

THE CHARACTER MAP IN DEFORMATION QUANTIZATION

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ABSTRACT. The third author recently proved that the Shoikhet–Dolgushev L_∞ -morphism from Hochschild chains of the algebra of smooth functions on a manifold to differential forms extends to cyclic chains. Localization at a solution of the Maurer–Cartan equation gives an isomorphism, which we call character map, from the periodic cyclic homology of a formal associative deformation of the algebra of functions to de Rham cohomology. We prove that the character map is compatible with the Gauss–Manin connection, extending a result of Calaque and Rossi on the compatibility with the cap product. As a consequence, the image of the periodic cyclic cycle 1 is independent of the deformation parameter and we compute it to be the A-roof genus of the manifold. Our results also imply the Tamarkin–Tsygan index Theorem.

1. INTRODUCTION

One of the consequences of the formality theorem for cyclic chains [17, 6] is the existence of a *character map* from the periodic cyclic homology of any formal associative deformation A_\hbar of algebras of functions on a manifold M to the de Rham cohomology $H^\bullet(M, \mathbb{R}[[\hbar]])$. By Kontsevich’s formality theorem for Hochschild cochains such deformations are classified by Poisson bivector fields in $\hbar\Gamma(M, \wedge^2 TM)[[\hbar]]$. A priori the character map depends on the choice of formality map for Hochschild cochains and cyclic chains. We consider here the Kontsevich L_∞ -morphism of cochains and the Shoikhet–Dolgushev L_∞ -morphism of modules from Hochschild chains to differential forms. The latter was shown by the third author [17] to be compatible with the Rinehart–Connes differential B and the de Rham differential and therefore extends to a morphism of L_∞ -modules from cyclic chains to the de Rham complex. In this paper we study the dependence of the character map on the Poisson bivector field and show that it is compatible with the Gauss–Manin connection. In particular, we show that the image of the class of the cyclic cycle 1 is independent of the Poisson bivector field and can thus be computed at the zero Poisson bracket, where it is given by the A-roof genus of M . This proves one part of Tsygan’s conjecture [15] on the character map (the independence on parameters) and disproves the other (about characteristic classes of foliations) for the Shoikhet–Dolgushev choice of formality map. Our main technical result is that the formality map on periodic chains is compatible with the Gauss–Manin connection. This result is an extension of the Calaque–Rossi theorem [1] on compatibility of the formality map on Hochschild chains with the cap product.

In the rest of the Introduction we recall the main notions (in 1.1) and describe the character map (in 1.2). We then state our main result on the compatibility of

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the character map with the Gauss–Manin connection (Theorem 1.3) and give the formula for the image of the cycle 1 (Corollary 1.5). The proof of the independence of the image of 1 on parameters is easier in the case of regular Poisson structures; we give it in Section 2. The evaluation of the image of 1 at the zero Poisson structure is discussed in Section 3. The case of general Poisson structures is discussed in Section 4, where we also apply our results to obtain a proof of the Tamarkin–Tsygan index Theorem.

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1.1. Cyclic chains as a module over Hochschild cochains and the Gauss–Manin connection. Here we recall some basic notions and fix sign conventions.

Let $C^\bullet(A) = \text{Hom}_k(\bar{A}^{\otimes \bullet}, A)$ be the normalized Hochschild cochain complex of a unital associative algebra A over $k = \mathbb{R}$ or $\mathbb{R}[[\hbar]]$. Here $\bar{A} = A/k1$. The degree-shifted complex $C^{\bullet+1}(A)$ with the Gerstenhaber bracket

$$[\phi, \psi] = \phi \bullet \psi - (-1)^{pq} \psi \bullet \phi, \quad \phi \in C^{p+1}(A), \quad \psi \in C^{q+1}(A),$$

$$\phi \bullet \psi(a_1, \dots, a_{p+q+1}) = \sum_{i=1}^{p+1} (-1)^{(i-1)q} \phi(a_1, \dots, a_{i-1}, \psi(a_i, \dots, a_{i+q}), \dots, a_{p+q+1}),$$

is a differential graded Lie algebra; the differential is the bracket $b = [\mu, \bullet]$ with the product $\mu \in C^2(A)$. The Lie bracket induces a Lie bracket on the cohomology $HH^{\bullet+1}(A)$ of $C^{\bullet+1}(A)$.

The complex of normalized Hochschild chains $C_\bullet(A) = A \otimes \bar{A}^{\otimes(-\bullet)}$ is a graded module over the dgla $C^\bullet(A)$. It is concentrated in non-positive degrees. The action of a cochain $\phi \in C^{p+1}(A)$ on a chain $a = (a_0, \dots, a_n) \in C_{-n}(A)$ is

$$\begin{aligned} L_\phi a &= (-1)^p \sum_{i=0}^{n-p} (-1)^{ip} (a_0, \dots, a_{i-1}, \phi(a_i, \dots, a_{i+p}), \dots, a_n) \\ &\quad + (-1)^p \sum_{i=n-p+1}^n (-1)^{in} (\phi(a_i, \dots, a_n, a_0, \dots, a_{p-n+i-1}), \dots, a_{i-1}). \end{aligned}$$

The Hochschild differential on $C_\bullet(A)$ is the action of the product $b = L_\mu$. The module property implies that $b \circ L_\phi - (-1)^p L_\phi \circ b = L_{b\phi}$ for all $\phi \in C^{p+1}(A)$, so that the action induces an action of the graded Lie algebra $HH^{\bullet+1}(A)$ on the homology $HH_\bullet(A)$ of (C_\bullet, b) . There is a second differential B of degree -1 on this complex (anti-)commuting with b : the Rinehart–Connes differential

$$B(a_0, \dots, a_n) = \sum_{i=0}^n (-1)^{in} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1}).$$

The induced action of $\phi \in HH^p(A)$ on $HH_\bullet(A)$ obeys the homotopy formula $L_\phi = B \circ I_\phi - (-1)^p I_\phi \circ B$, where $I_\phi: C_{-n}(A) \rightarrow C_{-n+p}(A)$ is the internal multiplication, defined on chains by

$$(1) \quad I_\phi(a_0, \dots, a_n) = (a_0 \phi(a_1, \dots, a_p), a_{p+1}, \dots, a_n).$$

Here is a version of the homotopy formula on chains due to Getzler [8], who extended Rinehart’s formula [12] for $p = 1$.

Lemma 1.1. *Let $\phi \in C^p(A)$ and let $H_\phi: C_{-n}(A) \rightarrow C_{-n+p-2}(A)$ be the map*

$$H_\phi(a_0, \dots, a_n) = \sum_{i=0}^{n-p} \sum_{j=0}^{n-p-i} (-1)^{i(n-p+1)+j(p+1)} (1, a_{n+1-i}, \dots, a_n, a_0, \dots, a_j, \phi(a_{j+1}, \dots, a_{j+p}), \dots, a_{n-i})$$

Then H_ϕ commutes with B and

$$L_\phi = [B, I_\phi] - H_{b\phi} + [b, H_\phi].$$

Here $[\ , \]$ denotes the graded commutator in $\text{End}_k(C_\bullet(A))$.

Proof. Both $H_\phi \circ B$ and $B \circ H_\phi$ vanish since $(1, 1, \dots) = 0$ in the normalized complex. The formula for L_ϕ can be proved by explicit calculations, see [12, 8]. \square

The periodic cyclic chain complex is the complex of Laurent series $PC_\bullet(A) = C_\bullet(A)((u))$ in an indeterminate u of degree 2 and u -linear differential $D = b + uB$. Lemma 1.1 and the identities $[b, I_\phi] = I_{b\phi}$, $[B, H_\phi] = 0$ imply that the u -linear extension of L_ϕ obeys

$$(2) \quad L_\phi = \frac{1}{u}([D, \hat{I}_\phi] - \hat{I}_{b\phi}), \quad \hat{I}_\phi = I_\phi + uH_\phi,$$

on the periodic cyclic complex.

Let (A_t) be a one-parameter family of algebras given by an algebra over $k[t]$ equal to $A[t]$ as a $k[t]$ -module. Then the product has the form $\mu + \gamma(t)$, where μ is the product in $A = A_0$ and $\gamma(t)$ is a solution of the Maurer–Cartan equation $b\gamma(t) + \frac{1}{2}[\gamma(t), \gamma(t)] = 0$. As noticed by Getzler [8], eq. (2) implies that there exists a connection, called Gauss–Manin connection, on the periodic cyclic cohomology viewed as a vector bundle over the parameter space. The parallel translation is given at the level of chains by the following formula.

Lemma 1.2. (Getzler) *Suppose that $c(t) \in PC_\bullet(A_t)$ is a solution of the differential equation*

$$\frac{d}{dt}c(t) + \frac{1}{u}\hat{I}_{\dot{\gamma}(t)}c(t) = 0,$$

and that $c(0)$ is a cyclic cycle. Then $c(t)$ is a cyclic cycle for all t and its class in $PH_\bullet(A_t)$ depends only on the class of $c(0)$.

The statement follows from (2) and the fact that $\dot{\gamma}(t) = \frac{d}{dt}\gamma(t)$ is a 2-cocycle in $C^2(A_t)$.

Getzler also showed that this formula defines a flat connection on the periodic cyclic cohomology of any family of algebras. Tsygan [16] extended the Gauss–Manin connection to a flat superconnection at the level of chains, see also [7].

1.2. The character map. Let now $k = \mathbb{R}$ and $A = C^\infty(M)$ be the algebra of smooth functions on a manifold M and $C^\bullet(A)$ be the complex of multidifferential Hochschild cochains. All operations described above are defined on this subcomplex of the Hochschild complex.

Kontsevich [9], in his proof of the formality theorem, constructed an L_∞ -quasi-isomorphism \mathcal{U} from the dgla $T^{\bullet+1}(M) = \Gamma(M, \wedge^{\bullet+1}TM) \simeq HH^{\bullet+1}(A)$, with

Schouten–Nijenhuis bracket¹ and zero differential, to $C^{\bullet+1}(A)$. The first component \mathcal{U}_1 is the Hochschild–Kostant–Rosenberg (HKR) quasi-isomorphism. Shoikhet [13] proved Tsygan’s formality conjecture for chains on $M = \mathbb{R}^n$ by giving a morphism \mathcal{V} of L_∞ -modules over $T^{\bullet+1}(M)$ from $C_\bullet(A)$ to the dg module of differential forms $\Omega^{-\bullet}(M, \mathbb{R})$ with zero differential. The action of $\gamma \in T^p(M)$ on $\Omega^{-\bullet}$ is given by the Lie derivative $L_\gamma = d \circ \iota_\gamma - (-1)^p \iota_\gamma \circ d$, where d is the de Rham differential and the internal multiplication of vector fields is extended to multivector fields by the rule $\iota_\gamma \iota_\eta = \iota_{\gamma \wedge \eta}$. The L_∞ -action of $T^{\bullet+1}(M)$ on Hochschild chains is the pull-back of the action of cochains on chains by Kontsevich’s morphism \mathcal{U} . Shoikhet’s result was extended to general manifolds by Dolgushev [5]. The first component \mathcal{V}_0 of the L_∞ -morphism \mathcal{V} is the HKR map $(a_0, \dots, a_n) \mapsto (1/n!)a_0 da_1 \cdots da_n$ and induces an isomorphism in homology if we interpret $C_{-n}(A)$ as the complex of ∞ -jets around the small diagonal of M^{n+1} . Finally, it was shown by the third author [17] that Shoikhet’s (local) morphism commutes with the Connes differential B on chains and the de Rham differential on differential forms. Globalizing, one obtains a morphism of L_∞ -modules over $T^{\bullet+1}(M)$ from the periodic cyclic complex $PC_\bullet(A)$ to the de Rham complex $(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])(u), ud)$. We will henceforth denote this morphism by \mathcal{V} , and refer to it as the cyclic Shoikhet–Dolgushev morphism. We advise the reader that this is slightly misleading. While in the local case our \mathcal{V} is indeed identical to Shoikhet’s morphism, the globalized version differs since the globalization procedure is sensitive to the change in the differential from $C_{-\bullet}(A)$ to $PC_\bullet(A)$.

For any solution of the Maurer–Cartan equation $[\pi, \pi] = 0$ in $\hbar T^2(M)[[\hbar]]$, i.e., a formal Poisson bivector field, we get a solution $\mathcal{U}_0^\pi = \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{U}_n(\pi, \dots, \pi)$ of the Maurer–Cartan equation $b\mathcal{U}_0^\pi + \frac{1}{2}[\mathcal{U}_0^\pi, \mathcal{U}_0^\pi] = 0$ in $C^2(A)[[\hbar]]$. The Maurer–Cartan equation is the associativity of the star product $f \star g = fg + \mathcal{U}_0^\pi(f \otimes g)$ on $A[[\hbar]]$. Let A_\hbar be the algebra $(A[[\hbar]], \star)$. The twists at π of \mathcal{U}, \mathcal{V} are L_∞ -morphisms $\mathcal{U}^\pi, \mathcal{V}^\pi$ defined by

$$\begin{aligned} \mathcal{U}_n^\pi(\gamma_1, \dots, \gamma_n) &= \sum_{m=0}^{\infty} \frac{1}{m!} \mathcal{U}_{n+m}(\gamma_1, \dots, \gamma_n, \pi, \dots, \pi), & n \geq 1, \\ \mathcal{V}_n^\pi(\gamma_1, \dots, \gamma_n) &= \sum_{m=0}^{\infty} \frac{1}{m!} \mathcal{V}_{n+m}(\gamma_1, \dots, \gamma_n, \pi, \dots, \pi), & n \geq 0. \end{aligned}$$

The twist \mathcal{U}^π is an \hbar -linear L_∞ -quasiisomorphism from $T^{\bullet+1}(M)[[\hbar]]$ with Poisson differential $[\pi, \]$ to $C^{\bullet+1}(M)[[\hbar]]$ with Hochschild differential $b_\star = b + L_{\mathcal{U}_0^\pi}$ calculated with the star product. Similarly \mathcal{V}^π is an L_∞ -morphism of modules over the dgla $(T^{\bullet+1}(M), [\pi, \])$ from $(PC_\bullet(A)[[\hbar]], b_\star + uB)$ to $(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])(u), L_\pi + ud)$. We refer to [20, 9, 15, 13, 5, 17] for more details.

Since $L_\pi = d \circ \iota_\pi - \iota_\pi \circ d$, the map $\alpha \mapsto e^{\iota_\pi/u} \alpha$ is an isomorphism of complexes

$$(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])(u), L_\pi + ud) \rightarrow (\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])(u), ud)$$

Passing to cohomology, we obtain the *character map*

$$PH_\bullet(A_\hbar) \rightarrow H^{-\bullet}(M, \mathbb{R}[[\hbar]])(u),$$

¹The Schouten–Nijenhuis bracket is the Lie bracket on vector fields and is extended to general multivector fields by the rule $[\alpha \wedge \beta, \gamma] = \alpha \wedge [\beta, \gamma] + (-1)^{pq} \beta \wedge [\alpha, \gamma]$, $\alpha \in T^p(M), \beta \in T^q(M)$. With this sign convention the HKR homomorphism sends $\xi_1 \wedge \cdots \wedge \xi_p$ to the Hochschild cocycle $a_1 \otimes \cdots \otimes a_p \mapsto (1/p!) \sum_{\sigma \in S_p} \text{sgn}(\sigma) \xi_{\sigma(p)}(a_1) \cdots \xi_{\sigma(1)}(a_p)$

of modules over the Poisson cohomology, induced by

$$\tilde{\mathcal{V}}_0^\pi = e^{\iota^\pi/u} \circ \mathcal{V}_0^\pi,$$

from the periodic cyclic cohomology of the deformed algebra A_\hbar to de Rham cohomology.

1.3. Dependence on parameters. Suppose we have a one-parameter family of solutions $\pi(t)$ of the Maurer–Cartan equation, with, say, polynomial dependence on t , for example $\pi(t) = t\pi$ for some Poisson structure π . We then obtain a one-parameter family of star-products whose reduction modulo \hbar^N depends polynomially on t for any N . The Gauss–Manin connection allows us to identify the periodic cyclic cohomology for different values of the parameter, so we can study the dependence on t of the character map.

Theorem 1.3. *Let A_t be the quantum algebra of functions on M associated with the Poisson bracket $\pi(t)$. Suppose $c(t)$ is a cycle in $PC_{-n}(A_t)$, horizontal with respect to the Gauss–Manin connection, i.e., such that*

$$\frac{d}{dt}c(t) + \frac{1}{u}\hat{I}_{\mathcal{U}_1^{\pi(t)}(\dot{\pi}(t))}c(t) = 0, \quad \hat{I}_\gamma = I_\gamma + uH_\gamma.$$

Then the class of $\tilde{\mathcal{V}}_0^{\pi(t)}(c(t))$ in $\oplus_m H^{n+2m}(M, \mathbb{R}[[\hbar]])u^m$ is independent of t .

The proof of this theorem is based on an identity extending the “compatibility with cap product” result of Calaque and Rossi [1]. To formulate the result we introduce the modified internal multiplication of a multivector field γ on differential forms:

$$\hat{\iota}_\gamma = \iota_\gamma + u\frac{1}{2}dL_\gamma.$$

It is clear that the Cartan formula $L_\gamma = [d, \hat{\iota}_\gamma]$ still holds.

Proposition 1.4. *Let π be a Poisson bivector field on M , $A = C^\infty(M)$ and $\gamma \in \Gamma(M, \wedge^p TM)$ such that $[\pi, \gamma] = 0$.*

(1) *The map*

$$\hat{\iota}_\gamma \circ \mathcal{V}_0^\pi - \mathcal{V}_0^\pi \circ \hat{I}_{\mathcal{U}_1^\pi(\gamma)} + u\mathcal{V}_1^\pi(\gamma)$$

is a map of complexes from the periodic cyclic complex $PC_\bullet(A_\hbar)$ (with differential $b_\star + uB$) to the complex $\Omega^{-\bullet}((u))[[\hbar]]$ (with differential $L_\pi + u d$). Furthermore this map induces the zero map on cohomology.

(2) *In the special case $M = \mathbb{R}^n$ and π, γ as before, there exist maps*

$$\mathcal{V}_I^\pi(\gamma): C_\bullet(A) \rightarrow \Omega^{-\bullet+p-1}(M, \mathbb{R}[[\hbar]]), \quad \mathcal{V}_H^\pi(\gamma): C_\bullet(A) \rightarrow \Omega^{-\bullet+p-3}(M, \mathbb{R}[[\hbar]]),$$

linear in γ , such that

$$\hat{\iota}_\gamma \circ \mathcal{V}_0^\pi - \mathcal{V}_0^\pi \circ \hat{I}_{\mathcal{U}_1^\pi(\gamma)} + u\mathcal{V}_1^\pi(\gamma) = (L_\pi + ud) \circ X^\pi(\gamma) + X^\pi(\gamma) \circ (b_\star + uB),$$

with homotopy $X^\pi(\gamma) = \mathcal{V}_I^\pi(\gamma) + u\mathcal{V}_H^\pi(\gamma)$.

Simple part of the proof. Let us compute the commutator with the differential

$$\begin{aligned} (b_\star + uB) \circ (\hat{\iota}_\gamma \circ \mathcal{V}_0^\pi - \mathcal{V}_0^\pi \circ \hat{I}_{\mathcal{U}_1^\pi(\gamma)} + u\mathcal{V}_1^\pi(\gamma)) - (-1)^p (\hat{\iota}_\gamma \circ \mathcal{V}_0^\pi - \mathcal{V}_0^\pi \circ \hat{I}_{\mathcal{U}_1^\pi(\gamma)} + u\mathcal{V}_1^\pi(\gamma)) \circ (ud) \\ = uL_\gamma \circ \mathcal{V}_0^\pi - u\mathcal{V}_0^\pi \circ L_\gamma + u(\mathcal{V}_0^\pi \circ L_\gamma - \mathcal{V}_0^\pi \circ L_\gamma) = 0. \end{aligned}$$

□

Proposition 1.4 is our main technical result and is proven in Section 4. The operators \mathcal{V}_I and \mathcal{V}_H are given explicitly in terms of integrals over configuration spaces. Given this proposition, the proof of the theorem is a simple calculation. With the notation $\pi = \pi(t)$, $\dot{\pi} = \frac{d}{dt}\pi(t)$, $b_\star(t) = b + L_{\pi(t)}$ we have:

$$\begin{aligned} \frac{d}{dt}\mathcal{V}_0^\pi + \frac{1}{u}\hat{\iota}_{\dot{\pi}} \circ \mathcal{V}_0^\pi &= \mathcal{V}_1^\pi(\dot{\pi}) + \frac{1}{u}\hat{\iota}_{\dot{\pi}} \circ \mathcal{V}_0^\pi \\ &= \frac{1}{u}(\mathcal{V}_0^\pi \circ \hat{I}_{\mathcal{U}_1^\pi(\dot{\pi})} + (L_\pi + ud) \circ X^\pi(\dot{\pi}) \\ &\quad + X^\pi(\dot{\pi}) \circ (b_\star(t) + uB)) \end{aligned}$$

If $c(t)$ is a cycle obeying the differential equation we get

$$\left(\frac{d}{dt} + \frac{1}{u}\hat{\iota}_{\dot{\pi}}\right)(\mathcal{V}_0^\pi c(t)) = \left(d + \frac{1}{u}L_\pi\right)(X^\pi(\dot{\pi})c(t))$$

This formula can be rewritten as

$$\frac{d}{dt}\left(e^{\hat{\iota}_{\dot{\pi}}/u}\mathcal{V}_0^\pi c(t)\right) = d\left(e^{\hat{\iota}_{\dot{\pi}}/u}X^\pi(\dot{\pi})c(t)\right).$$

Thus the de Rham class of $e^{\hat{\iota}_{\dot{\pi}}/u}\mathcal{V}_0^\pi c(t)$ is independent of t . Moreover, since $e^{\hat{\iota}_{\dot{\pi}}/u} = (1 - \frac{1}{2u}d\iota_\pi d)e^{\hat{\iota}_{\dot{\pi}}/u}$, and $e^{\hat{\iota}_{\dot{\pi}}/u}\mathcal{V}_0^\pi c(t)$ is a cocycle, we may replace $\hat{\iota}_{\dot{\pi}}$ by ι_π in the exponential.

Remark. The identity of Proposition 1.4 reduces at $u = 0$ to the identity

$$\iota_\gamma \circ \mathcal{V}_0^\pi - \mathcal{V}_0^\pi \circ I_{\mathcal{U}_1^\pi(\gamma)} = L_\pi \circ \mathcal{V}_I^\pi(\gamma) + \mathcal{V}_I^\pi(\gamma) \circ b_\star.$$

This is the statement of compatibility with the cap product ($I_\phi = \phi \cap$) proved in [1].

Remark. Geometrically one should think of the periodic cyclic complex $PC_\bullet(A_\hbar)$ as the fibre of a trivial vector bundle on the Maurer–Cartan variety of formal deformations of the product. This vector bundle carries the Gauss–Manin connection $d + \theta$, where θ is a one-form with values in the endomorphisms of C_\bullet . The value of this one-form on a tangent vector $\gamma \in \text{Ker}(b_\star)$ is

$$\theta_\gamma = \frac{1}{u}\hat{I}_\gamma = \frac{1}{u}I_\gamma + H_\gamma.$$

The statement of Proposition 1.4 is that the Shoikhet–Dolgushev quasi-isomorphism intertwines, up to homotopy, this connection and the connection $d + \frac{1}{u}\hat{\iota}$ on the trivial vector bundle on the Maurer–Cartan variety of Poisson structure with fibre $\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])((u))$.

1.4. The image of the cycle 1. For any unital algebra A , 1 is a cyclic cycle and thus also a periodic cyclic cycle.

Corollary 1.5. *Let $\pi \in \hbar\Gamma(M, \wedge^2 TM)[[\hbar]]$ be a formal Poisson bivector field. The image of the cyclic cycle $1 \in (C^\infty(M)[[\hbar]], \star)$ by the character map is $\hat{A}_0(M) + u\hat{A}_2(M) + u^2\hat{A}_4(M) + \dots$, where $\hat{A}_i(M) \in H^i(M)$ are the components of the A-roof genus of M .*

Indeed, $\pi(t) = t\pi$ is a Poisson bivector field for any t . By Theorem 1.3 we can evaluate the class of $\tilde{\mathcal{V}}_0^{t\pi}(1)$ at $t = 0$. We do this in Section 3.

2. SPECIAL CASE: REGULAR POISSON STRUCTURES

In this section, we will illustrate the above results for regular Poisson structures. In this special case, there is a simple proof of \hbar -independence by elementary means that does not require the more intricate Theorem 4.1. The results of this section are not needed anywhere else in the paper.

Proposition 2.1. *For \mathcal{V} the cyclic Shoikhet–Dolgushev morphism and π a regular Poisson structure, the de Rham cohomology class of*

$$\tilde{\mathcal{V}}_0(1) = e^{\hbar\iota_\pi/u} \mathcal{V}_0^\pi(1)$$

is independent of \hbar .

In general, let us introduce the notation $\tilde{\mathcal{V}}_j = e^{\hbar\iota_\pi/u} \mathcal{V}_j^\pi$. It is then easy to see that the first L_∞ relations reduce to the following

$$\begin{aligned} u d\tilde{\mathcal{V}}_0(\alpha) &= \tilde{\mathcal{V}}_0((b_\star + uB)\alpha) \\ u d\tilde{\mathcal{V}}_1(\gamma; \alpha) + (-1)^{|\gamma|} (L_\gamma + \hbar u_{[\pi, \gamma]}; \alpha) \tilde{\mathcal{V}}_0(\alpha) &= (-1)^{|\gamma|} \tilde{\mathcal{V}}_1(\gamma; (b_\star + uB)\alpha) \\ &\quad + \hbar \tilde{\mathcal{V}}_1([\pi, \gamma]; \alpha) + (-1)^{|\gamma|} \tilde{\mathcal{V}}_0(L_{\mathcal{U}(\gamma)}\alpha) \end{aligned}$$

For the proof of the proposition, first note that the subcomplex $(T_{\parallel}^\bullet, [\pi, \cdot]) \subset (T^\bullet, [\pi, \cdot])$ of multivector fields tangential to the symplectic foliation is locally acyclic. Concretely, in a small enough neighbourhood around any point, the manifold looks like a product of a symplectic space and one with trivial Poisson structure.

It follows that there is a good covering $\{U_i\}_{i \in I}$ of M , a collection of (locally defined) tangential vector fields $\xi = \{\xi_i\}_{i \in I}$, and a collection of (locally defined) functions $f = \{f_{ij}\}_{i \neq j \in I}$, $f_{ij} = -f_{ji}$, such that

$$\begin{aligned} \pi &= [\pi, \xi_i] && \text{on } U_i \\ \xi_i - \xi_j &= [\pi, f_{ij}] && \text{on } U_i \cap U_j. \end{aligned}$$

In other words, the class $[\pi]$ in Poisson cohomology can be represented by the cocycle $\check{\delta}f$ in the Poisson–Čech double complex, where $\check{\delta}$ is the Čech-differential.

Next compute on U_i :²

$$\begin{aligned} \frac{d}{d\hbar} \tilde{\mathcal{V}}_0(1) &= \frac{d}{d\hbar} e^{\hbar\iota_\pi/u} \sum_{j \geq 0} \frac{\hbar^j}{j!} \mathcal{V}_j(\pi, \dots, \pi; 1) \\ &= \iota_\pi \tilde{\mathcal{V}}_0(1)/u + \tilde{\mathcal{V}}_1(\pi; 1) \\ &= \iota_\pi \tilde{\mathcal{V}}_0(1)/u + \tilde{\mathcal{V}}_1([\pi, \xi_i]; 1) \\ &= \iota_\pi \tilde{\mathcal{V}}_0(1)/u + \left(-\frac{1}{\hbar} L_{\xi_i} - \iota_\pi/u\right) \tilde{\mathcal{V}}_0(1) + \frac{u}{\hbar} d\tilde{\mathcal{V}}_1(\xi_i; 1) \\ &= \frac{1}{\hbar} d \left(-\iota_{\xi_i} \tilde{\mathcal{V}}_0(1) + u \tilde{\mathcal{V}}_1(\xi_i; 1) \right) \end{aligned}$$

²Implicitly we use here that $\tilde{\mathcal{V}}$ is local, i.e., its value at a point depends only on the jets of its arguments at that point. Hence it makes sense to restrict $\tilde{\mathcal{V}}$ to U_i .

The class of this cocycle in Čech-de Rham cohomology is the same as that of the Čech-degree 1 chain (denoting $\xi_{ij} = \xi_i - \xi_j$)

$$\begin{aligned} \frac{1}{\hbar} \left(-\iota_{\xi_{ij}} \tilde{\mathcal{V}}_0(1) + u \tilde{\mathcal{V}}_1(\xi_{ij}; 1) \right) &= \frac{1}{\hbar} \left(-\iota_{\xi_{ij}} \tilde{\mathcal{V}}_0(1) + u \tilde{\mathcal{V}}_1([\pi, f_{ij}]; 1) \right) \\ &= \frac{u}{\hbar^2} \left(d\iota_{f_{ij}} \tilde{\mathcal{V}}_0(1) + u d\tilde{\mathcal{V}}_1(f_{ij}; 1) - \tilde{\mathcal{V}}_0(L_{\mathcal{U}_1(f_{ij})} 1) \right) \\ &= \frac{u}{\hbar^2} \left(d\iota_{f_{ij}} \tilde{\mathcal{V}}_0(1) + u d\tilde{\mathcal{V}}_1(f_{ij}; 1) \right. \\ &\quad \left. - \frac{1}{u} \tilde{\mathcal{V}}_0((uB + b_*) \mathcal{U}_1(f_{ij})) \right) \\ &= \frac{u}{\hbar^2} d \left(\iota_{f_{ij}} \tilde{\mathcal{V}}_0(1) + u \tilde{\mathcal{V}}_1(f_{ij}; 1) - \tilde{\mathcal{V}}_0(\mathcal{U}_1(f_{ij})) \right) \end{aligned}$$

This cochain in turn describes the same Čech-de Rham cohomology class as the Čech-degree 2 cochain (denoting $f_{ijk} = f_{ij} + f_{jk} + f_{ki}$)

$$\frac{u}{\hbar^2} \left(f_{ijk} \tilde{\mathcal{V}}_0(1) + u \tilde{\mathcal{V}}_1(f_{ijk}; 1) - \tilde{\mathcal{V}}_0(\mathcal{U}_1(f_{ijk})) \right)$$

We claim that this expression is zero, at least for suitable choices made in the globalization step of the construction of \mathcal{U} , \mathcal{V} .³ Concretely, it follows from the following lemma.

Lemma 2.2. *Let π be a regular Poisson structure, f be a Poisson central function, and a_0, \dots, a_n be arbitrary functions. Let \mathcal{U} be the Kontsevich formality morphism and \mathcal{V} the cyclic Shoikhet-Dolgushev morphism, and \mathcal{U}^π and \mathcal{V}^π their twisted versions as before. Then, for suitable choices taken in the globalization step of \mathcal{U} , \mathcal{V} the following hold:*

$$\begin{aligned} (3) \quad & \mathcal{V}_1^\pi(f; a_0, \dots, a_n) = 0 \\ (4) \quad & \mathcal{V}_0^\pi(f) = f \mathcal{V}_0^\pi(1, a_1, \dots, a_n) \\ (5) \quad & \mathcal{U}_1^\pi(f) = f. \end{aligned}$$

Sketch of proof. For the proof, we assume familiarity with M. Kontsevich's construction of the formality morphism \mathcal{U}^π and the construction of \mathcal{V}^π by V. Dolgushev and B. Shoikhet, see [9, 5, 13]. Both constructions proceed by first defining the morphisms in the local cases (i.e., $M = \mathbb{R}^n$), and then globalizing these morphisms. The globalization step is briefly recalled in section 4.3. It depends on the choice of a particular Fedosov connection. We will choose this connection in such a manner that the flat lift of π , say $\hat{\pi}$, is fiberwise constant. This is possible only for regular Poisson structures.

Let us start with the third assertion of the Lemma. Using the notation from section 4.3 below, \mathcal{U}^π is defined as the twisted version of the composition of quasi-isomorphisms:

$$T_{\text{poly}} \xrightarrow{F} T_{\text{poly}}^{\text{fml}} \longrightarrow D_{\text{poly}}^{\text{fml}} \xleftarrow{G} D_{\text{poly}}.$$

Here the right hand morphism must be inverted up to homotopy before composing. To see that $\mathcal{U}_1^\pi(f) = f$ for a Poisson central function $f \in T_{\text{poly}}^0$ one needs to trace it through these three morphisms. The first morphism F only has a single Taylor

³Note that, due to non-canonical choices to be made in the globalization step (see section 4.3), \mathcal{U} and \mathcal{V} are defined uniquely only up to homotopy.

component, and maps f to its flat lift \hat{f} . Furthermore $F(\pi) = \hat{\pi}$ is fiberwise constant by our choice of Fedosov connection. The second morphism in the composition is the fiberwise application of M. Kontsevich's (local) formality morphism. It is defined by a sum of terms labelled by graphs. Using that $\hat{\pi}$ is fiberwise constant, that $\hat{\pi}$ is a flat lift and that $[\hat{\pi}, \hat{f}] = 0$, it is an elementary exercise to show that all Kontsevich graphs except the trivial one yield zero contribution. Hence the image of f under the twisted version of the first two maps is \hat{f} . Finally one has to invert G , up to homotopy. There are explicit formulas for this inverse. Twisting and inserting \hat{f} one can check explicitly that \hat{f} is mapped to f .

Next consider the statements involving \mathcal{V} . It is defined through the zig-zag of quasi-isomorphisms of L_∞ modules (see again section 4.3 for the notation)

$$\Omega((u)) \longrightarrow \Omega^{fml}((u)) \longleftarrow PC^{fml} \longleftarrow PC.$$

The left hand morphism has to be inverted before composing. Let us start with the chain $f \otimes a_1 \otimes \cdots \otimes a_n$, as an element of the periodic cyclic chain complex PC on the right. The twisted version of the right hand map sends it to the flat lift $\hat{f} \otimes \hat{a}_1 \otimes \cdots \otimes \hat{a}_n$, plus some correction terms. The correction terms arise because the L_∞ morphism is not just the flat lift, but picks up higher components through the inversion of the map of L_∞ algebras $D_{\text{poly}} \rightarrow D_{\text{poly}}^{fml}$, see 4.3. The correction terms however can be seen to all have the form $\hat{f} \otimes (\cdots)$, i.e., the \hat{f} is never hit by any derivatives.

The second morphism in the composition is defined using Shoikhet graphs. An example graph contributing to \mathcal{V}^π is shown in Figure 1. However, note that all graphs in which an arrow hits the vertex $\bar{0}$, corresponding to \hat{f} , contribute zero. This can be shown using that f is locally constant along the symplectic leaves, and that $\hat{\pi}$ is fiber-wise constant. Hence the image of f under the first two maps is the image of $1 \otimes a_1 \otimes \cdots \otimes a_n$ times \hat{f} . Inverting the left hand map, one can check that it sends this product to $f\mathcal{V}_0^\pi(1 \otimes a_1 \otimes \cdots \otimes a_n)$ and the second statement of the lemma is shown.

Finally let us turn to the first statement of the lemma. The component \mathcal{V}_1^π can receive contributions from each of the three morphisms in the composition. We claim that all these contributions are zero. To check this for the first and last morphism is technical, and will be left to the reader. Let us check that the second morphism contributes zero. Consider a (potentially) contributing Shoikhet graph, with, say $k \geq 1$ type I vertices, one of which is the one corresponding to \hat{f} . Vertices can have either one or two outgoing edges, furthermore there can be outgoing edges from the central vertex. However, using that $\hat{\pi}$ is fiberwise constant one sees that the maximal number of edges is one less than the dimension of the relevant configuration space, and hence the weight of the graph will be zero. \square

3. THE CYCLIC SHOIKHET–DOLGUSHEV MAP FOR $\pi = 0$

This section is devoted to the following result, needed in the proof of Corollary 1.5.

Proposition 3.1. *For \mathcal{V} the cyclic Shoikhet–Dolgushev morphism, $\pi = 0$, and c a cyclic cycle in $PC_\bullet(A)$*

$$[\tilde{\mathcal{V}}_0^{\pi=0}(c)] = \hat{A}_u(M) Co(c).$$

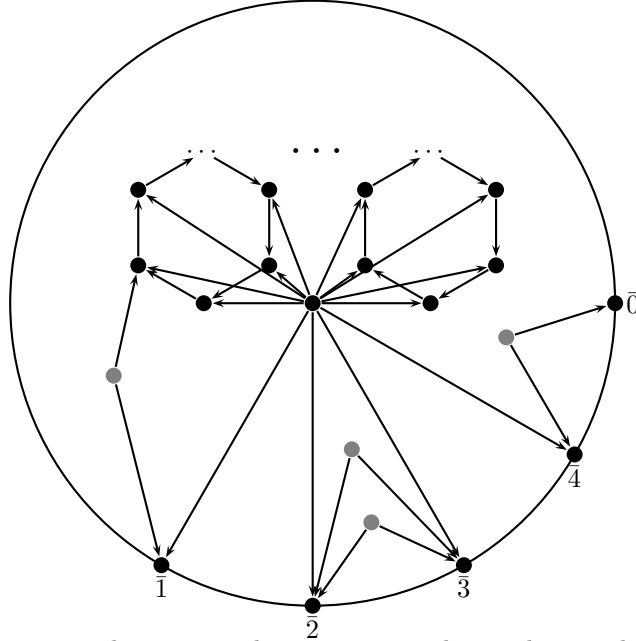


FIGURE 1. The picture shown an example graph contributing to \mathcal{V}_0^π in the regular case. The grey vertices correspond to the (fiberwise constant) Poisson structure $\hbar\pi$. Thus putting $\hbar = 0$ the only contributions come from graphs consisting entirely of the “wheels”.

Here

$$\hat{A}_u(M) := \hat{A}_0(M) + u\hat{A}_2(M) + u^2\hat{A}_4(M) + \dots$$

with $\hat{A}_i(M) \in H^i(M)$ being the components of the A-roof genus of M . Furthermore Co is the Connes map identifying periodic cyclic and de Rham cohomology, concretely on chains

$$Co(a_0, \dots, a_n) = a_0 da_1 \wedge \dots \wedge da_n.$$

Proof. The only contributing graphs are unions of wheels and edges directly connecting the center vertex with vertices on the boundary, as shown in Figure 1 (without the grey vertices). It is not hard to see that the latter (direct) edges produce the factor $Co(c)$. Let us turn to the wheels. Consider a single wheel with j vertices. It has a number (weight) w_j and a differential form α_j associated to it. The contribution of all combinations of wheels is hence

$$[\tilde{\mathcal{V}}_0(1)] = [e^{\sum_{j \geq 2} w_j \alpha_j}].$$

From the details of the globalization procedure (and tedious sign conventions) it can be seen that

$$\alpha_j = (-1)^{j(j-1)/2} u^j \text{tr}(R^j)$$

where R is the curvature 2-form of some affine torsion free connection on M . Furthermore, the weights w_j can be computed explicitly. A simple reflection argument shows that $w_j = 0$ if j is odd. For j even it was shown by M. van den Bergh [19]

(and the authors in [18]) that $w_j = -(-1)^{j(j-1)/2} \frac{B_j}{2^j j!}$.⁴ Hence it follows that

$$[\tilde{\mathcal{V}}_0(1)] = [e^{-\sum_{j \geq 1} \frac{B_{2j}}{4^j (2j)!} u^{2j} \text{tr}(R^{2j})}] = \det^{\frac{1}{2}} \left(\frac{uR/2}{\sinh(uR/2)} \right) = \hat{A}_u(M).$$

□

4. COMPATIBILITY OF THE CYCLIC SHOIKHET–DOLGUSHEV MAP WITH CAP PRODUCTS

The goal of this section is to define the morphisms \mathcal{V}_I^π and \mathcal{V}_H^π occurring in Proposition 1.4 above and give a proof of that proposition. Both morphisms will be defined as a sum of terms corresponding to Shoikhet graphs. In fact the formulas will be identical to those for the twisted version \mathcal{V}^π of B. Shoikhet’s morphism \mathcal{V} [13], except that the weights are computed slightly differently. Recall that the Taylor component \mathcal{V}_1^π of this morphism is defined by the formula

$$\begin{aligned} \mathcal{V}_1^\pi(\gamma; a_0, \dots, a_n) &= \\ &= \sum_{m \geq 0} \frac{\hbar^m}{m!} \sum_{\Gamma \in G(m+1, n)} \left(\int_{\pi_1^{-1}(U_1) \subset C_{m+1, n}} \omega_\Gamma \right) D_\Gamma(\gamma, \pi, \dots, \pi; a_0, \dots, a_n). \end{aligned}$$

Here $\gamma \in T_{\text{poly}}$ and a_0, \dots, a_n are smooth functions. The sum is over all Shoikhet graphs, see Appendix A or [13]. The form valued multi-differential operator D_Γ is defined in equation (9). The integral is over a certain subspace $\pi_1^{-1}(U_1)$ of the space $C_{m+1, n}$ (see Definition A.2), which is essentially B. Shoikhet’s space of configurations of $m + n + 3$ points in the unit disk. Here “essentially” means that our space $C_{m+1, n}$ is bigger by a factor S^1 , which we interpret as a choice of framing (direction) at the center of the disk. The map $\pi_1 : C_{m+1, n} \rightarrow C_{1, n}$ is the forgetful map, forgetting the positions of m points $2, \dots, m+1$. The subspace U_1 is obtained by fixing the framing to align with the position of point $\bar{0}$ on the boundary of the disk. For the notation used, see Appendix A, where also the differential form ω_Γ is defined.

The formulas for \mathcal{V}_I^π and \mathcal{V}_H^π are given by almost the same expression, except that the subspace U_1 is replaced by different subspaces. Concretely

$$\begin{aligned} \mathcal{V}_{Op}^\pi(\gamma; a_0, \dots, a_n) &= \\ &= \sum_{m \geq 0} \frac{\hbar^m}{m!} \sum_{\Gamma \in G(m+1, n)} \left(\int_{\pi_1^{-1}(U_{Op}) \subset C_{m+1, n}} \omega_\Gamma \right) D_\Gamma(\gamma, \pi, \dots, \pi; a_0, \dots, a_n) \end{aligned}$$

where Op stands for either I or H . The subspaces U_I and U_H corresponding to the operation I and H are depicted in Figure 2. For $U_I \subset U_1$ one fixes the framing to the position of point $\bar{0}$ on the boundary, and additionally restricts the point 1 in the interior to that line segment, connecting the center point 0 with $\bar{0}$. For U_H , there is no constraint on the inner framing, but the point 1 is restricted to the disk segment between (in clockwise sense) $\bar{0}$ and the framing at 0. This is the region colored gray in Figure 2. The operation \mathcal{V}_I has already been found by D. Calaque and C. Rossi [1], cf. the Remark before section 1.4. The main result of this section is the following theorem.

⁴This differs slightly from the formula in [19] since we included the symmetry factor j of the wheel in the weight.

Theorem 4.1. *The operation $\mathcal{V}_H^\pi(\gamma)$ satisfies*

$$\begin{aligned} \mathcal{V}_0^\pi H_{\mathcal{U}_1^\pi(\gamma)} - H_\gamma \mathcal{V}_0^\pi(-1)^{\deg_H} - \mathcal{V}_1^\pi(\gamma) + (-1)^{\deg_f} d\mathcal{V}_I^\pi(\gamma) + \mathcal{V}_I^\pi(\gamma)B(-1)^{\deg_H} = \\ = -(-1)^{|\gamma|} \mathcal{V}_H^\pi(\gamma)b_\star - L_\pi \mathcal{V}_H^\pi(\gamma) - \mathcal{V}_H^\pi([\pi, \gamma]) \end{aligned}$$

where \deg_H and \deg_f are the Hochschild and differential form grading operators.

Proof. This is a standard Kontsevich-Stokes proof. Concretely, one applies Stokes' Theorem to the integrals

$$\int_{\pi_1^{-1}(U_H)} d\omega_\Gamma = \int_{\partial\pi_1^{-1}(U_H)} \omega_\Gamma.$$

The left hand side vanishes since the weight forms are closed. Computing the right hand side yields some polynomial relations between weights, which are equivalent to the equation in the Theorem. In our case, the following codimension 1 boundary strata appear.

- (1) $k \geq 2$ interior vertices can approach each other in the interior. By a Kontsevich lemma, the contribution is zero unless $k = 2$. If the vertex 1 (corresponding to γ) is one of the two vertices collapsing, the result is the term $\mathcal{V}_H^\pi([\pi, \gamma])$. If not, the result is zero by the Jacobi identity $[\pi, \pi] = 0$.
- (2) Some interior vertices (except 1) approach the center 0. This produces the term $L_\pi \mathcal{V}_H^\pi(\gamma)$, see [13].
- (3) A cluster of k interior vertices, one of them is 1, approaches the center 0. If $k > 1$ this yields zero contribution by (a slight variation of) Kontsevich's Lemma. If $k = 1$, i.e., only the vertex 1 approaches 0, we claim that the contribution is $\pm H_\gamma \mathcal{V}_0^\pi$. One might take this as the definition of H_γ . But let us nevertheless show that the term produced equals $\pm \frac{1}{2} dL_\gamma$. Let α be the angle between the inner and outer framing. Start by integrating out the position of vertex 1. As in the previous stratum, this produces a term L_γ , but there remains a factor α in the weight form due to the restricted integration domain. Integrating out the inner framing yields an overall de Rham derivative (as in [17]), weighted by a factor $\frac{1}{2} = \int_0^1 \alpha d\alpha$.
- (4) The vertex 1 (corresponding to γ) can approach the line connecting 0 and $\bar{0}$. In this case one can easily integrate out the framing at the center vertex (see [17]) and obtains the term $d\mathcal{V}_I^\pi(\gamma)$.
- (5) The vertex 1 can approach the line corresponding to the framing at 0. The result is equal to $\mathcal{V}_I^\pi(\gamma)B$.
- (6) The inner and outer framing align. This leaves no restriction on the position of the vertex 1 and hence produces $\mathcal{V}_1^\pi(\gamma)$.
- (7) Vertices on the boundary circle and zero or more interior vertices approach each other. If the vertex 1 (corresponding to γ) is not in the approaching cluster, this produces the term $\mathcal{V}_H^\pi(\gamma)b_\star$.
- (8) If the vertex 1 is in the approaching cluster, the resulting term is equal to $\mathcal{V}_0^\pi H_{\mathcal{U}_1^\pi(\gamma)}$.

We apologize for not being very precise about signs. □

To globalize the above result, one has to replace multivector fields, multidifferential operators, differential forms and Hochschild chains by their Fedosov resolutions and apply the above map fiberwise, as usual. See, e.g., [5] for a thorough account of the standard globalization procedure.

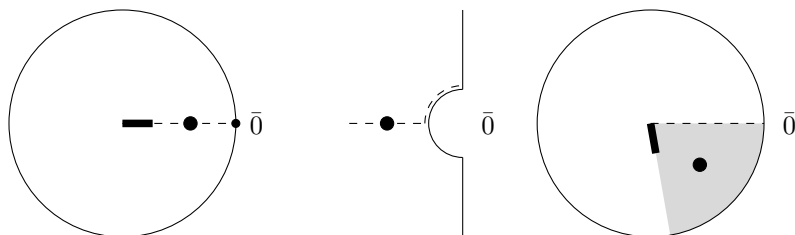


FIGURE 2. The subspaces U_I (left) and U_H (right). On the left, the inner and outer framings are aligned and the vertex is restricted to this common line. The middle picture shows a magnification of the left in an infinitesimal neighborhood of $\bar{0}$. On the right, the framings are not aligned and the vertex may move in the shaded region between them. Note that on the right the framings are not fixed, but part of the configuration.

4.1. **(Remainder of) Proof of part (2) of Proposition 1.4.** The above result shows that the equality in Proposition 1.4 is satisfied in linear order in u . For the sign, note that in our case $\gamma = \hat{\pi}$ is even, as are all forms and Hochschild chains involved.

The quadratic order in u of the equality in Proposition 1.4 asserts that the homotopy \mathcal{V}_H^π is again compatible with de Rham and Rinehart differential, i.e. that

$$d\mathcal{V}_H^\pi(\gamma) + \mathcal{V}_H^\pi(\gamma)B = 0.$$

Actually, both terms vanish separately.⁵ We restrict ourselves to the local case: Integrating out the inner framing in the explicit integral formula for $\mathcal{V}_H^\pi(\gamma)$ shows that $\mathcal{V}_H^\pi(\gamma) = d(\dots)$ and hence the first term vanishes. On the other hand, consider $\mathcal{V}_H^\pi(\gamma)B$. It consists of terms having 1 inserted at the marked point on the boundary of the disk (the outer framing). Hence there is no angle form depending on the outer framing and the integral vanishes. \square

4.2. **Globalization and the proof of part (1) of Proposition 1.4.** Let us next turn to the proof of part (2) of Proposition 1.4. By the previous arguments, we know that part (2) of that proposition holds. Hence we already know that part (1) holds in the local case, i.e., $M = \mathbb{R}^n$. What we need to check is that the statement survives globalization. Let us begin by introducing some notation, which will simplify the following discussion.

Definition 4.2. Let \mathfrak{g} be a dg Lie, or L_∞ algebra and let \mathfrak{m} be a dg Lie, or L_∞ module over $\mathbb{C}((u))$. We call \mathfrak{m} a *GM module*⁶ if it is additionally endowed with an operation

$$\hat{I}: \mathfrak{g} \otimes \mathfrak{m} \rightarrow \mathfrak{m}$$

⁵This could be derived from the *KS* relations, together with the facts $B \circ H = H \circ B = 0$ and the vanishing of the homotopy corresponding to the composition. However, we do it by hand here.

⁶GM here stands for Gauss-Manin. We made up the name just to convey the correct intuition in the present context, it is not historically motivated.

such that

$$(6) \quad \left[D, \hat{I}_\gamma \right] + \hat{I}_{b\gamma} = uL_\gamma$$

where D is the differential on \mathfrak{m} , L denotes the action of \mathfrak{g} on \mathfrak{m} and b is the differential on \mathfrak{g} .

Let \mathfrak{m}' be another GM module and suppose $\mathcal{V} : \mathfrak{m} \rightarrow \mathfrak{m}'$ is an L_∞ morphism. It is then easy to check that for any cocycle $\gamma \in \mathfrak{g}$, the operation

$$\Delta_\gamma := \left[\hat{I}_\gamma, \mathcal{V}_0 \right] + u\mathcal{V}_1(\gamma)$$

commutes with the differential. Hence it induces a map on homologies $H(\mathfrak{m}) \rightarrow H(\mathfrak{m}')$.

Definition 4.3. In the situation above, we say that \mathcal{V} is *compatible with the Gauss-Manin connection* if the operations Δ_γ vanish on homology, for all cocycles $\gamma \in \mathfrak{g}$.

The differential forms $\Omega((u))$ and the cyclic chains PC are both GM modules, with operations \hat{I} as introduced in the previous sections. Our goal is to show that the map $\mathcal{V}^\pi : PC[[\hbar]] \rightarrow \Omega((u))[[\hbar]]$ is compatible with the Gauss-Manin connection. Let us make some elementary observations we will need later.

Lemma 4.4. *Let \mathfrak{g} be a Lie or L_∞ algebra and let $\mathfrak{m}, \mathfrak{m}', \mathfrak{m}''$ be GM modules. Let $f : \mathfrak{m} \rightarrow \mathfrak{m}'$ and $g : \mathfrak{m}' \rightarrow \mathfrak{m}''$ be (L_∞) morphisms of modules, that are compatible with the Gauss-Manin connection. Then $g \circ f : \mathfrak{m} \rightarrow \mathfrak{m}''$ is also compatible with the Gauss-Manin connection.*

Lemma 4.5. *Let \mathfrak{g} be a Lie or L_∞ algebra, let $\mathfrak{m}, \mathfrak{m}'$ be GM modules, and let $f : \mathfrak{m} \rightarrow \mathfrak{m}'$ be an L_∞ morphism compatible with the Gauss-Manin connection. Let \mathfrak{g}' be another L_∞ algebra and $F : \mathfrak{g}' \rightarrow \mathfrak{g}$ be an L_∞ morphism. Then the pulled back L_∞ modules $F^*\mathfrak{m}, F^*\mathfrak{m}'$ over \mathfrak{g}' carry natural structures of GM modules, and the map $F^*f : F^*\mathfrak{m} \rightarrow F^*\mathfrak{m}'$ is compatible with the Gauss-Manin connection.*

Remark 4.6. The notations “GM module” and compatibility with the “Gauss-Manin connection” used here should not be taken too serious. In all cases we apply the above lemmas, \hat{I} will be the connection form of the Gauss-Manin connection, but in general, it is actually not justified to use this name.

4.3. Reminder on the Dolgushev-Fedosov globalization procedure. Let us briefly recall V. Dolgushev’s globalization procedure. For details, we refer to [5]. Fix an affine, torsion-free connection on M . Let $T_{\text{poly}}^{fml}, \Omega^{fml}, D_{\text{poly}}^{fml}$ and C^{fml} be V. Dolgushev’s resolutions of $T_{\text{poly}}, \Omega, D_{\text{poly}}$ and $C = C(A)$.⁷ The differentials on these spaces depend on the choice of connection. We will denote the part of the differential coming from the connection by ∇ , though this is slightly misleading. One has the following diagram

$$\begin{array}{ccccccc} (T_{\text{poly}}, 0) & \xrightarrow{F} & (T_{\text{poly}}^{fml}, \nabla) & \longrightarrow & (D_{\text{poly}}^{fml}, \nabla + b) & \xleftarrow{G} & (D_{\text{poly}}, b) \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ (\Omega, 0) & \xrightarrow{f} & (\Omega^{fml}, \nabla) & \xleftarrow{\mathcal{V}_f} & (C^{fml}, \nabla + b) & \xleftarrow{g} & (C, b). \end{array}$$

⁷In this subsection we will omit the symbol “•” from complexes to avoid cluttering the notation too much. E.g., we write Ω instead of $\Omega^{-\bullet}$.

Here the objects in the upper row are dg Lie algebras, connected by a chain of L_∞ quasi-isomorphisms. The left- and rightmost maps (F , and G) are the “flat lifts”, and are ordinary dg Lie algebra morphisms. The middle map is obtained by applying M. Kontsevich’s L_∞ -morphism fiberwise. The objects in the bottom row are dg Lie algebra modules, the double arrows shall symbolize the action. The arrows in the bottom row are L_∞ morphisms of modules. For example, the leftmost arrow in the bottom row denotes an L_∞ -morphism of L_∞ -modules over T_{poly} , where Ω^{fml} is considered T_{poly} -module by pulling back the T_{poly}^{fml} -module structure along the map $T_{\text{poly}} \rightarrow T_{\text{poly}}^{fml}$. Again the left- and rightmost maps (f and g) in the bottom row are the “flat lifts” and contain only a single non-vanishing Taylor component. The middle map is constructed by applying B. Shoikhet’s morphism of L_∞ -modules fiberwise. From this diagram it follows that there is an L_∞ -morphism $T_{\text{poly}} \rightarrow D_{\text{poly}}$ and an L_∞ morphism of modules $C \rightarrow \Omega$. However, to obtain it, one needs to invert the arrows f and G , up to homotopy.

Next let us recall the results of [17]. It is shown there that cyclic differential B and the de Rham differential d are intertwined by Shoikhet’s L_∞ -morphism. Furthermore, the “flat lifts” also commute with these operators. Hence one has the following diagram of L_∞ quasi-isomorphisms

$$\begin{array}{ccccccc} (T_{\text{poly}}, 0) & \xrightarrow{F} & (T_{\text{poly}}^{fml}, \nabla) & \longrightarrow & (D_{\text{poly}}^{fml}, \nabla + b) & \xleftarrow{G} & (D_{\text{poly}}, b) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\Omega((u)), ud) & \xrightarrow{f} & (\Omega^{fml}((u)), \nabla + ud_f) & \longleftarrow & (PC^{fml}, \nabla + b + uB_f) & \xleftarrow{g} & (PC, b + uB). \end{array}$$

Here PC (PC^{fml}) denotes the (resolution of the) periodic cyclic complex. The subscripts “f” in d_f and B_f shall indicate that the operators are applied fiberwise. Next one needs to invert the arrows f and G . By abstract reasons, there is an L_∞ quasi-isomorphism $\tilde{f} : \Omega^{fml}((u)) \rightarrow \Omega((u))$ such that $\tilde{f} \circ f$ is homotopic to the identity. For us, this means that there is an L_∞ quasi-isomorphism

$$\Phi : \Omega((u)) \rightarrow \Omega((u))[t, dt]$$

such that $\Phi(t = 0) = id$ and $\Phi(t = 1) = \tilde{f} \circ f$. Similarly, the morphism G can be inverted up to homotopy to an L_∞ quasi-isomorphism \tilde{G} , such that $G \circ \tilde{G}$ is homotopic to the identity. However, we need to simultaneously modify the morphism on modules g . Note that the naive guess, i.e., the pullback of g by \tilde{G} , write \tilde{G}^*g , will only produce a morphism of modules

$$PC \rightarrow (G \circ \tilde{G})^* PC^{fml}.$$

Here the notation means that the L_∞ -module structure on PC^{fml} is changed by pull-back along the endomorphism $G \circ \tilde{G}$ of D_{poly}^{fml} . The L_∞ modules $(G \circ \tilde{G})^* PC^{fml}$ and PC^{fml} are L_∞ -isomorphic, but one should be careful and choose the isomorphism as follows: The structures of an L_∞ algebra \mathfrak{g} and an L_∞ module \mathfrak{m} can be packaged into a single L_∞ structure on the semi-direct product $\mathfrak{g} \ltimes \mathfrak{m}$. In our case, $\mathfrak{g} = D_{\text{poly}}^{fml}$ and $\mathfrak{m} = PC^{fml}$. The homotopy linking $G \circ \tilde{G}$ to the identity can be extended to an L_∞ quasi-isomorphism

$$\Psi : \mathfrak{g} \ltimes \mathfrak{m} \rightarrow \mathfrak{g} \ltimes \mathfrak{m}[t, dt]$$

such that $\Psi(t = 0)$ is the identity and the \mathfrak{g} -component of $\Psi(t = 1)$ is $G \circ \tilde{G}$. Put an auxiliary grading on $\mathfrak{g} \times \mathfrak{m}$ by declaring \mathfrak{m} to be concentrated in degree 1 and \mathfrak{g} in degree 0. Then Ψ can be chosen to respect this grading. In other words its Taylor components vanish if more than one input is in \mathfrak{m} and have output in \mathfrak{m} iff exactly one input is in \mathfrak{m} .

The \mathfrak{m} -component of $\Psi(t = 1)$ gives an L_∞ isomorphism

$$PC^{fml} \rightarrow (G \circ \tilde{G})^* PC^{fml}$$

which can be inverted and composed with $(G \circ \tilde{G})^* g$ to yield the desired morphism, say \tilde{g} . The L_∞ morphism

$$\mathcal{V}: (PC, b + uB) \rightarrow (\Omega((u)), ud)$$

we use in the present paper is then the composition of the quasi-isomorphisms

$$\mathcal{V} = \tilde{f} \circ \mathcal{V}_f \circ \tilde{g}.$$

Note that there is no canonical choice for the inverses up to homotopy we had to pick above. Next we pick a formal Poisson structure $\pi \in \hbar T_{\text{poly}}[[\hbar]]$ and twist all occurring L_∞ algebras and modules. This yields the following diagram

$$(7) \quad \begin{array}{ccccccc} (T_{\text{poly}}[[\hbar]], [\pi, \cdot]) & \xrightarrow{F} & (T_{\text{poly}}^{fml}[[\hbar]], \nabla + [\pi, \cdot]) & \xrightarrow{\mathcal{U}^\pi} & (D_{\text{poly}}^{fml}[[\hbar]], \nabla + b_\star) & \xrightarrow{\tilde{G}^\pi} & (D_{\text{poly}}[[\hbar]], b_\star) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\Omega((u))[[\hbar]], L_\pi + ud) & \xleftarrow{\tilde{f}^\pi} & (\Omega^{fml}((u))[[\hbar]], \nabla + L_\pi + ud_f) & \xleftarrow{\mathcal{V}_f^\pi} & (PC^{fml}[[\hbar]], \nabla + b_\star + uB_f) & \xleftarrow{\tilde{g}^\pi} & (PC[[\hbar]], b_\star + uB). \end{array}$$

The composition of the lower arrows is \mathcal{V}^π . Our goal is to show that this morphism \mathcal{V}^π is compatible with the Gauss-Manin connection. By Lemmas 4.4 and 4.5 it is sufficient to check this statement for each of the three squares separately, i.e., show that \tilde{f}^π , \mathcal{V}_f^π and \tilde{g}^π are each compatible with the Gauss-Manin connection. This is done as follows:

- (1) Note that \tilde{f}^π is the inverse, up to homotopy, of the twisted version f^π of the morphism f . (In fact, the morphisms f and f^π are identical since f , the flat lift, has only a single non-vanishing Taylor component.) It is easy to see that f^π is compatible with the Gauss-Manin connection. Hence it suffices to show that compatibility with the Gauss-Manin connection is preserved under taking inverses up to homotopy. This statement is Corollary 4.9 of Section 4.5.
- (2) For the middle square, note that \mathcal{V}_f^π is obtained by applying B. Shoikhet's local L_∞ morphism fiberwise. Part (2) of Proposition 1.4 states that this local morphism is compatible with the Gauss-Manin connection, and even gives an explicit homotopy. Using the fiberwise version of this homotopy, one shows that \mathcal{V}_f^π is indeed compatible with the Gauss-Manin connection.
- (3) For the rightmost square, note that \tilde{g}^π is the composition

$$(G^\pi)^* PC[[\hbar]] \rightarrow (G^\pi \circ \tilde{G}^\pi)^* PC^{fml}[[\hbar]] \rightarrow PC^{fml}[[\hbar]].$$

The left hand morphism is a pullback of the flat lift, which is compatible with the Gauss-Manin connection on the nose. Hence, by Lemma 4.5 it is itself compatible with the Gauss-Manin connection. It will be shown in

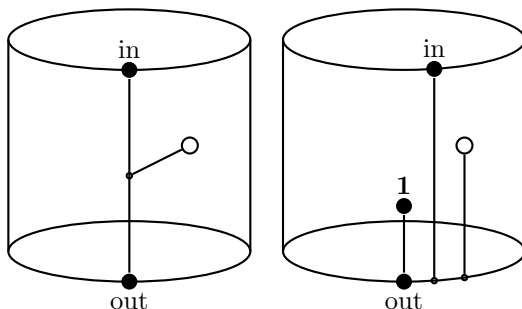


FIGURE 3. The operations I (cap product, left) and H (right) in the Kontsevich-Soibelman operad.

section 4.6 that the right hand morphism as well is compatible with the Gauss-Manin connection.

4.4. A conceptual remark, and the Kontsevich-Soibelman operad. The above proof is conceptually not very satisfying. The clean way to do this proof is to show that (i) M. Kontsevich's morphism and the Shoikhet-Dolgushev morphism can be extended to a morphism of algebras over the Kontsevich-Soibelman (KS) operad [11], up to homotopy, and (ii) that the L_∞ part of a KS map up to homotopy is compatible with the Gauss-Manin connection. Operations in the KS operad are given by certain graphs. For example, the graphs corresponding to the operations I and H above are depicted in Figure 3. To show the existence of a KS map up to homotopy, one has to construct a tower of homotopies, for the various operations in the KS operad. We construct here only a low order homotopy, corresponding to the operation H . The full construction will be published in a forthcoming paper.

4.5. Compatibility of the Gauss-Manin connection with L_∞ -homotopies.

Lemma 4.7. *Suppose \mathfrak{g} is an L_∞ algebra and $\mathfrak{m}, \mathfrak{m}'$ are GM modules. Suppose $\mathcal{V}, \mathcal{W} : \mathfrak{m} \rightarrow \mathfrak{m}'$ are two homotopic L_∞ morphisms of modules. Then \mathcal{V} is compatible with the Gauss-Manin connection iff \mathcal{W} is.*

Remark 4.8. For us, \mathcal{V}, \mathcal{W} are homotopic if there is an L_∞ morphism $f : \mathfrak{m} \rightarrow \mathfrak{m}'[t, dt]$ such that $f(t=0) = \mathcal{V}$, $f(t=1) = \mathcal{W}$.

Proof. Let f be as in the remark. It fits into a diagram, commutative in homology

$$\mathfrak{m} \xrightarrow{f} \mathfrak{m}'[t, dt] \rightrightarrows \mathfrak{m}'$$

where the two right hand arrows are the evaluations at $t=0$ and $t=1$ respectively. $\mathfrak{m}'[t, dt]$ is a GM module by t linear extension of the GM structure on \mathfrak{m}' . Let us denote the operator giving the GM structure also by \hat{I} . We know that for a closed $\gamma \in \mathfrak{g}$ the morphism

$$\Delta_\gamma := [\hat{I}_\gamma, f_0] + uf_1(\gamma)$$

commutes with the differential and hence defines an operation $[\Delta_\gamma]$ on homology. It fits into a commutative diagram

$$H(\mathfrak{m}) \xrightarrow{[\Delta_\gamma]} H(\mathfrak{m}'[t, dt]) \rightrightarrows H(\mathfrak{m}').$$

Hence the lower composition is zero iff the upper composition is. This proves the Lemma. \square

Corollary 4.9. Suppose \mathfrak{g} is an L_∞ algebra, $\mathfrak{m}, \mathfrak{m}'$ are GM modules and $\mathcal{V} : \mathfrak{m} \rightarrow \mathfrak{m}'$ is an L_∞ quasi-isomorphism, compatible with the Gauss-Manin connection. Then any inverse up to homotopy $\tilde{\mathcal{V}} : \mathfrak{m}' \rightarrow \mathfrak{m}$ is also compatible with the Gauss-Manin connection.

Proof. We have to show that

$$\Delta_\gamma := \left[\hat{I}_\gamma, \tilde{\mathcal{V}}_0 \right] + u\tilde{\mathcal{V}}_1(\gamma)$$

vanishes on homology. Since \mathcal{V} is a quasi-isomorphism, this is equivalent to saying that

$$\left(\left[\hat{I}_\gamma, \tilde{\mathcal{V}}_0 \right] + u\tilde{\mathcal{V}}_1(\gamma) \right) \circ \mathcal{V}_0 = \left[\hat{I}_\gamma, \tilde{\mathcal{V}}_0 \circ \mathcal{V}_0 \right] + u\tilde{\mathcal{V}}_1(\gamma) \circ \mathcal{V}_0 + u\tilde{\mathcal{V}}_0 \circ \mathcal{V}_1(\gamma) + [D, \dots]$$

vanishes on homology. For the equality we used that \mathcal{V} is compatible with the Gauss-Manin connection. But vanishing of the right hand expression on homology just means that the composition $\tilde{\mathcal{V}} \circ \mathcal{V}$ is compatible with the Gauss-Manin connection. But this follows from the Lemma since $\tilde{\mathcal{V}} \circ \mathcal{V}$ is homotopic to the identity by definition of “inverse up to homotopy”. \square

This Corollary in particular asserts that the map \tilde{F}^π in the left hand square of the diagram (7) is compatible with the Gauss-Manin connection.

4.6. Compatibility of the Gauss-Manin connection with algebra homotopies. Let again \mathfrak{g} be an L_∞ algebra, and \mathfrak{m} a GM module. Suppose we have an L_∞ endomorphism $\Psi : \mathfrak{g} \times \mathfrak{m} \rightarrow \mathfrak{g} \times \mathfrak{m}$ homotopic to the identity. We also assume, as usual, that Ψ and the homotopy respect the \mathfrak{m} degree, i.e., all Taylor components vanish when at least two of its arguments are in \mathfrak{m} , and the output is \mathfrak{m} if exactly one input is. In this case Ψ encodes an L_∞ morphism (of L_∞ algebras) $F^1 : \mathfrak{g} \rightarrow \mathfrak{g}$, together with a morphism of L_∞ modules $f^1 : \mathfrak{m} \rightarrow (F^1)^*\mathfrak{m}$.

Lemma 4.10. *In the situation above, f^1 is compatible with the Gauss-Manin connection.*

Proof. Since Ψ is homotopic to the identity, we can extend Ψ to a morphism $\tilde{\Psi} : \mathfrak{g} \times \mathfrak{m} \rightarrow \mathfrak{g} \times \mathfrak{m}[t, dt]$, such that the compositions with the evaluation at $t = 0$ and $t = 1$ are the identity and Ψ respectively. In particular there is an L_∞ morphism $F : \mathfrak{g} \rightarrow \mathfrak{g}[t, dt]$, and an L_∞ morphism of L_∞ \mathfrak{g} -modules $f : \mathfrak{m} \rightarrow F^*\mathfrak{m}[t, dt]$, such that the evaluations at $t = 0$ and $t = 1$ are the identities and F^1, f^1 respectively. Here, as before, $F^*\mathfrak{m}[t, dt]$ is $\mathfrak{m}[t, dt]$, endowed with a \mathfrak{g} -module structure by pulling back the natural $\mathfrak{g}[t, dt]$ module structure via F . Note that $F^*\mathfrak{g}[t, dt]$ is also a GM module, with the pulled back GM structure, and that the evaluation maps are compatible with the Gauss-Manin connection. Denote the GM structure on $\mathfrak{m}[t, dt]$ again by \hat{I} , and for a closed $\gamma \in \mathfrak{g}$ define as before

$$\Delta_\gamma := \left[\hat{I}_\gamma, f_0 \right] + u f_1(\gamma).$$

It induces a map $[\Delta_\gamma]$ on homology, whose composition with the evaluation at $t = 0$, i.e.,

$$H(\mathfrak{m}) \xrightarrow{[\Delta_\gamma]} H(\mathfrak{m}[t, dt]) \xrightarrow{ev_{t=0}} H(\mathfrak{m}').$$

is zero. Since the evaluations are quasi-isomorphisms it follows that $[\Delta] = 0$, i.e., f is compatible with the Gauss-Manin connection. But then also $f^1 = ev_{t=1} \circ f$ is compatible with the Gauss-Manin connection. \square

A twisted version of this Lemma produces the main result of this section, finishing the proof of Proposition 1.4.

Lemma 4.11. *The map \tilde{g}^π of section 4.3 is compatible with the Gauss-Manin connection.*

Proof. Using the notation of Section 4.3, the map \tilde{g}^π is a composition

$$(G^\pi)^* PC[[\hbar]] \rightarrow (G^\pi \circ \tilde{G}^\pi)^* PC^{fml}[[\hbar]] \rightarrow PC^{fml}[[\hbar]].$$

The left hand morphism is a pullback of the flat lift g , which is compatible with the Gauss-Manin connection on the nose. Hence, by Lemma 4.5 it is itself compatible with the Gauss-Manin connection. To show that the right hand morphism is compatible, it is sufficient to show that its homotopy inverse

$$PC^{fml}[[\hbar]] \rightarrow (F^{1,\pi})^* PC^{fml}[[\hbar]]$$

is. Here $F^{1,\pi}$ is the twisted version of $F^1 = G \circ \tilde{G}$. By construction of this map in Section 4.3, there are quasi-isomorphisms

$$PC^{fml}[[\hbar]] \rightarrow (F^\pi)^* PC^{fml}[t, dt][[\hbar]] \rightrightarrows PC^{fml}[[\hbar]]$$

such that the upper composition is the identity, and the lower is \tilde{G}^π . Here the two morphisms on the right are the evaluations at 0 and 1. Note that $PC^{fml}[[\hbar]]$ is endowed with a GM structure. Hence also $(F^m)^* PC^{fml}[t, dt][[\hbar]]$ is a GM algebra by t linear extension and pullback. Since all maps are quasi-isomorphisms and the diagram is commutative in homology, the statement of the Lemma then follows in exactly the same way as in the previous proof. \square

5. RELATION TO THE TAMARKIN -TSYGAN INDEX THEOREM

As observed by D. Tamarkin and B. Tsygan [14] the results on cyclic formality may be used to prove an algebraic index theorem as follows. Let π be a Poisson structure and $\pi_t = t\pi$ for $t \in \mathbb{R}$. Let A_t be the space $A = C^\infty(M)[[\hbar]]$ with the star product corresponding to π . Let $c \in PH_\bullet(A_1)$ be a cyclic homology class. By parallel transport wrt. the Gauss-Manin connection (see Theorem 1.3) one obtains a family of cyclic homology classes $c(t) \in PH_\bullet(A_t)$ such that $c(1) = c$. In particular $c(0) \in PH_\bullet(A)$. By Connes isomorphism (see section 3) $PH_\bullet(A) \cong H^\bullet(M)((u))$. We denote the image of $c(0)$ by $ch(c) \in H^\bullet(M)((u))$.

On the other hand one has a map from periodic cyclic to cyclic homology $PH_\bullet(A_1) \rightarrow CH_\bullet(A_1)$. On chains, this map is given by sending positive powers of u to zero. Furthermore, the zeroth cyclic homology is isomorphic to the zeroth Hochschild homology $CH_0(A_1) \cong HH_0(A_1)$. By the Shoikhet-Dolgushev morphism \mathcal{V}^π the latter is, in turn, isomorphic to zeroth Poisson homology $HP_0(M)$. Assume now that there is a volume form Ω on M and π is a unimodular with $\text{div}_\Omega \pi = 0$.⁸ Then there is a natural map $HP_0(M) \rightarrow \mathbb{C}$ by $f \mapsto \int_M f \Omega$.

⁸A Poisson structure is called unimodular if its divergence with respect to some (and hence every) volume form is a Hamiltonian vector field. If this is the case, rescaling the volume form by the exponential of the Hamiltonian function yields a volume form with respect to which the Poisson structure is divergence-free. If $\{ , \}$ denotes the Poisson bracket, then the last condition is equivalent to $\int_M \{f, g\} \Omega = 0, \forall f, g$.

Following Tamarkin and Tsygan, let us denote the composition by⁹

$$I: PH_0(A_1) \rightarrow CH_0(A_1) \xrightarrow{\mathcal{V}_0^\pi} HP_0(M) \rightarrow \mathbb{C}$$

$$I(c) = \int_M \mathcal{V}_0^\pi(c) \Omega.$$

By Theorem 1.3 we know that

$$[\mathcal{V}_0^\pi(c)] = [e^{-\iota_\pi/u} \mathcal{V}_0^{\pi=0}(c(0))].$$

Furthermore, by Proposition 3.1

$$\mathcal{V}_0^{\pi=0}(c(0)) = \hat{A}_u(M) ch(c).$$

Hence we arrive at the following Theorem, known as the Tamarkin-Tsygan index Theorem. The statement can be found in [14].

Theorem 5.1. *Let M be a compact manifold, $\pi \in \mathfrak{h}\Gamma(M, \wedge^2 TM)[[\hbar]]$ a formal Poisson structure, Ω a volume form on M with $\operatorname{div}_\Omega \pi = 0$ and $c \in PH_0(A_\hbar)$. Then*

$$I(c) = \int_M \hat{A}_u(M) ch(c) e^{\iota_\pi/u} \Omega.$$

APPENDIX A. GRAPHS AND WEIGHTS

We briefly recall here some standard facts about Kontsevich graphs and their weights. For more details, see [9].

Definition A.1. The set $G(m, n)$, $m, n \in \mathbb{N}_0$ of *Shoikhet graphs* consists of directed graphs Γ such that

- (1) The vertex set of Γ is

$$V(\Gamma) = \{0, \dots, m\} \cup \{\bar{0}, \dots, \bar{n}\}$$

where the vertices $\{1, \dots, m\}$ will be called the *type I* vertices and the vertices $\{\bar{0}, \dots, \bar{n}\}$ the *type II* vertices.

- (2) There are no double edges, i.e., edges (j, w) occurring twice in $E(\Gamma)$.
- (3) Every edge $e = (v, w) \in E(\Gamma)$ starts at a type I vertex, i.e., $v \in \{1, \dots, m\}$.
- (4) No edge ends at vertex 0.
- (5) For each type I vertex j , there is an ordering given on

$$\operatorname{Star}(j) = \{(j, w) \mid (j, w) \in E(\Gamma), w \in V(\Gamma)\}.$$

- (6) There are no tadpoles, i.e., edges of type (j, j) .

Let us next define the (Shoikhet) weight w_Γ of $\Gamma \in G(m, n)$. It is an integral of a certain differential form over a compact manifold with corners, the *configuration space* $C_{m,n}$.

$$(8) \quad w_\Gamma = \int_{C_\Gamma} \omega_\Gamma$$

⁹Note that by form degree reasons, terms with positive powers of u , though present in the integrand, do not contribute to I .

Definition A.2. The enlarged Shoikhet configuration space $\tilde{C}_{m,n}$ is the space of embeddings

$$(z_0, \dots, z_m, z_{\bar{0}}, \dots, z_{\bar{n}}) : V(\Gamma) \rightarrow D$$

of the set $\{0, \dots, m\} \cup \{\bar{0}, \dots, \bar{n}\}$ into the closed unit disk $D = \{z \in \mathbb{C}; |z| \leq 1\}$, together with a distinguished direction at vertex 0, such that

- (1) All type I vertices are mapped to the interior of D , i.e. $z_j \in D^\circ$ for $j = 0, \dots, m$.
- (2) All type II vertices are mapped to the boundary of D , i.e. $z_{\bar{j}} \in \partial D$ for $j = 0, \dots, n$.
- (3) The type II vertices occur in counterclockwise increasing order on the circle, i.e., $0 < \arg \frac{z_{\bar{1}}}{z_{\bar{0}}} < \dots < \arg \frac{z_{\bar{n}}}{z_{\bar{0}}} < 2\pi$.

The *Shoikhet configuration space* $C_{m,n}$ is the Fulton-MacPherson-Axelrod-Singer compactification of the quotient of $\tilde{C}_{m,n}$ under the action of the automorphism group of the unit disk $PSU(1, 1)$.

Remark A.3. Note that our configuration space is actually

$$\{\text{B. Shoikhet's original configuration space}\} \times S^1,$$

due to the additional data of a framing at 0. Hence the name is slightly inaccurate.

Remark A.4. Instead of working with equivalence classes of configurations under the $PSU(1, 1)$ action, we will generally work with a canonical representative, obtained by fixing the point z_0 at the center of the disk, i.e., $z_0 = 0$, and fixing the point $z_{\bar{0}}$ to be $z_{\bar{0}} = 1$.

The differential form ω_Γ that is integrated over configuration space can be expressed as a product of one-forms, one for each edge in Γ .

$$\omega_\Gamma = \bigwedge_{(0,v) \in E(\Gamma)} \alpha_1(v) \bigwedge_{j=1}^n \bigwedge_{(j,v) \in E(\Gamma)} \alpha(j, v)$$

Here the one-form $\alpha_0(v)$ is the differential of the (hyperbolic) angle between the framing at the vertex z_0 and the hyperbolic geodesic from z_0 to v . Similarly, the angle $\alpha(j, v)$ is the differential of the hyperbolic angle between the hyperbolic straight lines (z_j, v) and (z_j, z_0) .

To any graph $\Gamma \in G(m+1, n)$ as above one can associate a function D_Γ taking m vector fields $\gamma_1, \dots, \gamma_m$ and $n+1$ functions and returning a d -differential form where d is the degree of vertex 0 in Γ . It is defined such that for a constant d -vector field γ_0

$$\begin{aligned} (9) \quad & (-1)^d \iota_{\gamma_0} D_\Gamma(\gamma_1 \otimes \dots \otimes \gamma_m; a_1, \dots, a_n) = \\ & = \sum_{\varphi: E(\Gamma) \rightarrow [d]} \prod_{j=0}^m (\partial_{\varphi(f_1^j)} \partial_{\varphi(f_2^j)} \dots \gamma_j^{\varphi(e_1^j) \varphi(e_2^j) \dots}) \prod_{k=1}^n (\partial_{\varphi(f_1^k)} \partial_{\varphi(f_2^k)} \dots a_k). \end{aligned}$$

Here the sum runs over all maps φ from the edge set of Γ to the set $\{1, \dots, d\}$. The edges incoming to a vertex v are denoted by f_1^v, f_2^v, \dots and the edges outgoing by e_1^v, e_2^v, \dots .

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