundary that has the same properties. Since only these properties enter into the proof of the mQME, this produces a valid state. We will discuss this procedure and the question of uniqueness in more depth in a future paper.

With the second approach, choosing a propagator satisfying also 13 and 14, one can see that the only non-vanishing contributions from two-point diagrams come from diagrams 2c and 2e. Denoting the results by S_4^{eff} and S_5^{eff} respectively, we obtain

$$\begin{aligned}
S_0^{\text{eff}} &= -z^+_{1,k}\int_{\partial_1 M}\mathbb{A}^k - z^+_{2,k}\int_{\partial_1 M} dt\mathbb{A}^k,\\
S_1^{\text{eff}} &= \frac{1}{2}g_i^{jk}(z^{1i}z^+_{1j}z^+_{1k} + 2z^{2i}z^+_{1j}z^+_{2k}),\\
S_2^{\text{eff}} &= f^i_{jk}z^+_{1i}z^{2j}\int_{\partial_1 M} d\theta\mathbb{A}^k + f^i_{jk}(z^+_{1i}z^{1j} - z^+_{2i}z^{2j})\int_{\partial_1 M} dtd\theta\mathbb{A}^k,\\
S_3^{\text{eff}} &= \frac{1}{2}f^i_{jk}z^+_{1i}\int_{C_2(\partial_1 M)}\eta^T_{12}\mathbb{A}^j_1\mathbb{A}^k_2,\\
&\quad + \frac{1}{2}f^i_{jk}z^+_{2i}\int_{C_2(\partial_1 M)}\eta^T_{12}\frac{dt_1+dt_2}{2}\mathbb{A}^j_1\mathbb{A}^k_2,\\
S_4^{\text{eff}} &= f^i_{jk}f^j_{lm}z^+_{1i}z^{2l}\int_{C_2(\partial_1 M)} d\theta_1\eta^T_{12}\mathbb{A}^k_1\mathbb{A}^m_2 + f^i_{jk}f^j_{lm}(z^+_{1i}z^{1l} - z^+_{2i}z^{2l})\int_{C_2(\partial_1 M)} dt_1 d\theta_1\eta^T_{12}\mathbb{A}^k_1\mathbb{A}^m_2,\\
S_5^{\text{eff}} &= f^i_{jk}f^k_{lm}z^+_{1i}z^{2j}\int_{C_2(\partial_1 M)} d\theta_1\eta^T_{12}\mathbb{A}^l_1\mathbb{A}^m_2 + f^i_{jk}f^k_{lm}(z^+_{1i}z^{1j} - z^+_{2i}z^{2j})\int_{C_2(\partial_1 M)} dt_1 d\theta_1\eta^T_{12}\mathbb{A}^l_1\mathbb{A}^m_2,
\end{aligned}$$

where t denotes the parallel (longitudinal) and θ the meridian coordinate on the boundary torus (i.e. in the solid torus $[\mathrm{d}\theta] = 0$), and η^T is a propagator for abelian BF theory on the boundary torus.

4.2. **mQME.** Our goal in this section is to prove the modified Quantum Master Equation

$$(\hbar^2\Delta + \Omega)e^{\frac{i}{\hbar}S_{\text{eff}}} = 0,$$

ignoring terms of nonzero order in $\hbar$, more than two boundary vertices or more than second power in the interaction. Here Ω is given by the standard quantisation of

$$S^{\partial} = \int_{\partial M} \langle \mathsf{B}, \mathrm{d}\mathsf{A}\rangle + \frac{1}{2}\langle \mathsf{B}, [\mathsf{A},\mathsf{A}]\rangle + \frac{1}{2}\langle \mathsf{A}, [\mathsf{B},\mathsf{B}]\rangle,$$

which (on the solid torus) is

$$\Omega_{\mathrm{st}} = -\mathrm{i}\hbar \int_{\partial_1 M} \mathrm{d}\mathbb{A}^k \frac{\delta}{\delta \mathbb{A}^k} + \frac{1}{2} g_a^{bc} \int_{\partial_1 M} -\hbar^2 \mathbb{A}^a \frac{\delta}{\delta \mathbb{A}^b}\frac{\delta}{\delta \mathbb{A}^c} + \frac{1}{2} f^a_{bc} \int_{\partial_1 M} -\mathrm{i}\hbar \mathbb{A}^b\mathbb{A}^c \frac{\delta}{\delta \mathbb{A}^a}.$$

Remark 4.1. The second term containing two derivatives yields possibly singular results when applied to a single term in the effective action. Therefore the two derivatives are allowed to act only on different terms in a product of terms of the effective action. With this regularisation one can also check that $\Omega_{st}^2 = 0$.

One can check that $\Delta S_{\text{eff}} = 0$ and therefore $(\hbar^2\Delta + \Omega) e^{\frac{i}{\hbar}S^{\text{eff}}} = -\frac{1}{2}(S^{\text{eff}}, S^{\text{eff}}) e^{\frac{\mathrm{i}}{\hbar}S^{\text{eff}}} + \Omega e^{\frac{\mathrm{i}}{\hbar}S^{\text{eff}}}$. So we should check that $\frac{1}{2}(S^{\text{eff}}, S^{\text{eff}}) e^{\frac{\mathrm{i}}{\hbar}S^{\text{eff}}} = \Omega e^{\frac{\mathrm{i}}{\hbar}S^{\text{eff}}}$ up to higher order corrections.

4.2.1. *BV Bracket.* Let us compute first $(S^{\text{eff}}, S^{\text{eff}})$. Abbreviating $S_i^{\text{eff}} =: S_i$, we get that $(S^{\text{eff}}, S^{\text{eff}}) = \sum_i (S_i, S_i) + 2\sum_{i<j}(S_i, S_j)$.
We have that $(z^+_{1i}, z^{1j}) = \delta_{ij} = -(z^+_{2i}, z^{2j})$, and all other brackets vanish.
Since S_0 and S_3 only contain z^+ variables, we get that $(S_0, S_0) = (S_3, S_3) = (S_0, S_3) = 0$. Also, (S_2, S_3) contains three boundary fields, so we neglect it. The same is true for any bracket of S_4 with the rest, except (S_1, S_4), which is third power in the structure constants. So the only contributing brackets are $(S_0, S_1), (S_0, S_2), (S_1, S_1), (S_1, S_2), (S_1, S_3)$ and (S_2, S_2).

4.2.2. *Ω part.* Now let us compute $\Omega_{\text{st}} e^{\frac{\mathrm{i}}{\hbar}S^{\text{eff}}}$. At first, we will consider only contributions of order 0 in $\hbar$ and less than two $\mathbb{A}'s$. Let us split Ω into the following 3 terms:

$$\begin{aligned}
\Omega_0 &:= -\mathrm{i}\hbar \int_{\partial_1 M} \mathrm{d}\mathbb{A}^k \frac{\delta}{\delta\mathbb{A}^k},\\
\Omega_1 &:= -\frac{\mathrm{i}\hbar}{2} f^a_{bc} \int_{\partial_1 M} \mathbb{A}^b\mathbb{A}^c \frac{\delta}{\delta \mathbb{A}^a},\\
\Omega_2 &:= -\frac{\hbar^2}{2} g_a^{bc} \int_{\partial_1 M} \mathbb{A}^a \frac{\delta}{\delta\mathbb{A}^b}\frac{\delta}{\delta\mathbb{A}^c}.
\end{aligned}$$

By the usual rules of derivatives we will have

$$\Omega_{st} e^{\frac{\mathrm{i}}{\hbar}S^{\text{eff}}} = \left((\Omega_0 + \Omega_1)\frac{i}{\hbar} S^{\text{eff}} + \Omega_2 \left(\frac{i}{\hbar}\right)^2 \frac{1}{2} (S^{\text{eff}})^2 \right) e^{\frac{\mathrm{i}}{\hbar}S^{\text{eff}}}.$$

Let us look at the linear term first. Notice that $\Omega_0(S_0) = \Omega_0(S_1) = \Omega_0(S_2) = 0$, since we can integrate by parts, and the forms appearing in these integrals are closed. Also, since we are ignoring terms with more than two boundary fields, and $\Omega_1(S_1) = 0$, we only need to consider $\Omega_1(S_0)$ and $\Omega_1(S_2)$. Now we need to consider $\Omega_2\left(\frac{\mathrm{i}}{\hbar}\right)^2 \frac{1}{2!}(S^{\text{eff}})^2$. Since Ω_2 removes one $\mathbb{A}$, but adds one power in the interaction, we have to consider terms in $(S^{\text{eff}})^2$ with two or three $\mathbb{A}$'s and at most first power in the the interaction. One can easily check that the only products to consider are S_0^2, S_0S_2 and S_0S_3.

4.2.3. *Proving the mQME.*

Proposition 4.2. *To prove the mQME in the chosen degrees one can equivalently prove that*

$$(S_0, S_1) + (S_0, S_2) + \frac{1}{2}(S_1, S_1) + (S_1, S_2) + (S_1, S_3) + \frac{1}{2}(S_2, S_2) \tag{20}$$

$$= \frac{i}{\hbar}\left(\Omega_0(S_3) + \Omega_0(S_4) + \Omega_0(S_5) + \Omega_1(S_0) + \Omega_1(S_2)\right) + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 \Omega_2(S_0^2 + 2S_0S_2 + 2S_0S_3). \tag{21}$$

This can be shown using a direct computation, which we summarise as follows.

Lemma 4.3. *The following identities hold:*

i) $(S_0, S_1) = \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 \Omega_2(S_0^2)$,
ii) $(S_1, S_1) = 0$,
iii) $(S_0, S_2) = \frac{i}{\hbar}\left(\Omega_0(S_3) + \Omega_1(S_0)\right)$,
iv) $(S_1, S_2) = \left(\frac{i}{\hbar}\right)^2 \Omega_2(S_0 S_2)$,
v) $(S_1, S_3) = \left(\frac{i}{\hbar}\right)^2 \Omega_2(S_0 S_3)$,
vi) $(S_2, S_2) = \frac{i}{\hbar}\left(\Omega_0(S_4) + \Omega_0(S_5) + \Omega_1(S_2)\right)$.

Corollary 4.4. *The state defined by $\widehat{\psi} = e^{\frac{i}{\hbar}S^{\mathit{eff}}}$ satisfies the mQME on the solid torus at zeroth order in $\hbar$, considering terms with at most two boundary fields and at most second order in the interaction.*

4.3. **Change of data.** Now we will analyse how the state behaves under an infinitesimal change of gauge-fixing, i.e. the representatives of cohomology and the propagator. Such a change can be described by the action of a vector field X on M on these forms by the Lie derivative

$$\dot{\chi}_i = L_X\chi_i, \dot{\chi}^i = L_X\chi^i, \dot{\eta} = L_X\eta$$

(we will always write X to mean the vector field $(X, \ldots, X) \in TM \oplus \cdots \oplus TM \cong T(M \times \cdots \times M)$). Clearly we have

$$\frac{d}{dt}\widehat{\psi} = \frac{i}{\hbar}\frac{d}{dt}(S^{\mathrm{eff}})e^{\frac{i}{\hbar}S^{\mathrm{eff}}}.$$

Proposition 4.5. *If we expand S^{eff} as a sum of terms of the form*

$$S^{\mathit{eff}} = \sum \int_{C_n(\partial_1 M)} \gamma \pi_1^*\mathbb{A}\cdots\pi_n^*\mathbb{A},$$

then its time derivative is given by

$$\frac{d}{dt}(S^{\mathit{eff}}) = \sum \int_{C_n(\partial_1 M)} (L_{X^\partial}\gamma)\pi_1^*\mathbb{A}\cdots\pi_n^*\mathbb{A},$$

where X^∂ denotes restriction of X to the boundary.

Proof. S^{eff} is a sum of terms of the form

$$\int_{C_{m,n}(M,\partial_1 M)} \widehat{\gamma}\pi_1^*\mathbb{A}\cdots\pi_n^*\mathbb{A},$$

where $\widehat{\gamma}$ is a product of background fields and propagators on M. Since L_X is a derivation, we have $\frac{d}{dt}\widehat{\gamma} = L_X\widehat{\gamma}$. But the Lie derivative commutes with the integration over the bulk vertices, so we have proved the statement. □

We are now going to define a state ζ such that

$$(\hbar^2\Delta + \Omega)(\widehat{\psi}\zeta) = \frac{d}{dt}\widehat{\psi}$$

(as in (17)) for our example on the torus. Namely, we define $\gamma_i \in \Omega^{k_i}(C_{n_i}(\partial_1 M))$ by

$$S^{\mathrm{eff}} = \sum_i F_i(f, g, z, z^+)_{j_1\cdots j_{n_i}} \int_{C_{n_i}(\partial_1 M)} \gamma_i \pi_1^*\mathbb{A}^{j_1}\cdots\pi_{n_i}^*\mathbb{A}^{j_{n_i}}.$$

Then ζ is defined by

$$\zeta = \sum_i F_i(f, g, z, z^+)_{j_1\cdots j_{n_i}} \int_{C_{n_i}(\partial_1 M)} (\iota_{X^\partial}\gamma_i)\pi_1^*\mathbb{A}^{j_1}\cdots\pi_{n_i}^*\mathbb{A}^{j_{n_i}},$$

i.e. we replace every differential form γ_i by its contraction with X.

Proposition 4.6. *For the change of data described above and the effective action described in the last paragraph, we have that*

$$(\hbar^2\Delta + \Omega)(\widehat{\psi}\zeta) = \frac{d}{dt}\widehat{\psi}$$

at zeroth order in $\hbar$, considering only terms of at most two boundary fields and at most second power in the interaction.

Sketch of the proof. We have that

$$\Delta((\widehat{\psi}\zeta)) = \Delta(\widehat{\psi})\zeta \pm \widehat{\psi}\Delta(\zeta) \pm (\psi,\zeta) = \Delta(\widehat{\psi})\zeta \pm (\psi,\zeta),$$

since $\Delta(\zeta) = 0$. On the other hand, using that Ω_0 and Ω_1 are first-order differential operators and Ω_2 is a second-order differential operator,

$$\begin{aligned}\Omega(\widehat{\psi}\zeta) &= \Omega_0(\widehat{\psi}\zeta) + \Omega_1(\widehat{\psi}\zeta) + \Omega_2(\widehat{\psi}\zeta)\\ &= \Omega_0(\widehat{\psi})\zeta + \widehat{\psi}\Omega_0(\zeta) + \Omega_1(\widehat{\psi})\zeta + \widehat{\psi}\Omega_1(\zeta) + \Omega_2(\widehat{\psi})\zeta + \widehat{\psi}\Omega_2(\zeta) + (\widehat{\psi}\zeta)'\\ &= \Omega(\widehat{\psi})\zeta + \widehat{\psi}\Omega(\zeta) + (\widehat{\psi}\zeta)',\end{aligned}$$

where $(\widehat{\psi}\zeta)'$ denotes the term where one derivative in Ω_2 acts on $\widehat{\psi}$ and the other acts on ζ. By the mQME, terms where Δ and Ω act on ψ only cancel. Let us first consider the term where Ω acts on ζ only. After integrating by parts, $\Omega_0(\zeta)$ replaces $\iota_{X^\partial}\gamma_i$ by $\mathrm{d}\iota_{X^\partial}\gamma_i$, plus contributions from the boundary of the configuration space. As in the proof of the mQME, those are cancelled by $\Omega_1(\zeta)$. Since Ω_2 can only act on products of terms, $\Omega_2(\zeta) = 0$. Next, notice that by properties of BV brackets and derivatives we have

$$(\psi,\zeta) = (S^{\text{eff}},\zeta)\psi \qquad \text{and} \qquad (\psi\zeta)' = (S^{\text{eff}}\zeta)'\psi.$$

We are left to prove that $(S^{\text{eff}},\zeta) + (S^{\text{eff}}\zeta)'$ produces all the terms of the form $\iota_{X^\partial}\mathrm{d}\gamma$, then the result follows from Proposition 4.5 and Cartan's magic formula. We summarise this as follows.

Lemma 4.7. *Let S_i be the parts of the effective action as above. Denote by $\iota_{X^\partial}S_i, \mathrm{d}S_i$ the operation of replacing all differential forms γ appearing in S_i by $\iota_{X^\partial}\gamma$ or $\mathrm{d}\gamma$ respectively. Then the following identities hold:*

$$\Omega_2(S_0\iota_{X^\partial}S_0) = (S_1, \iota_{X^\partial}S_0), \tag{22}$$

$$\Omega_2(S_0\iota_{X^\partial}S_2) + \Omega_2(S_2\iota_{X^\partial}S_0) = (S_1, \iota_{X^\partial}S_2), \tag{23}$$

$$\Omega_2(S_0\iota_{X^\partial}S_3) + \Omega_2(S_3\iota_{X^\partial}S_0) = (S_1, \iota_{X^\partial}S_3), \tag{24}$$

$$(S_2, \iota_{X^\partial}S_0) + (S_0, \iota_{X^\partial}S_2) = \iota_{X^\partial}\mathrm{d}S_3, \tag{25}$$

$$(S_2, \iota_{X^\partial}S_2) = \iota_{X^\partial}\mathrm{d}S_4 + \iota_{X^\partial}\mathrm{d}S_5. \tag{26}$$

As in the proof of the mQME, these are all the relevant brackets and products for our choice of degrees. Since S_3 and S_4 are the only terms with differential forms that are not closed, all the terms we need are produced and we conclude the statement. □

5. Conclusions and outlook

We have shown that the BV-BFV formalism can be applied to split Chern-Simons theory and produces a non-trivial example. Using the method applied in Section 4.1.3 it is possible to make statements about the effective action to all orders. Furthermore, the structure of the identities in lemmata 4.3 and 4.7 seems to hint to the structure of the effective action being governed by the mQME alone, i.e. to the fact that one can recover the state in the perturbed theory from the state in the unperturbed theory requiring only that the mQME is satisfied. A natural question to consider would be: to what extent one can make such a statement rigorous, and in what generality one can prove it.

In another direction, the next step is to use the state on the solid torus to compute the Chern-Simons theta invariants of lens spaces via the gluing operation. The relatively simple expression for the effective action in terms of a propagator and the cohomology on the boundary should also allow for an extension to higher genus handlebodies and other background flat connections, and thereby the computation of the Chern-Simons invariants for all 3-manifolds.

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