GROUPOIDS AND POISSON SIGMA MODELS WITH BOUNDARY

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Abstract. This note gives an overview on the construction of symplectic groupoids as reduced phase spaces of Poisson sigma models and its generalization in the infinite dimensional setting (before reduction).

1. Introduction

Symplectic groupoids have been studied in detail since their introduction by Coste, Dazord and Weinstein [9] and they appear naturally in Poisson and symplectic geometry, as well as in some instances of the study of topological field theories. More precisely, in [4], it was proven that the reduced phase space of the Poisson sigma model under certain boundary conditions and assuming it is a smooth manifold, has the structure of a symplectic groupoid and it integrates the cotangent bundle of a given Poisson manifold M. This is a particular instance of the problem of integration of Lie algebroids, a generalized version of the **Lie third theorem**[12] . The general question can be stated as:

- Is there a Lie groupoid (G, M) such that its infinitesimal version corresponds to a given Lie algebroid (A, M)?

For the case where $A = T^*M$ and M is a Poisson manifold the answer is not positive in general, as there are topological obstructions encoded in what are called the monodromy groups [10]. A Poisson manifold is called integrable if such a Lie groupoid G exists. The properties of G are of special interest in Poisson geometry, since it is possible to equip G with a symplectic structure ω compatible with the multiplication map in such a way that G is a symplectic realization for (M, Π). For the integrable case, the symplectic groupoid integrating a given Poisson manifold (M, Π) is constructed explicitely in [4], as the phase space modulo gauge equivalence of the Poisson Sigma model (PSM), a 2-dimensional field theory.

In a more recent perspective (see [7, 8]), the study of the phase space before reduction plays a crucial role. This allows dealing with with nonintegrable Poisson structure, for which the reduced phase space is singular, on an equal footing as the integrable ones. This new approach differs from the stacky perspective of Zhu and Tseng (see [16]) and seems to be better adapted to symplectic geometry and to quantization.

In a paper in preparation [3], we introduce a more general version of a symplectic groupoid, called *relational symplectic groupoid*. In the case at hand, it

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corresponds to an infinite dimensional weakly symplectic manifold equipped with structure morphisms (canonical relations, i.e. immersed Lagrangian submanifolds) compatible with the Poisson structure of M. In this work, we prove that

(1) For any Poisson manifold M (integrable or not), the relational symplectic groupoid always exists.
(2) In the integrable case, the associated relational symplectic groupoid is equipped with a locally embedded Lagrangian submanifold.
(3) Conjecturally, given a regular relational symplectic groupoid $\mathcal{G}$ over M (a particular type of object that admits symplectic reduction), there exists a unique Poisson structure Π on M such that the symplectic structure ω on $\mathcal{G}$ and Π are compatible. This is still work in progress.

This paper is an overview of this construction and is organized as follows. Section 2 is a brief introduction to the Poisson sigma model and its reduced phase space. Section 3 deals with the version before reduction of the phase space and the introduction of the relational symplectic groupoid. An interesting issue concerning this construction is the treatment of non integrable Poisson manifolds: even if the reduction does not exists as a smooth manifold, the relational symplectic groupoid always exists. One natural question at this point is:

- Can there be a finite dimensional relational symplectic groupoid equivalent to the infinite dimensional one for an arbitrary Poisson manifold?

The answer to this question is work in progress and it will be treated in a subsequent paper. Section 4 contains some comments on the quantized version of the relational symplectic groupoid and its possible connection with geometric and deformation quantization.

Another aspect, which will be explored later, is the connection between the relational construction and the Poisson Sigma model with branes, where the boundary conditions are understood as choices of coisotropic submanifolds of the Poisson manifold. The relational symplectic groupoid seems to admit the existence of branes and would explain in full generality the idea of dual pairs in the Poisson sigma model with boundary [2, 5].
This new program might be useful for quantization as well. Using ideas from geometric quantization, what is expected as the quantization of the relational symplectic groupoid is an algebra with a special element, which fails to be a unit, but whose action is a projector in such a way that on the image of the projector we obtain a true unital algebra. Deformation quantization of a Poisson manifold could be interpreted in this way.

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2. PSM and its reduced phase space.

We consider the following data

(1) A compact surface Σ, possibly with boundary, called the source space.
(2) A finite dimensional Poisson manifold (M, Π), called the target space. Recall that a bivector field $\Pi \in \Gamma(TM \wedge TM)$ is called Poisson if the the

bracket $\{,\}: \mathcal{C}^\infty(M)\otimes\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$, defined by

$$\{f,g\}=\Pi(df,dg)$$

is a Lie bracket and it satisfies the Leibniz identity

$$\{f,gh\}=g\{f,h\}+h\{f,g\}, \forall f,g,h\in\mathcal{C}^\infty(M).$$

In local coordinates, the condition of a bivector Π to be Poisson reads as follows

$$\Pi^{sr}(x)(\partial_r)\Pi^{lk}(x)+\Pi^{kr}(x)(\partial_r)\Pi^{sl}(x)+\Pi^{lr}(x)(\partial_r)\Pi^{ks}(x)=0, \tag{1}$$

that is, the vanishing condition for the Schouten-Nijenhuis bracket of Π.

The space of fields for this theory is denoted with Φ and corresponds to the space of vector bundle morphisms between $T\Sigma$ and T^*M. This space can be parametrized by the pair (X,η), where X is a $\mathcal{C}^{k+1}$-map from Σ to M and $\eta\in\Gamma^k(\Sigma, T^*\Sigma\otimes X^*T^*M)$, where $k\in\{0,\,1,\,\cdots\}$ denotes the regularity type of the map.
On Φ, the following first order action is defined:

$$S(X,\eta):=\int_\Sigma\langle\eta,\,dX\rangle+\frac{1}{2}\langle\eta,\,(\Pi^\#\circ X)\eta\rangle,$$

where

- $\Pi^\#$ is the map from $T^*M\to TM$ induced from the Poisson bivector Π.
- dX and η are regarded as elements in $\Omega^1(\Sigma,X^*(TM))$ and $\Omega^1(\Sigma,X^*(T^*M))$, respectively.
- $\langle\,,\,\rangle$ denotes the pairing between $\Omega^1(\Sigma,X^*(TM))$ and $\Omega^1(\Sigma,X^*(T^*M))$ induced by the natural pairing between T_xM and T_x^*M, for all $x\in M$.

The integrand, called the Lagrangian, will be denoted by $\mathcal{L}$. Associated to this action, the corresponding variational problem $\delta S=0$ induces the following space

$$\text{EL}=\{\text{Solutions of the Euler-Lagrange equations}\}\subset\Phi,$$

where, using integration by parts

$$\delta S=\int_\Sigma\frac{\delta\mathcal{L}}{\delta X}\delta X+\frac{\delta\mathcal{L}}{\delta\eta}\delta\eta+\text{boundary terms}.$$

The partial variations correspond to:

$$\frac{\delta\mathcal{L}}{\delta X} = dX+(\Pi^\#\circ X)\eta=0 \tag{2}$$

$$\frac{\delta\mathcal{L}}{\delta\eta} = d\eta+\frac{1}{2}\langle(\partial\Pi^\#\circ X)\eta,\eta\rangle=0. \tag{3}$$

Now, if we restrict to the boundary, the general space of boundary fields corresponds to

$$\Phi_\partial:=\{\text{vector bundle morphisms between } T(\partial\Sigma) \text{ and } T^*M\}.$$

Following [6], Φ_∂ is endowed with a symplectic form and a surjective submersion $p:\Phi\to\Phi_\partial$. We define

$$L_\Sigma:=p(EL).$$

Finally, we define C_Π as the set of fields in Φ_∂ which can be completed to a field in $L_{\Sigma'}$, with $\Sigma^{'}:=\partial\Sigma\times[0,\varepsilon]$, for some ε.

It turns out that Φ_∂ can be identified with $T^*(PM)$, the cotangent bundle of the path space on M and that

$$C_\Pi := \{(X,\eta) | dX = \pi^{\#}(X)\eta,\ X : \partial\Sigma \to M,\ \eta \in \Gamma(T^*I \otimes X^*(T^*M))\}.$$

Furthermore the following proposition holds

Proposition 1. [4]. *The space C_Π is a coisotropic submanifold of Φ_∂.*

In fact, the converse of this proposition also holds in the following sense. If we define $S(X,\eta)$ and C_Π in the same way as before, without assuming that Π satisfies Equation (1) it can be proven that

Proposition 2. [2, 3]. *If C_Π is a coisotropic submanifold of Φ_∂, then Π is a Poisson bivector field.*

The geometric interpretation of the Poisson sigma model will lead us to the connection between Lie algebroids and Lie groupoids in Poisson geometry. First we need some definitions.
A pair (A,ρ), where A is a vector bundle over M and ρ (called the anchor map) is a vector bundle morphism from A to TM is called a *Lie algebroid* if

(1) There is Lie bracket $[,]_A$ on $\Gamma(A)$ such that the induced map $\rho_* : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.
(2) *Leibniz identity:*

$$[X, fY]_A = f[X,Y] + \rho_*(X)(f)Y, \forall\, X, Y \in \Gamma(A), f \in \mathcal{C}^\infty(M).$$

Lie algebras, Lie algebra bundles and tangent bundles appear as natural examples of Lie algebroids. For our purpose, the cotangent bundle of a Poisson manifold T^*M, where $[,]_{T^*M}$ is the Koszul bracket for 1-forms, that is defined for exact forms by

$$[df, dg] := d\{f,g\}, \forall f, g \in \mathcal{C}^\infty(M),$$

whereas for general 1-forms it is recovered by Leibniz and the anchor map given by $\Pi^{\#} : T^*M \to TM$, is a central example of Lie algebroids. To define a morphism of Lie algebroids we consider the complex $\Lambda^\bullet A^*$, where A^* is the dual bundle and a differential δ_A is defined by the rules

(1)
$$\delta_A f := \rho^* df,\ \forall f \in \mathcal{C}^\infty(M).$$
(2)
$$\langle \delta_A \alpha, X \wedge Y\rangle := -\langle \alpha,\ [X,Y]_A\rangle + \langle \delta\langle \alpha, X\rangle, Y\rangle - \langle \delta\langle \alpha, Y\rangle, X\rangle,\ \forall X, Y \in \Gamma(A), \alpha \in \Gamma(A^*),$$

where $\langle,\rangle$ is the natural pairing between $\Gamma(A)$ and $\Gamma(A^*)$.

A vector bundle morphism $\varphi : A \to B$ is a Lie algebroid morphism if

$$\delta_A \varphi^* = \varphi^* \delta_B.$$

This condition written down in local coordinates gives rise to some PDE's the anchor maps and the structure functions for $\gamma(A)$ and $\Gamma(B)$ should satisfy. In particular, for the case of Poisson manifolds, C_Π corresponds to the space of Lie algebroid morphisms between $T(\partial\Sigma)$ and T^*M where the Lie algebroid structure on the left is given by the Lie bracket of vector fields on $T(\partial\Sigma)$ with identity anchor map and on the right is the one induced by the Poisson structure on M.

As it was mentioned before, it can be proven that this space is a coisotropic submanifold of T^*PM. Its symplectic reduction, i.e. the space of leaves of its characteristic foliation, called the reduced phase space of the PSM, when is smooth, has a particular feature, it is a symplectic groupoid over M [4]. More precisely, a groupoid is a small category with invertible morphisms. When the spaces of objects and morphisms are smooth manifolds, a Lie groupoid over M, denoted by $G \rightrightarrows M$, can be rephrased as the following data[1]

$$G \times_{(s,t)} G \xrightarrow{\ \mu\ } G \xrightarrow{\ i\ } G \underset{t}{\overset{s}{\underset{\varepsilon}{\leftrightarrows}}} M$$

where s, t, ι, μ and ε denote the source, target, inverse, multiplication and unit map respectively, such that the following axioms hold (denoting $G_{(x,y)} := s^{-1}(x) \cap t^{-1}(y)$):

(A.1) $s \circ \varepsilon = t \circ \varepsilon = id_M$

(A.2) If $g \in G_{(x,y)}$ and $h \in G_{(y,z)}$ then $\mu(g,h) \in G_{(x,z)}$

(A.3) $\mu(\varepsilon \circ s \times id_G) = \mu(id_G \times \varepsilon \circ t) = id_G$

(A.4) $\mu(id_G \times i) = \varepsilon \circ t$

(A.5) $\mu(i \times id_G) = \varepsilon \circ s$

(A.6) $\mu(\mu \times id_G) = \mu(id_G \times \mu)$.

A Lie groupoid is called *symplectic* if there exists a symplectic structure ω on G such that

$$Gr_\mu := \{(a,b,c) \in G^3 |\, c = \mu(a,b)\}$$

is a lagrangian submanifold of $G \times G \times \overline{G}$, where $\overline{G}$ denotes the sign reversed symplectic strucure on G. Finally, we can state the following

Theorem 1. [4]. *For the Poisson sigma model with source space homeomorphic to a disc, the symplectic reduction $\underline{C_\Pi}$ of C_Π (the space of leaves of the characteristic foliation), if it is smooth, is a symplectic groupoid over M.*

The smoothness of the reduced phase space has particular interest. In [10], the necessary and suficient conditions for integrability of Lie algebroids, i.e. whether a Lie groupoid such that its *infinitesimal version* corresponds to a given Lie algebroid exists, are stated. In [11], these conditions have been further specialized to the Poisson case. It turns out that the reduced phase space of the PSM coincides with the space of equivalent classes of what are called $\mathcal{A}-$paths modulo $\mathcal{A}-$ homotopy [10], with $\mathcal{A} = T^*M$.

3. The version before reduction.

The main motivation for introducing the relational symplectic groupoid construction is the following. In general, the leaf space of a characteristic foliation is not a smooth finite dimensional manifold and in this particular situation, the smoothness of the space of reduced boundary fields is controlled by the integrability conditions stated in [10]. In this paper, we define a groupoid object in the extended symplectic category, where the objects are symplectic manifolds, possibly infinite dimensional, and the morphisms are immersed Lagrangian submanifolds. It is important to remark here that this is extended category is not properly a category! (The composition of morphisms is not smooth in general). However, for our construction, the corresponding morphisms will be composable.

We restrict ourselves to the case when $\mathcal{C}$ is the sometimes called *Extended Symplectic Category*, denoted by Sym^{Ext} and defined as follows:

[1] $G \times_{(s,t)} G$ is a smooth manifold whenever s (or t) is a surjective submersion.

Definition 1. *Sym^{Ext} is a category in which the objects are symplectic manifolds and the morphisms are immersed canonical relations.* [2] *Recall that $L : \mathcal{M} \nrightarrow \mathcal{N}$ is an immersed canonical relation between two symplectic manifolds $\mathcal{M}$ and $\mathcal{N}$ by definition if L is an immersed Lagrangian submanifold of $\bar{\mathcal{M}} \times \mathcal{N}$.* [3] *$Sym^{Ext}$ carries an involution $\dagger : (Sym^{Ext})^{op} \to Sym^{Ext}$ that is the identity in objects and in morphisms, for $f : A \nrightarrow B$, $f^\dagger := \{(b,a) \in B \times A | (a,b) \in f\}$.*

This category extends the usual symplectic category in the sense that the symplectomorphisms can be thought in terms of canonical relations.

Definition 2. *A* ***relational symplectic groupoid*** *is a triple $(\mathcal{G}, L, I)$ where*

- *$\mathcal{G}$ is a weak symplectic manifold.* [4]
- *L is an immersed Lagrangian submanifold of $\mathcal{G}^3$.*
- *I is an antisymplectomorphism of $\mathcal{G}$*

satisfying the following axioms

- **A.1** L is cyclically symmetric (i.e. if $(x,y,z) \in L$, then $(y,z,x) \in L$)
- **A.2** I is an involution (i.e. $I^2 = Id$).
  Notation. L is a canonical relation $\mathcal{G} \times \mathcal{G} \nrightarrow \bar{\mathcal{G}}$ and will be denoted by L_{rel}. Since the graph of I is a Lagrangian submanifold of $\mathcal{G} \times \mathcal{G}$, I is a canonical relation $\bar{\mathcal{G}} \nrightarrow \mathcal{G}$ and will be denoted by I_{rel}.
  L and I can be regarded as well as canonical relations
  $$\bar{\mathcal{G}} \times \bar{\mathcal{G}} \nrightarrow \mathcal{G} \text{ and } \mathcal{G} \nrightarrow \bar{\mathcal{G}}$$
  respectively and will be denoted by $\overline{L_{rel}}$ and $\overline{I_{rel}}$. The transposition
  $$\begin{aligned} T : \mathcal{G} \times \mathcal{G} &\to \mathcal{G} \times \mathcal{G} \\ (x,y) &\mapsto (y,x) \end{aligned}$$
  induces canonical relations
  $$T_{rel} : \mathcal{G} \times \mathcal{G} \nrightarrow \mathcal{G} \times \mathcal{G} \text{ and } \overline{T_{rel}} : \bar{\mathcal{G}} \times \bar{\mathcal{G}} \nrightarrow \bar{\mathcal{G}} \times \bar{\mathcal{G}}.$$
  The identity map $Id : \mathcal{G} \to \mathcal{G}$ as a relation will be denoted by $Id_{rel} : \mathcal{G} \nrightarrow \mathcal{G}$ and by $\overline{Id_{rel}} : \overline{\mathcal{G}} \nrightarrow \overline{\mathcal{G}}$.
- **A.3** $I_{rel} \circ L_{rel} = \overline{L}_{rel} \circ \overline{T}_{rel} \circ (\overline{I_{rel}} \circ \overline{I_{rel}}) : \mathcal{G} \times \mathcal{G} \nrightarrow \mathcal{G}$.
  Remark 1. Since I and T are diffeomorphisms, both sides of the equaliy correspond to immersed Lagrangian submanifolds.

  Define
  $$L_3 := I_{rel} \circ L_{rel} : \mathcal{G} \times \mathcal{G} \nrightarrow \mathcal{G}.$$
  As a corollary of the previous axioms we get that

  Corollary 1. $\overline{I_{rel}} \circ L_3 = \overline{L_3} \circ \overline{T_{rel}} \circ (\overline{I_{rel}} \times \overline{I_{rel}})$.

[2]This is not exactly a category because the composition of canonical relations is not in general a smooth manifold

[3]Observe here that usually one considers embedded Lagrangian submanifolds, but we consider immersed ones.

[4]A weak symplectic manifold M has a closed 2-form ω such that the induced map $\omega^\# : T^*M \to TM$ is injective. For finite dimensional manifolds, the notion of weak symplectic and symplectic manifolds coincides.

- **A.4** $L_3 \circ (L_3 \times Id) = L_3 \circ (Id \times L_3) : \mathcal{G}^3 \nrightarrow \mathcal{G}$ is an immersed Lagrangian submanifold.

  The fact that the composition is Lagrangian follows from the fact that, since I is an antisymplectomorphism, its graph is Lagrangian, therefore L_3 is Lagrangian, and so $(Id \times L_3)$ and $(L_3 \times Id)$. The graph of the map I, as a relation $* \nrightarrow \mathcal{G} \times \mathcal{G}$ will be denoted by L_I.

- **A.5** $L_3 \circ L_I$ is an immersed Lagrangian submanifold of $\mathcal{G}$.
  Remark 2. It can be proven that Lagrangianity in these cases is automatical if we start with a finite dimensional symplectic manifold $\mathcal{G}$.
  Let $L_1 := L_3 \circ L_I : * \nrightarrow \mathcal{G}$. From the definitions above we get the following

  Corollary 2.
  $$\overline{I_{rel}} \circ L_1 = \overline{L_1},$$
  that is equivalent to
  $$I(L_1) = \overline{L_1},$$
  where L_1 is regarded as an immersed Lagrangian submanifold of $\mathcal{G}$.

  Corollary 3.
  $$L_3 \circ (L_1 \times L_1) = L_1.$$

- **A.6** $L_3 \circ (L_1 \times Id)$ is an immersed Lagrangian submanifold of $\overline{\mathcal{G}} \times \mathcal{G}$.
  We define
  $$L_2 := L_3 \circ (L_1 \times Id) : \mathcal{G} \nrightarrow \mathcal{G}.$$

  Corollary 4.
  $$L_2 = L_3 \circ (Id \times L_1).$$

  Corollary 5. *L_2 leaves invariant L_1, L_2 and L_3, i.e.*
  $$\begin{aligned} L_2 \circ L_1 &= L_1 \\ L_2 \circ L_2 &= L_2 \\ L_2 \circ L_3 &= L_3. \end{aligned}$$

  Corollary 6.
  $$\overline{I_{rel}} \circ L_2 = \overline{L_2} \circ \overline{I_{rel}} \text{ and } L_2^{\dagger} = L_2.$$

The next set of axioms defines a particular type of relational symplectic groupoids, in which the relation L_2 plays the role of an equivalence relation and it allows to study the case of symplectic reductions.

Definition 3. *A relational symplectic groupoid $(\mathcal{G}, L, I)$ is called* ***regular*** *if the following axioms are satisfied. Consider $\mathcal{G}$ as a coisotropic relation $* \nrightarrow \mathcal{G}$ denoted by $\mathcal{G}_{rel}$.*

- **A.7** $L_2 \circ \mathcal{G}_{rel}$ is an immersed coisotropic relation.
  Remark 3. Again in this case, the fact that this is a coisotropic relation follows automatically in the finite dimensional setting.

  Corollary 7. *Setting $C := L_2 \circ \mathcal{G}_{rel}$ the following corollary holds.*

(1)
$$C^* = \mathcal{G}^* \circ L_2$$
(2) *L_2 defines an equivalence relation on C.*
(3) *This equivalence relation is the same as the one given by the characteristic foliation on C.*

- **A.8** The reduction $\underline{L_1} = L_1/L_2$ is a finite dimensional smooth manifold. We will denote $\underline{L_1}$ by M.
- **A.9** $S := \{(c, [l]) \in C \times M : \exists l \in [l], g \in \mathcal{G} | (l, c, g) \in L_3\}$ is an immersed submanifold of $\mathcal{G} \times M$.

Corollary 8.
$$T := \{(c, [l]) \in C \times M : \exists l \in [l], g \in \mathcal{G} | (c, l, g) \in L_3\}$$
is an immersed submanifold of $\mathcal{G} \times M$.

The following conjectures (this is part of work in progress) give a connection between the symplectic structure on $\mathcal{G}$ and Poisson structures on M.

Conjecture 1. *Let $(\mathcal{G}, L, I)$ be a regular relational symplectic groupoid. Then, there exists a unique Poisson structure on M such that S is coisotropic in $\mathcal{G} \times M$.*

Conjecture 2. *Assume $G := C/L_2$ is smooth. Then G is a symplectic groupoid on M with structure maps $s := S/L_2$, $t := T/L_2, \mu := L_{rel}/L_2, \iota = I, \varepsilon = L_1/L_2$.*

Definition 4. *A **morphism between relational symplectic groupoids** (G, L_G, I_G) and (H, L_H, I_H) is a map F from G to H satisfying the following properties:*
(1) *F is a Lagrangian subspace of $G \times \bar{H}$.*
(2) *$F \circ I_G = I_H \circ F$.*
(3) *$F^3(L_G) = L_H$.*

Definition 5. *A morphism of relational symplectic groupoids $F : G \to H$ is called an **equivalence** if the transpose canonical relation $F^\dagger$ is also a morphism.*

Remark 4. For our motivational example, it can be proven that
(1) Different differentiability degrees (the $\mathcal{C}^k-$ type of the maps X and η) give raise to equivalent relational symplectic groupoids.
(2) For regular relational symplectic groupoids, $\mathcal{G}$ and G are equivalent.

3.1. **Examples.** The following are natural examples of relational symplectic groupoids.

3.1.1. *Symplectic groupoids:* Given a Lie symplectic groupoid G over M, we can endow it naturally with a relational symplectic structure:
$$\begin{aligned} \mathcal{G} &= G. \\ L &= \{(g_1, g_2, g_3) | (g_1, g_2) \in G \times_{(s,t)} G,\, g_3 = \mu(g_1, g_2)\}. \\ I &= g \mapsto g^{-1},\, g \in G. \end{aligned}$$

Remark 5. In connection to the construction in the Poisson sigma model, we can conclude that when M is integrable, the reduced space of boundary fields $\underline{C_\Pi}$ is a relational symplectic groupoid.

3.1.2. *Symplectic manifolds with a given immersed Lagrangian submanifold:* Let (G,ω) be a symplectic manifold and $\mathcal{L}$ an immersed Lagrangian submanifold of G. We define

$$\begin{aligned} \mathcal{G} &= G. \\ L &= \mathcal{L}\times\mathcal{L}\times\mathcal{L}. \\ I &= \{\text{identity of } G\}. \end{aligned}$$

It is an easy check that this construction satisfies the relational axioms and furthermore

Proposition 3. *The previous relational symplectic groupoid is equivalent to the zero dimensional symplectic groupoid (a point with zero symplectic structure and empty relations).*

Proof: It is easy by checking that L is an equivalence from the zero manifold to $\mathcal{G}$.

3.1.3. *Powers of symplectic groupoids:* Let us denote $G_{(1)} = G$, $G_{(2)}$ the fiber product $G \times_{(s,t)} G$, $G_{(3)} = G \times_{(s,t)} (G \times_{(s,t)} G)$ and so on . It can be proven the following

Lemma 1. [3]. *Let $G \rightrightarrows M$ be a symplectic groupoid.*

1. *$G_{(n)}$ is a coisotropic submanifold of G^n.*
2. *The reduced spaces $\underline{G_{(n)}}$ are symplectomorphic to G. Furthermore, there exists a natural symplectic groupoid structure on $\underline{G_{(n)}}$ coming from the quotient, isomorphic to the groupoid structure on G.*

We have natural canonical relations $P_n : \underline{G_{(n)}} \to G^n$ defined as:

$$P := \{(x,\alpha,\beta) | x \in \underline{G_{(n)}},\ [\alpha] = [\beta] = x\},$$

satisfying the following relations:

$$P^\dagger \circ P = Gr(Id_G),\ P \circ P^\dagger = \{(g,h) \in G^n | [g] = [h]\}.$$

It can be checked that

Proposition 4. *$G_{(i)}$ is equivalent to $G_{(j)}, \forall i,\, j \geq 1$ and the equivalence is given by $P_i \circ P_j^\dagger$.*

3.1.4. *The cotangent bundle of the path space of a Poisson manifold.* This is the motivational example for the construction of relational symplectic groupoids. In this case, the coisotropic submanifold C_Π is equipped with an equivalence relation, called T^*M- homotopy [10], and denoted by $\sim$. More precisely, to points of C_Π are $\sim$- equivalent if they belong to the same leaf of the characteristic foliation of C_Π. We get the following relational symplectic groupoid (where L is the restriction to the boundary of the solutions of the Euler-Lagrange equations in the bulk)

$$\begin{aligned} \mathcal{G} &= T^*(PM). \\ L &= \{(X_1,\eta_1),(X_2,\eta_2),(X_3,\eta_3) \in C_\Pi^3 | (X_1 * X_2, \eta_1 * \eta_2) \sim (X_3 * \eta_3)\}. \\ I &= (X,\eta) \mapsto (\phi^* X, \phi^* \eta)\}. \end{aligned}$$

Here $*$ denotes path concatenation and

$$\begin{aligned} \phi : [0,1] &\to [0,1] \\ t &\mapsto 1-t \end{aligned}$$

Theorem 2. [3]. *The relational symplectic groupoid $\mathcal{G}$ defined above is regular.*

The improvement of Theorem 1 in terms of the relational symplectic groupoids can be summarized as follows. L_1 can be understood as the space of T^*M- paths that are T^*M- homotopy equivalent to the trivial T^*M- paths and

$$\overline{L_1} := \cup_{x_0 \in M} T^*_{\overline{(X,\eta)}} PM \cap L_1,$$

where $\overline{(X,\eta)} = \{(X,\eta) | X \equiv X_0, \eta \in \ker \Pi^{\#}\}$, we can prove the following

Theorem 3. [3]. *If the Poisson manifold M is integrable, then there exists a tubular neigborhood of the zero section of T^*PM, denoted by $N(\Gamma_0(T^*PM))$ such that $\overline{L_1} \cap N(\Gamma_0(T^*PM))$ is an embedded submanifold of T^*PM.*

Theorem 4. [3]. *If M is integrable, then $L_1 \cap N(\Gamma_0(T^*PM))$, $L_2 \cap N(\Gamma_0(T^*PM))^2$ and $L_3 \cap N(\Gamma_0(T^*PM))^3$ are embedded Lagrangian submanifolds.*

4. Quantization.

The structure of relational symplectic groupoid may be reformulated in the category *Hilb* of Hilbert spaces. We define a *preunital Frobenius algebra* as the following data

- A Hilbert space H, equipped with an inner product $\langle\, , \rangle$ and an associative map
$$m : H \otimes H \to H.$$
- Defining
$$\langle a, b\rangle_H := \langle \overline{a}, b\rangle,$$
where $^-$ denotes complex conjugation, the following axioms holds:
  (1) *Cyclicity or Frobenius condition:*
  $$\langle m(a,b), c\rangle_H = \langle a, m(b,c)\rangle_H$$
  (2) *Projectability:* Choosing an orthonormal basis $\{e_i\}$ of H and assuming that
  $$e := \sum_i m(e_i, \bar{e_i})$$
  is a well defined element in H, the operator
  $$\begin{array}{rcl} P : H & \to & H \\ a & \mapsto & m(a,e) \end{array}$$
  is an orthogonal projection.

Remark 5. Under the assumptions above, $(H, m, \langle\, , \rangle_H)$ is not an unital algebra, however, the image of the operator P, called the *reduced algebra*, is unital.

The relational symplectic groupoid may be seen as the dequantization of this structure. The hard problem consists in going the other way around, namely, in quantizing a relational symplectic groupoid.

In the finite dimensional examples, methods of geometric quantization might be available, the problem being that of finding an appropriate polarization compatible with the structures. This question, in the case of a symplectic groupoid, has been addressed by Weinstein [17] and Hawkins [13]. The relational structure might allow more flexibility.

In the infinite-dimensional case, notably the example in 3.1.4., perturbative functional integral techniques might be available. The reduced algebra should give back a deformation quantization of the underlying Poisson manifold.

Finally notice that quantization might require weakening a bit the notion of preunital Frobenius algebra, allowing for example nonassociative products. However, one expects that the reduced algebra should always be associative.

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