

POISSON SIGMA MODEL AND SEMICLASSICAL QUANTIZATION OF INTEGRABLE SYSTEMS.

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Dedicated to the memory of L.D. Faddeev.

ABSTRACT. In this paper we outline the construction of semiclassical eigenfunctions of integrable models in terms of the semiclassical path integral for the Poisson sigma model with the target space being the phase space of the integrable system. The semiclassical path integral is defined as a formal power series with coefficients being Feynman diagrams. We also argue that in a similar way one can obtain irreducible semiclassical representations of Kontsevich's star product.

INTRODUCTION

This paper is motivated by two problems. The first one is how to describe the full semiclassical expansion of eigenfunctions of a complete set of quantum Hamiltonians for a quantum integrable system. More generally, the problem is how to describe full semiclassical asymptotics of the scalar products of two integrable systems on the same space of geometric quantization, or corresponding cyclic amplitudes. The second problem is how to construct an irreducible semiclassical representation of Kontsevich's star product on the algebra of functions on a Poisson manifold.

It turns out that both questions can be answered in terms the semiclassical quantization of the Poisson sigma model with special boundary conditions. This connects the structure of the semiclassical eigenfunctions of any given integrable system to the quantization of a very special topological gauge theory.

This paper is a research announcement. An extended version of this paper with proofs and mathematically sound construction of Feynman diagrams will be published separately.

We would like to dedicate this paper to the memory of L.D. Faddeev. He made fundamental contributions to both, quantization of gauge theories and quantization of integrable systems [9][8]. As we demonstrate here they are related in a very direct way. For other relations between gauge theories and integrable systems see [7][12][10][13].

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1. SEMICLASSICAL ASYMPTOTIC IN INTEGRABLE SYSTEMS AND TOPOLOGICAL QUANTUM MECHANICS

1.1. Semiclassical asymptotic in integrable systems. Let (M, ω) be a symplectic manifold. Fix the geometric quantization data which consist of the following:

- A line bundle L (a prequantization line bundle) with Hermitian structure on fibers and a Hermitian connection α on L such that the symplectic form ω is the curvature of α , i.e. $d\alpha = \omega$ and the Hermitian product is covariantly constant.
- A real polarization $P \subset TM$ which is an integrable tangent distribution on M such that each generic leaf is a Lagrangian submanifold in M . We assume that the space of leaves $B = M/P$ is almost everywhere smooth, i.e. that the polarization is a Lagrangian fibration $\pi : M \rightarrow B$ (generic fibers are Lagrangian).

The space of geometric quantization $H_P^{(1/2)}$ is the space of half-densities on M which are covariantly constant (with respect to the connection α) along P . Locally it can be identified with functions on M/P .

Let $C_h(M)$ be quantized algebra of functions on M . Assume it acts on the space $H_P^{(1/2)}$.

A classical integrable system on M is a Lagrangian fibration¹ $\pi : M \rightarrow B$ which defines a Poisson commuting subalgebra $C(M, B) \subset C(M)$ in the algebra of functions on M , $C(M, B) = \pi^*(C(B))$. Assume that the subalgebra $C(M, B)$ is quantized, i.e. deformed into a maximal commutative subalgebra $C_h(M, B) \subset C_h(M)$, where $C_h(M)$ is a deformation quantization of $C(M)$ given by a star product (see [11] and references therein). For $2n$ -dimensional M such subalgebra has rank n . For the purpose of this paper one can think that $M = T^*Q$ for some smooth n -dimensional manifold Q , the algebra $C_h(M)$ is the algebra of differential operators where derivatives are multiplied by $-i\hbar$ and the algebra $C(M)$ is the algebra of smooth functions on Q which are polynomials in the cotangent directions. If \hbar is a formal variable we assume that vector spaces $C_h(M)$ and $C(M)[[\hbar]]$ are isomorphic and the isomorphism is given by placing derivatives to the right of the smooth coefficient functions. The commutative subalgebra $C_h(M, B)$ is the algebra of commuting differential operators with the principal symbol from $\pi^*(C(B))$.

Now, assume that we have a real polarization P with the corresponding Lagrangian fibration $\pi : M \rightarrow B = M/P$. Assume we have two integrable systems corresponding to the Lagrangian fibrations $\pi_{1,2} : M \rightarrow B_{1,2}$ ². Assume that fibers of both projections are generically transverse and generically transverse with leaves of P . Assume that the algebra $C_h(M)$ acts on the space $H_P^{(1/2)}$. When $M = T^*Q$, P is the real polarization given by the Lagrangian fibration $T^*Q \rightarrow Q$ and the space $H_P^{(1/2)}$ can be naturally identified with functions on Q .

We will say a vector $\psi_\chi \in H_P^{(1/2)}$ is an eigenvector of $C_h(M, B_2)$ corresponding to the character $\chi : C_h(M, B_2) \rightarrow \mathbb{R}$ if $a\psi_\chi = \chi(a)\psi_\chi$ for any $a \in C_h(M, B_2)$. Semiclassically, as $\hbar \rightarrow 0$ the set of characters can be identified with B_2 . Similarly, we have eigen-half-densities for $C_h(M, B_1)$. Semiclassically, characters of $C_h(M, B_1)$ can be identified with points of

¹The generic fibers are Lagrangian submanifolds.

²In other words we have three mutually transverse polarizations.

B_1 . Denote by $\psi_{b_1}^{(1)}$ and $\psi_{b_2}^{(2)}$ semiclassical eigen-half-densities of commutative subalgebras $C_h(M, B_1)$ and $C_h(M, B_2)$ acting in $H_P^{(1/2)}$.

Conjecture 1. [14] *Semiclassical asymptotics of the scalar product of eigen-half-densities of $C_h(M, B_2)$ and $C_h(M, B_1)$ in the space $H_P^{(1/2)}$ has the following structure:*

$$(1) \quad (\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)}) = \frac{C}{(2\pi\hbar)^{n/2}} \sum_{c \in \mathcal{L}_{b_1}^{(1)} \cap \mathcal{L}_{b_2}^{(2)}} e^{\frac{i}{\hbar} S_{\gamma_1, \gamma_2}(c, b_1, b_2) + \frac{i\pi}{2} \mu_{\gamma_1, \gamma_2}(c)} \left| \det \left(\frac{\partial^2 S_{\gamma_1, \gamma_2}(c, b_1, b_2)}{\partial b_1^i \partial b_2^j} \right) \right|^{1/2} (1 + O(\hbar)) \sqrt{|db_1 db_2|}$$

Here as we made a choice of reference points $x_0^{(i)} \in \mathcal{L}_{b_i}^{(i)}$, $S_{\gamma_1, \gamma_2}(c, b_1, b_2) = S_{\gamma_1} - S_{\gamma_2}$ where $S_{\gamma_i} = \int_{\gamma_i \subset \mathcal{L}^{(i)}} \alpha$ and α is the prequantization connection, γ_1 and γ_2 are paths in $\mathcal{L}_{b_1}^{(1)}$ and $\mathcal{L}_{b_2}^{(2)}$ respectively, connecting corresponding reference points³ and c . Lagrangian submanifolds $\mathcal{L}_{b_1}^{(1)}$ and $\mathcal{L}_{b_2}^{(2)}$ are fibers over $b_1 \in B_1$ and $b_2 \in B_2$ respectively. We assume that $\mathcal{L}_{b_1}^{(1)}$ and $\mathcal{L}_{b_2}^{(2)}$ are transverse. The number $\mu_{\gamma_1, \gamma_2}(c)$ is the weighted number of points along γ_2 between x_2 and c where the tangent space to $\mathcal{L}_{b_2}^{(2)}$ intersect a fiber of π_1 over a line. The point counts with plus if it is crossed in the positive direction and with the minus if it is crossed in the negative direction, i.e. μ is the Maslov index. When Bohr-Sommerfeld quantization conditions for b_1 and b_2 hold, the exponent does not depend on the choice of γ_1 and γ_2 . The Hessian and the higher order terms do not depend on the choice of reference points.

Note that to write the formula (1) we made a choice of a reference point on every fiber of π_1 and of π_2 . Consistent choice of such reference points can be inferred, e.g., from fixing a Lagrangian submanifold Λ intersecting fibers of π_1 and π_2 generically transversally. A change of Λ changes the formula for the scalar product by a constant factor (each term in the sum changes by the same factor). Thus, the eigenfunctions depend on Λ only projectively. The choice of Λ is also convenient for making the expression (1) invariant with respect to automorphisms of the prequantization line bundle. Let us multiply the scalar product by the factor $\exp(\frac{i}{\hbar} \int_{\gamma \in \Lambda} \alpha)$ where the path γ connects intersection points $\Lambda \cap \mathcal{L}_1$ and $\Lambda \cap \mathcal{L}_2$ in Λ . From the spectral point of view this is just the multiplication of eigenfunctions by constants. However, it makes semiclassical scalar products invariant with respect to automorphisms of the prequantization line bundle.

³Here we assume that paths are contained in a neighborhood where the bundle is trivialized and α is a 1-form on M . Otherwise we should lift paths to the prequantization bundle. The result gives the parallel transport in the prequantization line bundle which is consistent with the fact that globally, $\psi_{b_a}^{(a)}$ are not functions in q and b_a , but sections of the prequantization line bundle. Globally, the exponent in the formula (1) is also not a function in b_1 and b_2 , but a section of a line bundle.

Changing $\alpha \mapsto \alpha + df$ is an automorphism of the prequantization line bundle. It corresponds to the shift $-i\hbar \frac{\partial}{\partial q} \mapsto i\hbar \frac{\partial}{\partial q} + \frac{\partial f}{\partial q}$ in the deformation quantization of $C(T^*Q)$ by differential operators. In (1) it results in the phase change $(\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)}) \mapsto (\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)}) \exp(\frac{i}{\hbar} (f(x_1) - f(x_2)))$.

A natural object which does not depend on the choice of Λ is the transition probability, i.e. the square of the absolute value of the scalar product:

$$(2) \quad |(\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)})|^2 = \frac{1}{2\pi h^n} \sum_{a,c \in \mathcal{L}_{b_1}^{(1)} \cap \mathcal{L}_{b_2}^{(2)}} \left| \det \left(\frac{\partial^2 S_\gamma(c, b_1, b_2)}{\partial b_1^i \partial b_2^j} \right) \det \left(\frac{\partial^2 S_\gamma(a, b_1, b_2)}{\partial b_1^i \partial b_2^j} \right) \right|^{1/2} \\ \exp \left(\frac{i}{h} S_{\gamma_1, \gamma_2}(b_1, b_2) + \frac{i\pi}{2} (\mu_{\gamma_1, \gamma_2}(c) - \mu_{\gamma_1, \gamma_2}(a)) \right) (1 + O(h)) |db_1 db_2|$$

Here γ_1 is a path in $\mathcal{L}_{b_1}^{(1)}$ connecting points a and c and γ_2 is a path in $\mathcal{L}_{b_2}^{(2)}$ connecting these points. The exponent $S_{\gamma_1, \gamma_2}(b_1, b_2) = \int_{D_{a,c}} \omega$, where $D_{a,c}$ is a disc bounded by γ_1 and γ_2 , does not depend on the choice of paths if Bohr-Sommerfeld quantization conditions hold. The Hessian also does not depend on the choice of γ 's.

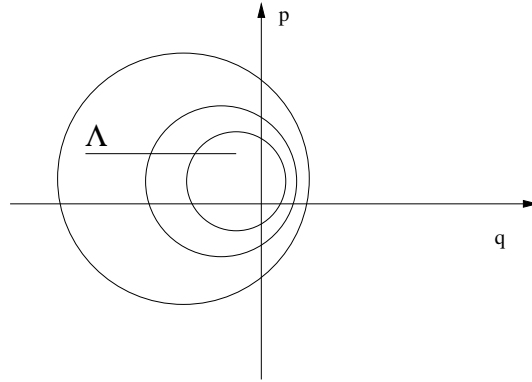


FIGURE 1. Choice of reference points corresponding to an integrable system on $\mathbb{R}^2 = T^*\mathbb{R}$ with $\alpha = pdq$ and the standard Lagrangian fibration $T^*\mathbb{R} \rightarrow \mathbb{R}$. In this case $\Lambda = \{p = 0\}$. Locally, Λ is a section of the Lagrangian fibration given by level curves of the Hamiltonian.

More generally, if $\pi_a : M \rightarrow B_a$ are Lagrangian fibrations, the cyclic product

$$(\psi_{b_1}^{(1)}, \psi_{b_k}^{(k)}) \cdots (\psi_{b_3}^{(3)}, \psi_{b_2}^{(2)}) (\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)})$$

does not depend on the choice of $U(1)$ -ambiguities of normalized eigenfunctions. We will call such products *cyclic amplitudes*. For the semiclassical asymptotic of a cyclic amplitude we have:

$$(3) \quad (\psi_{b_1}^{(1)}, \psi_{b_k}^{(k)}) \cdots (\psi_{b_3}^{(3)}, \psi_{b_2}^{(2)}) (\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)}) = \\ \frac{1}{2\pi h^{kn/2}} \sum_{c_1 \in \mathcal{L}_{b_1}^{(1)} \cap \mathcal{L}_{b_2}^{(2)}, \dots, c_k \in \mathcal{L}_{b_k}^{(k)} \cap \mathcal{L}_{b_1}^{(1)}} \left| \prod_{a=1}^k \det \left(\frac{\partial^2 S}{\partial b_a^i \partial b_{a+1}^j} \right) \right|^{1/2} \\ \exp \left(\frac{i}{h} S(c_1, \dots, c_k) + \frac{i\pi}{2} \mu \right) (1 + O(h)) \prod_{a=1}^k |db_a|$$

Here $S(c_1, \dots, c_k) = \sum_{a=1}^k \int_{\gamma_{a,a+1}} \alpha|_{\mathcal{L}_{b_a}^{(a)}}$ where $\gamma_{a,a+1}$ is a path in $\mathcal{L}_{b_a}^{(a)}$ connecting c_a and c_{a+1} and μ is the corresponding Maslov index. The exponent does not depend on the choice of γ 's if Bohr-Sommerfeld conditions hold. By the Stokes theorem it is clear that $S(c_1, \dots, c_k) = \int_D \omega$ where D is a disk bounded by paths $\gamma_{a,a+1}$.

1.2. Topological quantum mechanics.

1.2.1. *Scalar product as a path integral for topological quantum mechanics.* We want to write the formula (1) as the semiclassical expansion of a path integral. The natural candidate is the topological quantum mechanics

$$(4) \quad (\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)}) = C \int_{\gamma(0) \in \mathcal{L}_1, \gamma(1) \in \mathcal{L}_2} e^{\frac{i}{\hbar} (J_\gamma \alpha + f_{b_1}^{(1)}(\gamma(0)) - f_{b_2}^{(2)}(\gamma(1)) + \int_{\gamma_0 \subset \Lambda} \alpha)} D\gamma$$

Here γ_0 connects reference points x_1 and x_2 ⁴ and $\mathcal{L}_a = \mathcal{L}_{b_a}^{(a)}$ where $a = 1, 2$ and $f_{b_a}^a$ are boundary corrections with defining property

$$(5) \quad df_{b_a}^{(a)} = \iota_a^*(\alpha)$$

here $\iota_a : \mathcal{L}_a \rightarrow M$ are embeddings of Lagrangian submanifolds to M . These functions can be chosen as follows. For $x_a \in \mathcal{L}_a$ fix a path γ_a on \mathcal{L}_a connecting x with a reference point $x_a \in \mathcal{L}_a$. Define

$$f_{b_a}^{(a)}(x) = \int_{\gamma_a} \alpha$$

This function satisfies (5) and the exponent in (4) does not depend on γ_a if Bohr-Sommerfeld quantization conditions hold. An arbitrary constant C changes by the factor $\exp(\frac{i}{\hbar} (\int_{x_1}^{y_1} \alpha - \int_{x_2}^{y_2} \alpha))$ when we change reference points from x to y .

This integral should be understood semiclassically, as a formal power series supported at critical points of the exponent. Changing the reference points x_a changes the integral by a constant factor. A consistent choice of reference points on fibers of Lagrangian fibrations $\pi_a : M \rightarrow B_a$ is given a Lagrangian submanifold Λ intersecting transversally each fiber.

It is easy to see that critical points are exactly the intersection points $\mathcal{L}_1 \cap \mathcal{L}_2$. Indeed if we choose local coordinates x^i so that $\alpha = \sum_i \alpha_i(x) dx^i$ and choose a parametrization of γ , we will have the following formula for the variation of the integral in the exponent:

$$\delta \int_\gamma \alpha = \int_0^1 \sum_{ij} \left(\frac{\alpha_i}{\partial x^j} - \frac{\alpha_j}{\partial x^i} \right) \delta x^j(t) \frac{dx^i}{dt}(t) dt + \sum_i (\alpha_i(x(1)) \delta x^i(1) - \alpha_i(x(0)) \delta x^i(0))$$

This implies that $\frac{dx^i}{dt} = 0$, i.e. critical points are constant trajectories and therefore, taking into account boundary conditions for γ , we conclude that the critical points of the exponent in (4) are the intersection points $\mathcal{L}_1 \cap \mathcal{L}_2$. However, to develop the perturbation theory for such path integrals it is convenient to reformulate them in terms of two dimensional topological field theory. So, the next step is to reformulate the integral (4) as the partition function for the Poisson sigma model with special boundary conditions. We will do this in the section 2.

1.2.2. *Transition probabilities and cyclic amplitudes.* Transition probabilities can also be written in terms of topological quantum mechanics. They do not depend on the normalization of eigenfunctions. Semiclassically this leads to an unambiguous semiclassical formula

⁴The last summand in the exponent is making it to be invariant with respect to automorphisms of the prequantization line bundle. This means that the integral and the expression (1) with this factor depend only on the curvature $\omega = d\alpha$ of α .

(2) for these probabilities which does not require a choice of reference points. It is natural to expect the following path integral for (2):

(6)

$$|(\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)})|^2 = \int_{\gamma_1(0), \gamma_2(0) \in \mathcal{L}_1; \gamma_1(1), \gamma_2(1) \in \mathcal{L}_2} \exp\left(\frac{i}{\hbar} \left(\int_{\gamma_1} \alpha - \int_{\gamma_2} \alpha + \int_{\sigma_1} \alpha - \int_{\sigma_2} \alpha \right)\right) D\gamma D\sigma$$

Here path σ_1 connecting $\gamma_1(0)$ and $\gamma_2(0)$ in \mathcal{L}_1 and σ_2 connecting $\gamma_1(1)$ and $\gamma_2(1)$ in \mathcal{L}_2 . As for the amplitudes, this path integral should be understood semiclassically. We argue that it reproduces the formula (2).

Similarly we expect that the cyclic amplitudes can be written as the path integral

$$(7) \quad (\psi_{b_1}^{(1)}, \psi_{b_k}^{(k)}) \dots (\psi_{b_3}^{(3)}, \psi_{b_2}^{(2)}) (\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)}) = \int \exp\left(\frac{i}{\hbar} \sum_{a=1}^k \left(\int_{\gamma_a} \alpha + \int_{\sigma_a} \alpha \right)\right) \prod_{a=1}^k D\gamma_a D\sigma_a$$

Here $\gamma_a : I = [0, 1] \rightarrow M$, such that $\gamma_a(0) \in \mathcal{L}_a$, $\gamma_a(1) \in \mathcal{L}_{a+1}$ and σ_a is a path in \mathcal{L}_a such that $\sigma_a(0) = \gamma_{a-1}(1)$, $\sigma_a(1) = \gamma_a(0)$.

1.2.3. Two-dimensional version of the path integral for the topological quantum mechanics.

Here we reformulate the topological classical mechanics as a two-dimensional topological field theory. As before, M is a symplectic manifold with a prequantization line bundle with connection α such that $\omega = d\alpha$. Consider the classical field theory on a disk D with fields $X : D \rightarrow M$ and with the action functional⁵

$$S[X] = \frac{1}{2} \int_D X^*(\omega) = \frac{1}{2} \int_D \omega_{ij}(X) dX^i \wedge dX^j$$

Using Stokes theorem we can write it as:

$$\int_D X^*(d\alpha) = \int_{\partial D} X^*(\alpha),$$

where α is the prequantization connection. Now let us organize the boundary conditions in such a way that the integral over the boundary ∂D would give the exponent in (4). We assume that Λ is a Lagrangian submanifold defining the reference points. It should be transversal to \mathcal{L}_{b_1} and \mathcal{L}_{b_2} for generic (b_1, b_2) .

Assume the boundary is partitioned in four intervals $\partial D = I \sqcup I_1 \sqcup I_2 \sqcup I_{12}$ with no restriction on X on I and

$$X|_{I_1} \in \mathcal{L}_1, \quad X|_{I_2} \in \mathcal{L}_2, \quad X|_{I_{12}} \in \Lambda$$

Then the integral over the boundary can be written as

$$\int_{\partial D} \alpha = \int_I \alpha + \int_{I_1} \alpha + \int_{I_2} \alpha + \int_{I_{12}} \alpha = \int_I \alpha + f_{b_1}^{(1)}(\gamma(0)) - f_{b_2}^{(2)}(\gamma(1)) + \int_{I_{12}} \alpha$$

where f_1 and f_2 are generating functions for the connection α restricted to the Lagrangian submanifolds \mathcal{L}_1 and \mathcal{L}_2 respectively as in (4), with reference points $x_i = \mathcal{L}_i \cap \Lambda$. The last term corresponds to an arbitrary constant C in (5). Denote by $\gamma = \{X(s) \in M | s \in I\}$ the parameterized path in M corresponding to $I \subset \partial D$, then we can write

$$\int_I X^*(\alpha) = \int_{\gamma} \alpha, \quad f_{b_i}^{(i)}(x) = \int_{\gamma_{x, x_i}} \alpha,$$

⁵This action is also the topological part of the A-model [15].

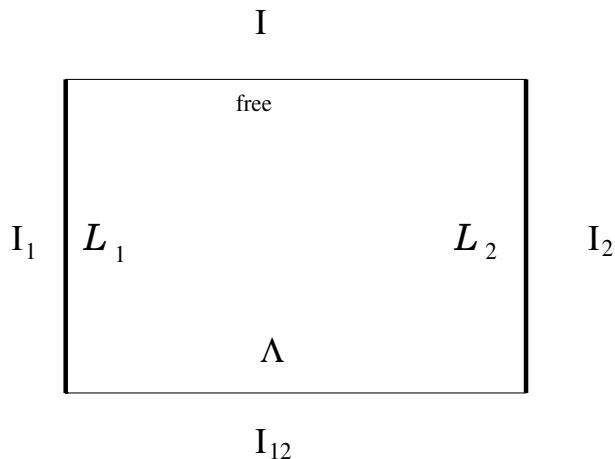


FIGURE 2. Partitioning the boundary of a disk and the boundary conditions for the field X corresponding to amplitudes in topological quantum mechanics.

Here $x \in \mathcal{L}_i$, $x_i = \mathcal{L}_i \cap \Lambda$.

Critical points of $S[X]$ are mappings which are constant on I and satisfy the boundary conditions on the other parts of ∂D .

The path integral representing the scalar product (4) can be written then up to (an infinite) constant as

$$(8) \quad (\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)}) = \int_{X:D \rightarrow M} e^{\frac{i}{\hbar} \int_D X^*(\omega)} DX$$

with the boundary conditions described above.

This path integral should be treated as a semiclassical path integral given by oscillatory exponential contributions and power series with coefficients being Feynman diagrams. It is a second order topological quantum field theory. To deal with the boundary conditions using the Hamiltonian framework, we will reformulate it as the first order Poisson sigma model.

Note that we expect to reproduce the formula (1) where we initially fixed the reference points $x_a \in \mathcal{L}_{b_a} \cap \Lambda$. It means that we should fix $X(u_a) = x_a$, where $u_a = I_a \cap I_{12}$, in the formula (8). The path integral then is defined in terms of Feynman diagram contributions for each critical point of the action with fixed x_1 and x_2 . Varying x_1 and x_2 results in an overall $U(1)$ factor and does not change the Feynman diagram contributions.

Similarly the path integral for the topological quantum mechanics describing the cyclic amplitudes can be written as

$$(9) \quad (\psi_{b_1}^{(1)}, \psi_{b_k}^{(k)}) \dots (\psi_{b_3}^{(3)}, \psi_{b_2}^{(2)}) (\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)}) = \int_{X:D \rightarrow M} e^{\frac{i}{\hbar} \int_D \omega} DX$$

where the boundary of the disc is partitioned to intervals I_1, \dots, I_k and J_1, \dots, J_k , see Fig.3 with boundary conditions $X|_{I_a} \in \mathcal{L}_a$ and X is free on intervals J_a .

2. TOPOLOGICAL QUANTUM MECHANICS AS THE POISSON SIGMA MODEL

2.1. Classical Poisson sigma model corresponding to topological quantum mechanics.

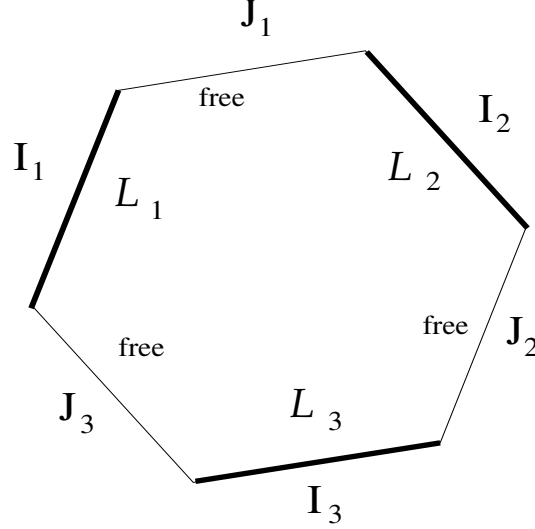


FIGURE 3. A partitioning of the boundary of a disk and the boundary conditions for the X -field corresponding to cyclic amplitudes, $k = 3$.

2.1.1. *Poisson sigma model and boundary conditions.* Recall that in the Poisson sigma model, the fields are pairs (X, η) representing components of the mapping of bundles $(X, \eta) : TD \rightarrow T^*M$. Because the main difficulty of the problem is not in the geometric structure of M , we assume that $M = \mathbb{R}^N$ for some even N , equipped with possibly nonconstant symplectic structure. Then $X : D \rightarrow \mathbb{R}^N$ and $\eta \in \Omega^1(D) \otimes \mathbb{R}^N$. Here we have identified $\mathbb{R}^N \simeq (\mathbb{R}^N)^*$. We will denote the components of these fields $X^i(u) \in \Omega^0(D)$ and $\eta_j(u) \in \Omega^1(D)$.

The action is:

$$S[X, \eta] = \int_D (\eta_i \wedge dX^i + \frac{1}{2} \omega^{ij}(X) \eta_i \wedge \eta_j)$$

Here $\omega^{ij}(X)$ is the Poisson tensor corresponding to ω_{ij} , i.e. the inverse matrix to $\omega_{ij}(X)$. We will follow the agreement of summing up over repeated contravariant and covariant indices.

As we will see below, the following boundary conditions for the field (X, η) correspond to the amplitudes (4) in topological quantum mechanics:

- For $s \in I$, $(X, \eta) \in M \subset T^*M$ where M is regarded as a zero-section Lagrangian submanifold of T^*M , i.e. no condition on X and $\eta|_I = 0$.
- For $s \in I_1$, $X(s) \in \mathcal{L}_1$ and $\eta(s) \in N_{X(s)}^* \mathcal{L}_1 \otimes T_s^* D$ where $N^* \mathcal{L}_1$ is the conormal bundle to \mathcal{L}_1 . Its fiber over $X(s)$ is the Lagrangian subspace in $T_{X(s)}^* M$, defined as $\{\eta(s) \in T_{X(s)}^* M \mid \langle a, \eta(s) \rangle|_{I_1} = 0\}$ for all $a \in T_{X(s)} \mathcal{L}_1$. By restriction of a form to the boundary we always mean the pull-back. Here $\langle -, - \rangle$ is the pairing of tangent and cotangent vectors,
- Similarly, for $s \in I_2$, $X(s) \in \mathcal{L}_2$ and $\eta(s) \in N_{X(s)}^* \mathcal{L}_2 \otimes T_s^* D$.
- For $s \in I_{12}$, $X(s) \in \Lambda$ and $\eta(s) \in N_{X(s)}^* \Lambda \otimes T_s^* D$.

2.1.2. *The relation to topological quantum mechanics.* The topological sigma model described above is related to the second order topological field theory introduced earlier as follows. Consider the critical point of $S[X, \eta]$ in η . The functional is quadratic in η , therefore such a critical point is unique and it is given by

$$(10) \quad \bar{\eta}_i = -\omega_{ij} dX^j$$

Let V be an isotropic submanifold M . It is clear that if the boundary condition for X is chosen in such a way that $X(s) \in V$ when $s \in I \subset \partial D$ then $\bar{\eta}(s) \in N^*V$ when $s \in I$. Thus, this critical point agrees with boundary conditions above. Computing the action of the Poisson sigma model at the critical point for η we recover the second order action for the X -field:

$$S[X, \bar{\eta}] = \frac{1}{2} \int_D \omega_{ij}(X) dX^i \wedge dX^j$$

Now let us describe the critical points of the action of Poisson sigma model in both X and η . The Euler-Lagrange equations are

$$dX^i + \omega^{ij}(X)\eta_j = 0, \quad d\eta_i + \frac{1}{2}\partial_i\omega^{kl}\eta_k \wedge \eta_l = 0$$

It is easy to see that the first equation implies the second one. Note that here we used the invertibility of the Poisson tensor, i.e. the fact that the Poisson sigma model has a symplectic manifold as the target space.

If Euler-Lagrange equations hold, the variation of the action is given only by boundary terms:

$$\delta S[X, \eta]_{\text{boundary}} = \int_{\partial D} \eta_i \delta X^i$$

This integral vanishes because of the boundary conditions. Indeed, $\delta X|_{I_a} \in T_X \mathcal{L}_a$, $\eta|_{I_a} \in N_X^* \mathcal{L}_a$, which implies $\eta_i \delta X^i|_{I_a} = 0$, similarly for I_{12} with \mathcal{L}_a being replaced by Λ , and on I the boundary term vanishes because $\eta = 0$.

Thus, the critical points are given by (10) with an arbitrary smooth function $X : D \rightarrow M$ satisfying the boundary conditions described above. Boundary conditions on η , together with the Euler-Lagrange equations, imply

$$X(s) = c, \quad s \in I, \quad c \in \mathcal{L}_1 \cap \mathcal{L}_2$$

Thus, a critical point of $S[X, \eta]$ is a pair $(\bar{X}, \bar{\eta})$ where $\bar{\eta}$ is given by (10) and \bar{X} is any smooth function on D satisfying the boundary conditions and $X|_I = c, c \in \mathcal{L}_1 \cap \mathcal{L}_2$.

2.1.3. Gauge invariance. The Poisson sigma model is gauge invariant with respect to the following infinitesimal gauge transformations, i.e. vector fields on the space of fields

$$\delta_\beta X^i = \omega^{ij} \beta_j, \quad \delta_\beta \eta_i = -d\beta_i - \partial_i \omega^{jk} \eta_j \beta_k$$

Here $\beta \in \Omega^0(D) \otimes \mathbb{R}^N$ (in general $\beta(u) \in T_{X(u)}^* M$). The Lie subalgebra with $\beta|_I = 0$, $\beta(s) \in N_{X(s)}^* \mathcal{L}_1$ when $s \in I_1$, $\beta(s) \in N_{X(s)}^* \mathcal{L}_2$ when $s \in I_2$ and $\beta(s) \in N_{X(s)}^* \Lambda$ when $s \in I_{12}$ preserves the boundary conditions corresponding to the topological quantum mechanics.

The gauge symmetry of the Poisson sigma model is directly related to the reparametrization invariance of topological quantum mechanics.

2.2. Amplitudes. Thus, up to constants we expect the following formula for scalar products (amplitudes) in terms of the Poisson sigma model:

$$(\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)}) = \int_{(X, \eta): D \rightarrow T^*M} e^{\frac{i}{\hbar} S(X, \eta)} DX D\eta$$

Here we assume the boundary conditions described in section 2.1.1 and the path integral should be understood semiclassically, as a formal power series of Feynman diagrams. Because the action is gauge invariant and therefore highly degenerate, to make sense of such a path

integral we should do the gauge fixing. We will do it later using Lorenz gauge fixing and an extra gauge fixing on the boundary.

As it was discussed before, changing the auxiliary Lagrangian submanifold Λ will only produce a constant $U(1)$ -factor.

2.3. Cyclic amplitudes. We can do the same to the path integral describing cyclic amplitudes as we did for amplitudes. This gives the following representation of cyclic amplitudes in terms of the Poisson sigma model:

$$(11) \quad (\psi_{b_1}^{(1)}, \psi_{b_k}^{(k)}) \cdots (\psi_{b_3}^{(3)}, \psi_{b_2}^{(2)}) (\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)}) = \int_{(X,\eta):D \rightarrow T^*M} e^{\frac{i}{\hbar} S(X,\eta)} DX D\eta$$

Here we impose boundary conditions

- $\eta = 0$ and X is not fixed on J_a ,
- $X(s) \in \mathcal{L}_a$ and $\eta(s) \in N_{X(s)}^* \mathcal{L}_a \otimes T_s^* I_a$, when $s \in I_a$.

The natural question about the formula (11) is why does the path integral factorize into the product of path integrals describing the amplitudes? Roughly speaking, this happens because the field theory is topological. This implies that we can deform the disc on Fig.3 into a semi-infinite strip with the side corresponding to the auxiliary Lagrangian Λ (which we have to choose) at infinity. In terms of the gauge fixing this means choosing a very special metric on D . In this limit the critical points are simply $X(u) = c \in \mathcal{L}_1 \cap \mathcal{L}_2$ and $\eta = 0$. Feynman diagrams vanish if any of their edges extend to infinity. Thus, all Feynman diagrams are supported at the end of the semi-infinite strip near the boundary segment I .

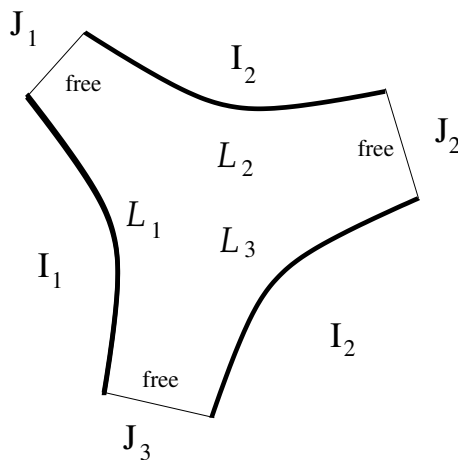


FIGURE 4. Deformation of the disc into a star with infinitely long tentacles for $k = 3$.

For the same reasons, a disc with alternating boundary conditions (corresponding to cyclic amplitudes) can be deformed into a “star” with semi-infinite “tentacles”, as it is shown on Fig. 4. In this limit critical points can be chosen as constant maps $X(u) = c \in \mathcal{L}_a \cap \mathcal{L}_{a+1}$ near the end of the star’s tentacles. Also, in this limit only the Feynman diagrams localized near the ends of the star will contribute and these contributions are exactly the amplitudes of the topological quantum mechanics discussed above. Thus, in this limit the path integral (11) factorizes as expected. But since the field theory is topological, this factorization is exact.

This is the heuristics of the factorization. The factorization in terms of Feynman diagrams involves direct manipulations with the diagrams (as in case of the associativity of the star product in [11]) and will be done in a separate publication.

3. CLASSICAL BV EXTENDED POISSON SIGMA MODEL AND GAUGE FIXING

3.1. Classical BV-BFV Poisson sigma model. Recall that the BV extended Poisson sigma model is an AKSZ model where fields on D which can be naturally organized in supermultiplets [2]:

$$\tilde{X}^i = X^i + \eta^{\dagger,i} + \beta^{\dagger,i}, \quad \tilde{\eta}_i = \beta_i + \eta_i + X_i^\dagger$$

The space of fields \mathcal{F}_D on the bulk D is \mathbb{Z} -graded by the ghost number: $gh(X) = gh(\eta) = 0$, $gh(\eta^\dagger) = gh(X^\dagger) = -1$, $gh(\beta) = 1$, $gh(\beta^\dagger) = -2$. The fields are differential forms $X, \beta \in \Omega^0(D, \mathbb{R}^N)$, $\eta, \eta^\dagger \in \Omega^1(D, \mathbb{R}^N)$ and $\beta^\dagger, X^\dagger \in \Omega^2(D, \mathbb{R}^N)$. The space of fields $\mathcal{F}_{\partial D}$ on the boundary ∂D consists of pull backs of $X, \eta, \eta^\dagger, \beta$.

The space of fields on D can be naturally regarded as a shifted cotangent bundle to the space of fields X, η and β .

Both \mathcal{F}_D and $\mathcal{F}_{\partial D}$ have natural symplectic structures:

$$\begin{aligned} \omega_D &= \int_D \delta X_i^\dagger \wedge \delta X^i + \int_D \delta \eta^{\dagger,i} \wedge \delta \eta_i + \int_D \delta \beta^{\dagger,i} \wedge \delta \beta_i \\ \omega_{\partial D} &= \int_{\partial D} \delta \eta_i \wedge \delta X^i + \int_D \delta \beta_i \wedge \delta \eta^{\dagger,i} \end{aligned}$$

The action functional of the BV extended Poisson sigma model is

$$(12) \quad S_D = \int_D (\tilde{\eta}_i \wedge d\tilde{X}^i + \frac{1}{2} \omega^{ij}(\tilde{X}) \tilde{\eta}_i \wedge \tilde{\eta}_j)$$

The boundary action

$$S_{\partial D} = \int_{\partial D} (\tilde{\eta}_i \wedge d\tilde{X}^i + \frac{1}{2} \omega^{ij}(\tilde{X}) \tilde{\eta}_i \wedge \tilde{\eta}_j) = \int_{\partial D} \beta_i dX^i + \omega^{ij}(X) \beta_i \eta_j + \frac{1}{2} \partial_k \omega^{ij}(X) \eta^{\dagger,k} \beta_i \beta_j$$

together with cohomological vector fields

$$Q_D = \int_D (d\tilde{X}^i + \omega^{ij}(\tilde{X}) \tilde{\eta}_j) \wedge \frac{\delta}{\delta \tilde{X}^i} + \int_D (d\tilde{\eta}_i + \frac{1}{2} \partial_i \omega^{kl}(\tilde{X}) \tilde{\eta}_k \wedge \tilde{\eta}_l) \wedge \frac{\delta}{\delta \tilde{\eta}_i}$$

and

$$Q_{\partial D} F = \{S_{\partial D}, F\}_{\partial D}$$

defines the BV-BFV structure of the models [5].

Now let us discuss boundary conditions for the BV extended model.

3.2. Boundary conditions corresponding to the topological quantum mechanics.

Boundary conditions for BV extended field theories are Q_{∂} -invariant Lagrangian submanifolds in the space of boundary BFV fields [5].

Boundary conditions corresponding to amplitudes in topological quantum mechanics are:

- On I , no conditions on \tilde{X} and $\tilde{\eta} = 0$, i.e. fields are in the zero section Lagrangian.
- When $s \in I_1$, $X(s) \in \mathcal{L}_1$, $\eta^\dagger(s)|_{I_1} \in T_{X(s)}[-1]\mathcal{L}_1 \otimes T_s^* I_1$, $\eta(s)|_{I_1} \in N_{X(s)}^* \mathcal{L}_1 \otimes T_s^* I_1$, and $\beta(s)|_{I_1} \in N_{X(s)}^*[1]\mathcal{L}_1 \otimes T_s^* I_1$.
- When $s \in I_2$, $X(s) \in \mathcal{L}_2$, $\eta^\dagger(s)|_{I_2} \in T_{X(s)}[-1]\mathcal{L}_2 \otimes T_s^* I_2$, $\eta(s)|_{I_2} \in N_{X(s)}^* \mathcal{L}_2 \otimes T_s^* I_2$, and $\beta(s)|_{I_2} \in N_{X(s)}^*[1]\mathcal{L}_2 \otimes T_s^* I_2$.

- And similarly, when $s \in I_{12}$, $X(s) \in \Lambda$, $\eta^\dagger(s)|_{I_{12}} \in T_{X(s)}[-1]\Lambda \otimes T_s^*I_{12}$, $\eta(s)|_{I_{12}} \in N_{X(s)}^*\Lambda \otimes T_s^*I_{12}$, and $\beta(s)|_{I_{12}} \in N_{X(s)}^*[1]\Lambda \otimes T_s^*I_{12}$.

Here, as before $\mathcal{L}_a = \mathcal{L}_{b_a}^{(a)}$. Note that boundary conditions for β and η^\dagger do not follow from boundary conditions for X and η . Our choice of boundary conditions is such that the boundary Lagrangian submanifold is tangent to the vector field $Q_{\partial D}$. In other words gauge transformations preserve this boundary condition.

Boundary conditions corresponding to cyclic amplitudes on intervals J_a are the same as above on I , and on intervals I_a , are the same as above for I_1 , I_2 or I_{12} for corresponding Lagrangian submanifolds. For more details see [3][4].

3.3. Gauge fixing. Recall that in the BV framework gauge fixing means a choice of a Lagrangian submanifold in the odd-symplectic space of bulk fields. In case when the gauge symmetry is the result of a gauge group action, gauge fixing is a choice of a section of the projection from the space of fields to gauge classes of fields. This is equivalent to choosing a Lagrangian submanifold in a BV extended field theory. In the case of Poisson sigma model, where the gauge symmetry is not given by a group action, choosing such a Lagrangian submanifold is the analog of choosing a gauge fixing section.

We choose the Lorenz gauge fixing which means we fix the following Lagrangian submanifold in \mathcal{F}_D :

$$L_{gf} = \{X_i^\dagger = \beta^{\dagger,i} = 0, \quad \eta^{\dagger,i} = *d\gamma^i, \quad d^*\eta_i = 0\}$$

In particular, this means $d^*\tilde{\eta}_i = 0$ and $d^*\tilde{X}^i = 0$. Because D is simply-connected, the condition $d^*\eta_i = 0$ implies that $\eta_i = *df_i$ for some function f .

Let us check that this subspace with the boundary conditions corresponding to the topological quantum mechanics is indeed Lagrangian. Consider the restriction of the symplectic form

$$\omega_D = \int_D \delta X_i^\dagger \wedge \delta X^i + \int_D \delta \eta^{\dagger,i} \wedge \delta \eta_i + \int_D \delta \beta^{\dagger,i} \wedge \delta \beta_i$$

to L_{gf} . The first term is zero because $X^\dagger = 0$ on L_{gf} , the third term is zero because on this submanifold $\beta^\dagger = 0$. The second term restricted to L_{gf} can be written as :

$$\int_D \delta \eta^{\dagger,i} \wedge \delta \eta_i = \int_D *d\delta\gamma^i \wedge \delta \eta_i = - \int_D \delta\gamma^i d^* \delta \eta_i + \int_{\partial D} \delta\gamma^i * \delta \eta_i$$

The first integral is zero on L_{gf} . In order for the boundary term in this integral to vanish, one needs to put an extra boundary condition $\gamma(s) \in T_{X(s)}L_a$ for $s \in I_a$ and $\gamma = 0$ on I (where $\eta = 0$). In particular, this condition implies that the normal component of η^\dagger is in $T_{X(s)}L_a$ for $s \in I_a$ and vanishes for $s \in I^6$.

⁶In order for the space of fields, subject to the boundary conditions corresponding to topological quantum mechanics to be closed under d and d^* , one needs to refine (or “strictify”) the boundary conditions.

On I , one should impose the ultra-Dirichlet boundary condition for $\tilde{\eta}$ and ultra-Neumann for \tilde{X} , cf. [6], Appendix A.1 (see also Appendix A.3.2 in the same reference for the logic of strictification of the boundary conditions). To refine boundary conditions for $s \in I_a$ one should split T^*M into the direct sum of $N^*\mathcal{L}_a$ and a complementary vector space. This can be achieved by choosing an almost complex structure J on TM . The refined boundary conditions can be chosen then as even normal derivatives of $\tilde{X}|_{I_a}$ and odd normal derivatives of $(*\tilde{X})|_{I_a}$ are in $T\mathcal{L}_a$; odd normal derivatives of $\tilde{X}|_{I_a}$ and even normal derivatives of $(*\tilde{X})|_{I_a}$ are in $J(T\mathcal{L}_a)$; even normal derivatives of $\tilde{\eta}|_{I_a}$ and odd normal derivatives of $(*\tilde{\eta})|_{I_a}$ are in $N^*\mathcal{L}_a$; odd normal derivatives of $\tilde{\eta}|_{I_a}$ and even normal derivatives of $(*\tilde{\eta})|_{I_a}$ are in $J^*(N^*\mathcal{L}_a)$.

To be more precise, we need these boundary conditions for the fluctuations around classical solutions, but we will defer this discussion to the extended version of this note.

The gauge fixing submanifold L_{gf} is a Lagrangian submanifold of conormal type if we will impose the “strictification” of boundary conditions discussed above. One can define an equivalent gauge fixing using the “gauge fixing fermion”. For this one needs to introduce a quadruplet of auxiliary fields γ , $\bar{\nu}$ (Lagrange multiplier), γ^\dagger and ν^\dagger . Such extra fields are typical for the BV extended version of Faddeev-Popov gauge fixing with Lagrange multipliers. For details, see [2].

3.4. Solutions to the Euler-Lagrange equations. The gauge fixed action is the restriction of the action to L_{gf} which was computed in [2]:

$$S_{gf}[\tilde{X}, \tilde{\eta}] = \int_D (\eta_i \wedge dX^i + \frac{1}{2} \omega^{ij}(X) \eta_i \wedge \eta_j - \eta^{\dagger,i} \wedge (d\beta_i + \partial_i \omega^{kl}(X) \eta_k \beta_l) - \frac{1}{4} \eta^{\dagger,i} \wedge \eta^{\dagger,j} \partial_i \partial_j \omega^{kl} \beta_k \beta_l)$$

In this action $\eta_i = *df_i$ and $\eta^{\dagger,i} = *d\gamma^i$. For the variation we have:

$$(13) \quad \delta S_{gf}[\tilde{X}, \tilde{\eta}] = \text{bulk terms} + \int_D (\eta_i \delta X^i + \eta^{\dagger,i} \delta \beta)$$

As usual, it consists of two terms: the bulk term and the boundary term. The bulk terms vanishes if and only if the Euler-Lagrange equations hold:

$$\begin{aligned} d^*(dX^i + \omega^{ij}(X) \eta_j - \eta^{\dagger,l} \wedge \partial_l \omega^{ij}(X) \beta_j) &= 0, \\ d\eta_i + \frac{1}{2} \partial_i \omega^{kl} \eta_k \wedge \eta_l - \eta^{\dagger,l} \wedge \partial_l \partial_i \omega^{jk}(X) \eta_j \beta_k - \frac{1}{4} \eta^{\dagger,k} \wedge \eta^{\dagger,l} \partial_k \partial_l \partial_i \omega^{st} \beta_s \beta_t &= 0 \\ d^*(d\beta_i + \partial_i \omega^{jk}(X) \eta_j \beta_k + \frac{1}{2} \eta^{\dagger,j} \partial_i \partial_j \omega^{km}(X) \beta_k \beta_m) &= 0, \\ d\eta^{\dagger,i} + \eta^{\dagger,j} \wedge \partial_j \omega^{ki}(X) \eta_k + \frac{1}{2} \eta^{\dagger,l} \wedge \eta^{\dagger,j} \partial_l \partial_j \omega^{ki}(X) \beta_k &= 0 \end{aligned}$$

The boundary terms in (13) vanish because of the boundary conditions described earlier. These equations with boundary conditions described earlier still have infinitely many solutions because the gauge symmetry on the boundary is not yet fixed, see the discussion below.

3.5. The case of constant symplectic structure on \mathbb{R}^{2n} . From now on we will focus on the simplest but already non-trivial case when $M = \mathbb{R}^{2n}$ with constant symplectic structure. In this case the Euler-Lagrange equations become:

$$d^*(dX^i + \omega^{ij} \eta_j) = 0, \quad d\eta_j = 0, \quad d^*d\beta_i = 0, \quad d\eta^{\dagger,i} = 0$$

We also should take into account the gauge fixing conditions $\eta^{\dagger,i} = *d\gamma^i$, $d^*\eta = 0$ and the boundary conditions. This gives

$$(1) \quad d^*dX^i = 0, \quad X|_{I_\alpha} \in \mathcal{L}_\alpha, \quad X|_{I_{12}} \in \Lambda, \quad dX^i|_I = 0$$

Here and below $\alpha = 1, 2$.

(2) For η we have

$$d\eta = 0$$

with boundary conditions $\eta|_I = 0$, $\eta(s)|_{I_\alpha} \in N_{X(s)}^* \mathcal{L}_\alpha \otimes T_s^* I_\alpha$ when $s \in I_\alpha$ and $\eta(s)|_{I_{12}} \in N_{X(s)}^* \Lambda \otimes T_s^* I_{12}$ when $s \in I_{12}$.

(3) For β_i we have the linear boundary value problem

$$d^*d\beta_i = 0, \quad \beta_i|_I = 0,$$

with $\beta(s)|_{I_\alpha} \in N_{X(s)}^*\mathcal{L}_\alpha \otimes T_s^*I_\alpha$ when $s \in I_\alpha$, $\beta(s)|_{I_{12}} \in N_{X(s)}^*\Lambda \otimes T_s^*I_{12}$ when $s \in I_{12}$, and $\beta|_I = 0$.

(4) For fields γ^i we obtain linear problem:

$$d^*d\gamma^i = 0$$

with boundary conditions $*d\gamma(s)|_{I_\alpha} \in T_{X(s)}\mathcal{L}_\alpha \otimes T_s^*I_\alpha$ when $s \in I_\alpha$, $*d\gamma(s)|_{I_{12}} \in T_{X(s)}\Lambda \otimes T_s^*I_{12}$ when $s \in I_{12}$, and free boundary conditions on I .

Note that Euler-Lagrange equations are linear, but boundary conditions are not. It is clear that these boundary conditions are incomplete: we still have infinitely many solutions to Euler-Lagrange equations with these boundary conditions. This is due to remaining gauge symmetry on the boundary. To observe the remaining gauge symmetry let us focus on fields with $gh = 0$, i.e. on X and η . When the symplectic structure is constant, infinitesimal gauge transformations are

$$\delta X^i = \omega^{ij}\beta_j, \quad \delta\eta_i = -d\beta_i$$

Such a vector field is parallel (tangent) to the Lorenz gauge fixing section $d*\eta = 0$ only if $d^*d\beta_i = 0$, i.e. when β is harmonic. Boundary values of β are constrained by conditions on X . Since $X(s) \in \mathcal{L}_\alpha$ when $s \in I_\alpha$ we have $\delta X(s) \in T_{X(s)}\mathcal{L}_\alpha$ and therefore $\beta \in N_{X(s)}^*\mathcal{L}_\alpha$. This agrees with boundary conditions on η . However, there are infinitely many such harmonic functions. They form the remaining boundary gauge symmetry. We call it boundary gauge symmetry because a harmonic function is determined by its boundary value.

It is easy to see that the refined boundary conditions discussed in the previous section fix this boundary gauge symmetry.

4. QUANTIZATION OF THE BV EXTENDED POISSON SIGMA MODEL

Here we will focus on the Poisson sigma model with boundary conditions corresponding to amplitudes and cyclic amplitudes in the topological quantum mechanics.

According to the general philosophy of BV quantization, we should choose a gauge fixing Lagrangian submanifold in the space of bulk BV fields in such a way that the BV extended action has isolated critical points on this Lagrangian. Then the perturbative partition function is defined by the sum of corresponding Feynman diagrams. Thus, our main object, should be

$$\int_{L_{gf}} \exp\left(\frac{i}{\hbar} S_{gf}[\tilde{X}, \tilde{\eta}]\right) D\tilde{X} D\tilde{\eta}$$

with boundary conditions described above.

However, in our case Lorenz gauge fixing does not provide isolated critical points because of the remaining gauge transformations which are parallel to the Lagrangians \mathcal{L}_a (unless we consider refined boundary conditions). In addition to this, we have nonlinear boundary conditions. One way to tackle both problems is to treat the boundary conditions as boundary constraints using Lagrange multipliers.

The main idea in the rest of this section is to introduce Lagrange multipliers λ for the boundary constraints on X , keep X free on the boundary and impose boundary conditions

$\eta = 0^7$. The usual mechanism with Lagrange multipliers will constrain field X to \mathcal{L}_a on I_a . Lagrange multipliers λ parameterize the “true” boundary values of η . Thus, Lagrange multipliers will effectively change boundary condition $\eta = 0$ and free X on each I_a to the correct boundary condition corresponding to the Lagrangian submanifold \mathcal{L}_a . We conjecture that this gives correct Feynman diagram expansion in the path integral. Here we consider the case of the amplitude in the asymptotic metric, so $a = 1, 2$.

Assume that Lagrangian fibers $\mathcal{L}_{b_a}^{(a)}$ are the level surfaces of the Poisson commuting functions $H_\alpha^{(a)} \in C(M)$, where $\alpha = 1, \dots, \dim(M)/2$. This is a reasonable assumption in the context of integrable systems where these functions are commuting Hamiltonians.

In the BV framework we want to introduce an extra term \tilde{S}_∂ in the action with Lagrange multipliers and their BV counterparts in such a way that the total action satisfies the classical master equation. In order to do this, introduce boundary fields $\lambda_\alpha^a \in \Omega^1(I_a)$, $\lambda_\alpha^{a,\dagger} \in \Omega^0(I_a)[-1]$, $c_\alpha^a \in \Omega^0(I_a)[1]$, $c_\alpha^{a,\dagger} \in \Omega^1(I_a)[-2]$, $\bar{c}_\alpha^a \in \Omega^1(I_a)[-1]$, $\bar{c}_\alpha^{a,\dagger} \in \Omega^0(I_a)$, $\mu_\alpha^a \in \Omega^1(I_a)$, $\mu_\alpha^{a,\dagger} \in \Omega^0(I_a)[-1]$. Denote this space of fields $\tilde{F}_{\partial D}$. Naturally

$$\tilde{F}_{\partial D} = \oplus_a \tilde{F}_{I_a}$$

Each space \tilde{F}_{I_a} has a natural symplectic structure of degree -1 :

$$\tilde{\omega}_{I_a} = \int_{I_a} \sum_\alpha (\delta \lambda_\alpha^{a,\dagger} \wedge \delta \lambda_\alpha^a + \delta c_\alpha^{a,\dagger} \wedge \delta c_\alpha^a + \delta \bar{c}_\alpha^{a,\dagger} \wedge \delta \bar{c}_\alpha^a + \delta \mu_\alpha^{a,\dagger} \wedge \delta \mu_\alpha^a)$$

Note that the symplectic space $(\tilde{F}_{\partial D}, \tilde{\omega}_{\partial D})$ is not the space of BFV fields, but the space of BV extended Lagrange multipliers.

Denote by $F_D^{(0)}$ the space of bulk fields with boundary conditions $\eta|_{\partial D} = 0$ and $X|_{\partial D}$ is free and define the total space of fields as $\mathcal{F}_D = F_D^{(0)} \oplus \tilde{F}_{\partial D}$. This space has a natural symplectic form of degree -1 which is the direct sum of symplectic forms on $F_D^{(0)}$ and $\tilde{F}_{\partial D}$.

Define the boundary correction to the action as

$$\tilde{S}_\partial = \sum_a \sum_\alpha \int_{I_a} (\lambda_\alpha^a (H_a^{(\alpha)} - b_a^\alpha) + \lambda_\alpha^{a,\dagger} d c_\alpha^a + c_\alpha^a \partial_i H_a^{(\alpha)} \eta^{i,\dagger} + \mu_\alpha^{a,\dagger} \bar{c}_\alpha^a)$$

Again, this is not the BFV action, this is a BV extended term with Lagrange multiplier enforcing the boundary conditions on X .

It is easy to check that the modified action $S + \tilde{S}_\partial$ satisfies the classical master equation.

$$\{S + \tilde{S}_\partial, S + \tilde{S}_\partial\}_D = 0$$

where the Poisson bracket is defined by the symplectic structure on the total space of BV fields \mathcal{F}_D .

We will choose Lorenz gauge fixing in $F_D^{(0)}$, as in [2]. The additional gauge fixing for BV Lagrange multipliers can be chosen as a version of the Lorenz gauge fixing. Corresponding Lagrange submanifold in $\tilde{\mathcal{F}}_\partial$ is the graph of the function

$$\psi = \sum_a \int_{I_a} \sum_\alpha \bar{c}_\alpha^a d^* \lambda_\alpha^a$$

⁷To be more precise, we should do this only to fluctuations, but we will discuss this, as many other missing details in this note in its extended version.

The Lagrangian submanifold is defined as $a^\dagger = \frac{\delta\psi}{\delta a}$ where a are boundary fields which translates to:

$$\begin{aligned}\lambda_a^{\alpha,\dagger} &= -d^*\bar{c}_a^\alpha, & c_a^{\alpha,\dagger} &= 0 \\ \bar{c}_\alpha^{a,\dagger} &= d^*\lambda_\alpha^a, & \mu_\alpha^{a,\dagger} &= 0\end{aligned}$$

For the restriction of the boundary action to this Lagrangian submanifold we have

$$\tilde{S}_{\partial, gf} = \sum_a \sum_\alpha \int_{I_a} (\lambda_\alpha^a (H_a^{(\alpha)} - b_a^\alpha) - d^*\bar{c}_a^\alpha dc_a^\alpha + c_a^\alpha \partial_i H_a^{(\alpha)} \eta^{i,\dagger} + \mu_\alpha^a d^*\lambda_\alpha^a)$$

Note that if we ignore all boundary fields except λ , Euler-Lagrange equations for $S + \tilde{S}_{\partial D}$ induce Euler-Lagrange equations discussed before in the bulk. On the boundary, on $I_a \in \partial D$ we will have $H_a^{(\alpha)}(X) = b_a^\alpha$ which means $X \in \mathcal{L}_a$ and we will also have $\eta_i = \sum_\alpha \lambda_\alpha \partial_i H_a^{(\alpha)}$ which means $\eta(s) \in N_{X(s)}^* \mathcal{L}_a$.

4.1. Gluing. Two Lagrangian fibrations $\pi_a : M \rightarrow B_a$ on M define two real polarizations P_1, P_2 . Denote by $U_{P_2, P_1} : H_{P_1}^{1/2} \rightarrow H_{P_2}^{1/2}$ the integral operator with the integral kernel $(\psi_{b_2}^{(2)}, \psi_{b_1}^{(1)})$. Here by an integral operator we mean a formal integral operator where the integration is given by the formal power series in Feynman diagrams. This is a semiclassical version of the Blattner-Kostant-Sterberg kernel in geometric quantization.

Path integral arguments naturally lead to the composition rule

$$U_{P_2, P} * U_{P, P_1} = U_{P_2, P_1} \exp\left(\frac{i\pi}{4} \mu(P_1, P, P_2)\right)$$

where $\mu(P_1, P, P_2)$ is the Maslov index. Here the composition is the formal stationary phase integral over B given by contributions from critical points. We conjecture that this identity holds as an identity of formal powers series with coefficients being Feynman diagrams for the Poisson sigma model with appropriate boundary conditions. In terms of the path integral, it is a two-step argument. First because our field theory is topological, we can deform the gauge fixing choice of the metric to a metric where the interval I shrinks to a point. Then integrating over B allows all possible values of X on the interval I and thus, removes the constraint.

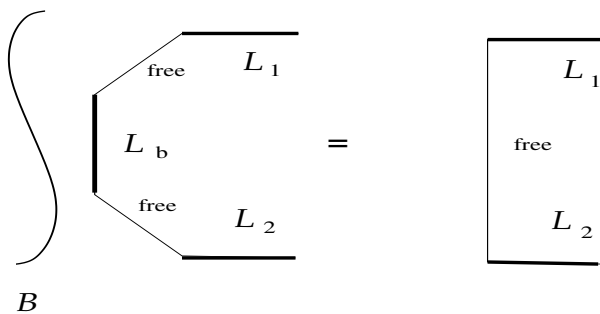


FIGURE 5. Composing two BKS kernels.

Pictorially this identity corresponds to gluing identity from Fig. 5.

5. GEOMETRIC REPRESENTATION OF KONTSEVICH'S STAR PRODUCTS FOR POISSON MANIFOLDS

Here we will denote by \mathcal{P} a Poisson manifold, i.e. a smooth manifold with a bivector field $p \in \Gamma(\wedge^2 T^* \mathcal{P})$ such that the bracket

$$\{f, g\} = p(df \wedge dg)$$

defines a Lie algebra structure on the space $C(\mathcal{P})$ of smooth functions on \mathcal{P} .

Recall that a star product of two smooth functions on \mathcal{P} is an associative product on $C(\mathcal{P})[[\hbar]]$ defined as

$$f * g = fg + \frac{i\hbar}{2} \{f, g\} + \sum_{n \geq 2} m_n(f, g) (i\hbar)^n$$

where coefficients m_n are bidifferential operators. The associativity of the star product is equivalent to certain bilinear identities for coefficients m_n . Kontsevich in [11] gave an explicit construction of such star product where the bidifferential operators m_n are described explicitly in terms of integrals on configuration spaces.

5.1. Kontsevich's star-product as Poisson sigma model. In this section, as above, we assume that D is an oriented disc. We think of D as a disc in \mathbb{R}^2 and assume the counter clock wise orientation.

The Poisson σ -model with boundary conditions $\eta = 0$ and X free [2] describes Kontsevich's star product [11] on the space of smooth functions on a Poisson manifold \mathcal{P} . Recall this construction. The star product of two smooth functions f and g can be regarded as the path integral

$$f * g = \int_{X(\bullet)=x} \exp\left(\frac{i}{\hbar} S[X, \eta]\right) f(X(0)) g(X(1)) DX D\eta$$

Here $\bullet, 0, 1$ are three points on the boundary of the disc (see Fig. 6). The integral is understood as a formal power series of Feynman integral contributions from fluctuations around the critical point $X(u) = x, \eta(u) = 0$. The description of Feynman diagrams in the Lorenz gauge in BV framework was derived in [2].

The associativity of this star product can be proven directly as in [11]. It has a simple heuristic path integral interpretation. The associativity of the star product can be interpreted as the computation of the path integral in two different ways as it is shown on Fig. 7 where all path integrals should be understood perturbatively, i.e. as formal series in \hbar with coefficients given by Feynman diagrams. Note that, of course, this path integral interpretation should be regarded only as a guideline for how to prove the identity between the Feynman diagrams, but not as a proof itself. The proof remains to be the direct proof given in [11].

Making formal deformations of ω in Kontsevich's star product one can obtain all non-equivalent $*$ -products. Changing coordinates on the target space by formal diffeomorphisms we obtain all equivalent star products.

5.2. Poisson sigma model with Lagrangian boundary conditions. Let $S \subset \mathcal{P}$ be a symplectic leaf. Assume we have three Lagrangian fibrations on S , $\pi : S \rightarrow B$ and $\pi_{1,2} : S \rightarrow B_{1,2}$.

Thus, as in the first section, we can define the vector space of $1/2$ -densities $H_P^{1/2}$ and two integrable systems corresponding to two other Lagrangian fibrations. We want to define the

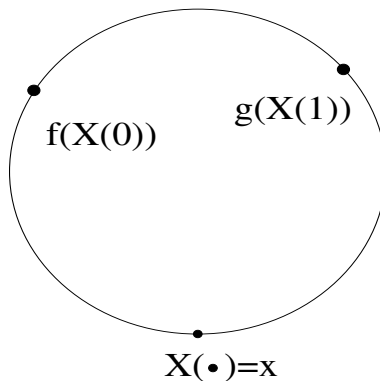


FIGURE 6. The disc with three points $\bullet, 0, 1$ on the boundary and insertions of functions f and g evaluated at $X(0)$ and $X(1)$ in the integral.

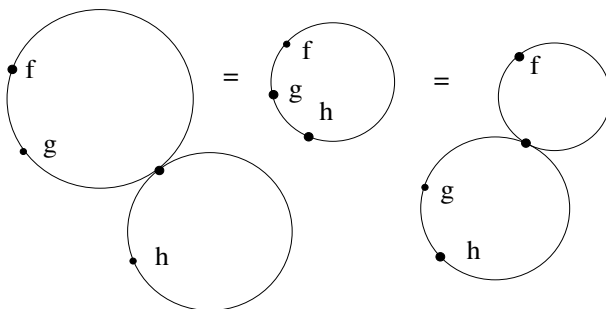


FIGURE 7. The path integral interpretation of the associativity.

action of the associative algebra $(C(\mathcal{P})[[\hbar]], *)$ on the vector space $H_P^{1/2}$. That is, to each $f \in C(\mathcal{P})[[\hbar]]$ we want to assign $\pi(f) : H_P^{1/2} \rightarrow H_P^{1/2}$, in such a way that $\pi(f * g) = \pi(f)\pi(g)$.

Define the matrix elements of $\pi(f)$ between semiclassical eigen-half-densities as the path integral

$$(14) \quad (\psi_{b_2}^{(2)}, \pi(f)\psi_{b_1}^{(1)}) = \int \exp\left(\frac{i}{\hbar}S[X, \eta]\right) f(X(\bullet)) DX D\eta$$

Here the integral is taken over fields of the Poisson sigma model with the target space \mathcal{P} with boundary conditions $X|_{I_a} \in \mathcal{L}_a \subset S \subset \mathcal{P}$, $\eta|_{I_a} \in N_X^* \mathcal{L}_a$, $a = 1, 2$ where $\mathcal{L}_a = \mathcal{L}_{b_a}^{(a)}$ are Lagrangian fibers of $\pi_{1,2}$. On the lower part of the boundary we have the same boundary condition with \mathcal{L}_a being replaced by Λ and on the upper part of the boundary conditions are $\eta = 0$ and X is any. These boundary conditions and the insertion of the function f are illustrated on Fig. 8.

Note that matrix elements (14) do not depend on the choice of α . In particular, for $M = T^*Q$ a combination of a formal diffeomorphism and a formal deformation of ω brings this star product to the one given by differential operators (the one we used in the first section).

Boundary conditions corresponding to the dual map $\pi(f)^*$ are shown on Fig. 9. It is natural to consider (14) as the kernel of an integral operator $\pi_{21}(f) : H_{P_1}^{1/2} \rightarrow H_{P_2}^{1/2}$. In particular, when $P_1 = P_2 = P$, $\pi(F)$ is an endomorphism of $H_P^{1/2}$. But in this case the

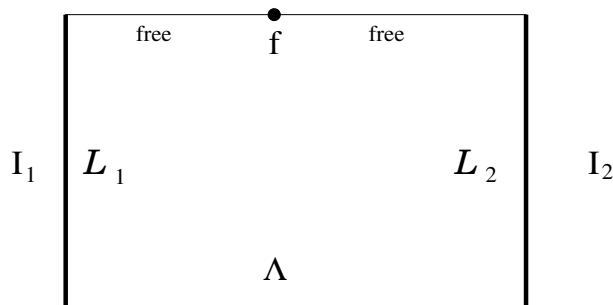


FIGURE 8. The boundary conditions for the path integral representing matrix elements $(\psi_{b_2}^{(2)}, \pi(f)\psi_{b_1}^{(1)})$.

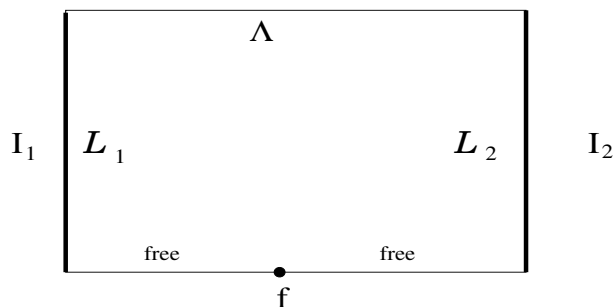


FIGURE 9. The boundary conditions for the path integral representing matrix elements $(\psi_{b_1}^{(1)}, \pi(f)^*\psi_{b_2}^{(2)}) = (\psi_{b_2}^{(2)}, \pi(f)\psi_{b_1}^{(1)})$.

semiclassical limit becomes quite singular. It is clear that the following identities hold:

$$U_{P_3P_2}\pi_{21}(f) = \pi_{31}(f) = \pi_{32}(f)U_{P_2P_1}$$

where $U_{P_2P_1}$ is the Blattner-Kostant-Sternberg kernel given by the scalar product which was the subject of previous sections. For the semiclassical asymptotic of (14) we have

$$(15) \quad \pi_{21}(f)(b_2, b_1) = \frac{C}{(2\pi\hbar)^{n/2}} \sum_{c \in \mathcal{L}_{b_1}^{(1)} \cap \mathcal{L}_{b_2}^{(2)}} e^{\frac{i}{\hbar} S_{\gamma_1, \gamma_2}(c, b_1, b_2) + \frac{i\pi}{2} \mu_{\gamma_1, \gamma_2}(c)} \left| \det \left(\frac{\partial^2 S_{\gamma_1, \gamma_2}(c, b_1, b_2)}{\partial b_1^i \partial b_2^j} \right) \right|^{1/2} f(c) (1 + O(\hbar)) \sqrt{|db_1 db_2|}$$

where everything has the same meaning as in (1).

The path integral arguments used in the composition law of amplitudes also imply the following important property of matrix elements (14). If f is constant on fibers $\mathcal{L}_{b_1}^{(1)}$ and g is constant on fibers $\mathcal{L}_{b_2}^{(2)}$ then

$$\pi_{21}(f)(b_2, b_1) = U_{P_2P_1}(b_2, b_1) f(b_1) \quad \pi_{21}(g)(b_2, b_1) = g(b_2) U_{P_2P_1}(b_2, b_1)$$

Now we should check the homomorphism property:

$$(16) \quad \pi_{21}(f * g)(b_2, b_1) = \int_B \pi_{20}(f)(b_2, b) \pi_{01}(g)(b, b_1)$$

where index 0 corresponds to the Lagrangian fibration P and $*$ is the Kontsevich's star product.

This identity has the following path integral interpretation. The partition function corresponding to Fig. 10 is equal to the partition function corresponding to Fig. 11 and Fig. 12. The arguments are the same as in case of the gluing properties of amplitudes.

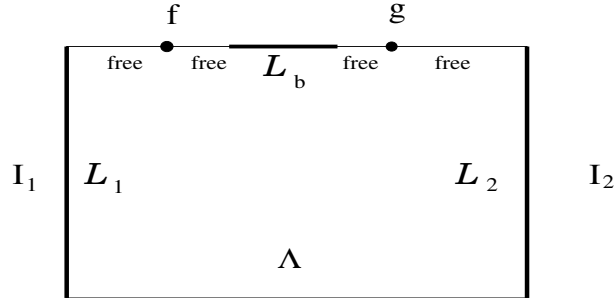


FIGURE 10. Boundary conditions for the path integral interpretation of the integrand in (16).

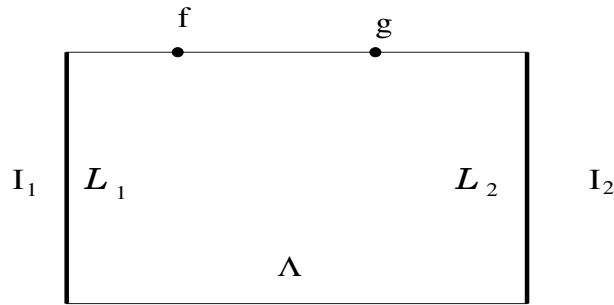


FIGURE 11. Boundary conditions for the path integral interpretation of the result of integration in the right side of (16).

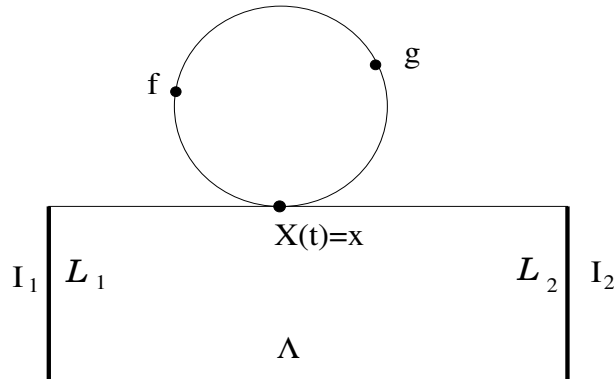


FIGURE 12. Boundary conditions for the path integral interpretation of the left side of (16).

According to the general philosophy of quantization of Poisson manifolds, symplectic leaves correspond to irreducible representations⁸. Thus, we expect that representations constructed in this way are irreducible representations of $C_h(\mathcal{P})$.

6. CONCLUSION

Throughout this paper we use real polarizations which are given by Lagrangian fibrations. Such polarizations have clear semiclassical meaning in terms real symplectic geometry and analytical mechanics.

However such polarizations are typically quite singular. Complex polarizations have richer mathematical structures, are less singular, allow tools from complex geometry, and in many important cases, from algebraic geometry. This is why complex polarizations are more in use in mathematical literature on geometric quantization. In physics literature, the corresponding representations of quantum observables are known as the holomorphic realization of quantum mechanics. Topological quantum mechanics in holomorphic representation is closely related the A-model in string theory [15] (see also [1]).

Note that the path integral formula for (1) suggests that the torsion for the Poisson sigma model on a disc with boundary conditions corresponding to TQM is given by the semiclassical Hessian in (1).

This article is only an outline. The detailed description of Feynman diagrams with proofs of basic statements will be done in a separate publication.

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⁸When \mathcal{P} is the dual space to a Lie algebra, this correspondence is at heart of the orbit method, which assigns irreducible representations of a Lie group to co-adjoint orbits. This is known as Kirillov-Kostant method.

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