INTEGRATION OF TWISTED POISSON STRUCTURES

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Abstract. Poisson manifolds may be regarded as the infinitesimal form of symplectic groupoids. Twisted Poisson manifolds considered by Ševera and Weinstein [14] are a natural generalization of the former which also arises in string theory. In this note it is proved that twisted Poisson manifolds are in bijection with a (possibly singular) twisted version of symplectic groupoids.

1. Introduction

Poisson manifolds may be regarded as the infinitesimal form of symplectic groupoids [6], i.e., Lie groupoids endowed with a multiplicative symplectic form. Up to singularities, Poisson manifolds may be integrated to symplectic groupoids as described in [2] (conditions under which integration with no singularities is possible are given in [4]). In this paper we generalize this result to the case when the two structures (of symplectic groupoid and of Poisson manifold) are twisted by a closed 3-form.

Let M be a smooth manifold. A pair (π, ϕ), where π is a bivector field and ϕ is a closed 3-form, is called a twisted Poisson structure if it satisfies the equation

$$[\pi\,,\,\pi] = \frac{1}{2} \wedge^3 \pi^{\#}\phi, \tag{1.1}$$

where $[\ ,\ ]$ denotes the Schouten–Nijenhuis bracket and $\pi^{\#}$ is the vector bundle homomorphism $T^*M \to TM$ induced by π (viz., $\pi^{\#}(x)(\sigma) := \pi(x)(\sigma, \bullet)$, with $x \in M$, $\sigma \in T^*_xM$). According to [14], one also says that π is a ϕ-Poisson tensor. In the case $\phi = 0$ one recovers the usual notions of Poisson tensor and Poisson manifold. Twisted Poisson structures have been extensively studied in the physics literature, e.g., [11, 5, 9].

As explained in [14], a twisted Poisson structure induces a Lie algebroid structure on T^*M with anchor map $\pi^{\#}$ and Lie bracket of

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sections σ and τ defined by

$$[\,\sigma\,,\,\tau\,] := \mathrm{L}_{\pi^\#\sigma}\tau - \mathrm{L}_{\pi^\#\tau}\sigma - \mathrm{d}\pi(\sigma,\tau) + \phi(\pi^\#\sigma, \pi^\#\tau, \bullet). \tag{1.2}$$

In particular, $\forall f, g \in C^\infty(M)$ we have:

$$[\mathrm{d}f, \mathrm{d}g] = \mathrm{d}\{f,g\} + \phi(X_f, X_g, \bullet) \tag{1.3}$$

and

$$[X_f, X_g] = X_{\{f,g\}} + \pi^\#(\phi(X_f, X_g, \bullet)), \tag{1.4}$$

with $\{f,g\} = \pi(\mathrm{d}f, \mathrm{d}g)$ and $X_f = \pi^\#\mathrm{d}f$.

We will denote this Lie algebroid by $T^*M_{(\pi,\phi)}$. Sections of its exterior algebra are ordinary differential forms. One may define a derivation δ deforming the de Rham differential by ϕ; viz., we define a graded derivation $\delta\colon \Omega^*(M) \to \Omega^{*+1}(M)$ by setting $\delta f = \mathrm{d}f$ if $f \in C^\infty(M)$ and

$$\delta\sigma = \mathrm{d}\sigma - \iota_{\pi^\#\sigma}\phi,$$

if $\sigma \in \Omega^1(M)$. It turns out that

$$\delta[\,\sigma\,,\,\tau\,] = [\,\delta\sigma\,,\,\tau\,] + [\,\sigma\,,\,\delta\tau\,], \qquad \forall\sigma,\tau \in \Omega^1(M),$$

and that $\delta^2 = [\,\phi\,,\,\bullet\,]$ (where we have extended the Lie bracket to the whole of $\Omega^*(M)$ as a biderivation). So $(T^*M_{(\pi,\phi)}, \delta)$ constitutes an example of a quasi Lie bialgebroid [12, 8], a generalization of Drinfeld's quasi Lie bialgebras [7, 10].

If $T^*M_{(\pi,\phi)}$ may be integrated to a Lie groupoid $(G \rightrightarrows M, \alpha, \beta)$ (i.e., if it exists a Lie groupoid G whose Lie algebroid is $T^*M_{(\pi,\phi)}$), the differential δ induces extra structure on G. Namely, denoting by α and β the source and target maps of G, then G may be endowed with a non-degenerate, multiplicative 2-form ω that satisfies

$$\mathrm{d}\omega = \alpha^*\phi - \beta^*\phi.$$

In other words, (ω, ϕ) is a 3-cocycle for the double complex $\Omega^*(G^{(*)})$, where $G^{(0)} = M$, $G^{(1)} = G$ and elements of $G^{(k)}$ are k-tuples of elements of G that may be multiplied (in the given order). One differential is de Rham and the other is the groupoid-complex differential. Observe that in the true Poisson case (i.e., $\phi = 0$), ω is closed, so G is an ordinary symplectic groupoid. In the general case, G is called a non-degenerate twisted symplectic groupoid, and the non-degenerate 2-form ω is said to be relatively ϕ-closed. The main theorem of the paper (conjectured in [14]) is

Theorem. *There is a bijection between integrable twisted Poisson structures and source-simply connected non-degenerate twisted symplectic groupoids.*

Here "integrable twisted Poisson structure" means that the associated Lie algebroid is integrable.

In Sect. 2 we give an introduction to non-degenerate twisted symplectic groupoids and prove that they induce twisted Poisson structures on the base manifolds (Theorem 2.6 on page 6).

In Sect. 5 we prove the Theorem, though in a more general setting. In fact, as shown in the generaliztion [3] (see also [13]) of the construction given in [2]), to any Lie algebroid A one can associate a topological source-simply-connected groupoid $G(A)$, which is the Lie groupoid integrating A whenever A is integrable. The topological groupoid $G(A)$ is defined as the leaf space of a smooth foliation, as we recall in Sect. 3; so it makes sense to define on it a notion of smooth functions and forms. In the case when A is $T^*M_{(\pi,\phi)}$, we prove that $G(A)$ may always be endowed with a non-degenerate, multiplicative, relatively ϕ-closed 2-form ω. The construction is a modification, described in Sect. 4, of the method developed in [2], where the true Poisson case (i.e., $\phi = 0$) was dealt with.

As a final remark, we mention that general multiplicative 2-forms, their infinitesimal counterparts and their integrations are being treated in [1].

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2. Non-degenerate twisted symplectic groupoids

Definition 2.1. A non-degenerate twisted symplectic groupoid is a Lie groupoid $(G \rightrightarrows M, \alpha, \beta)$ equipped with a non-degenerate 2-form $\omega \in \Omega^2(G)$ and a 3-form $\phi \in \Omega^3(M)$ such that

(1) $\mathrm{d}\phi = 0$;
(2) $\mathrm{d}\omega = \alpha^*\phi - \beta^*\phi$;
(3) ω is multiplicative, i.e., the 2-form $(\omega, \omega, -\omega)$ vanishes when being restricted to the graph of the groupoid multiplication $\Lambda \subset G \times G \times G$.

Let π_G denote the bivector field on G corresponding to ω. Then (π_G, Ω), where $\Omega = \alpha^*\phi - \beta^*\phi$, defines a twisted Poisson structure on G in the sense of [14].

For any $\xi \in \Gamma(A)$, by $\overrightarrow{\xi}$ and $\overleftarrow{\xi}$ we denote its corresponding right and left invariant vector fields on the groupoid G, respectively. The following properties can be easily verified.

Proposition 2.2.

(1) $\epsilon^*\omega = 0$, where $\epsilon\colon M \to G$ is the natural embedding;
(2) $i^*\omega = -\omega$, where $i\colon G \to G$ is the groupoid inversion;
(3) for any $\xi, \eta \in \Gamma(A)$, $\omega(\overrightarrow{\xi}, \overrightarrow{\eta})$ is a right invariant function on G, and $\omega(\overleftarrow{\xi}, \overleftarrow{\eta})$ is a left invariant function on G;
(4) $\omega(\overrightarrow{\xi}, \overleftarrow{\eta}) = 0$;
(5) $\omega(\overrightarrow{\xi}, \overrightarrow{\eta})(x) = -\omega(\overleftarrow{\xi}, \overleftarrow{\eta})(x^{-1})$.

Proof. The proof is standard, and essentially follows from the multiplicativity of ω.

(1) For any $\delta'_m,\ \delta''_m \in T_mM$, since $(\delta'_m, \delta'_m, \delta'_m),\ (\delta''_m, \delta''_m, \delta''_m) \in T\Lambda$, it follows that $\omega(\delta'_m, \delta''_m) = 0$.
(2) $\forall x \in G$ and $\forall \delta'_x,\ \delta''_x \in T_xG$, it is clear that $(\delta'_x, i_*\delta'_x, \alpha_*\delta'_x),\ (\delta''_x, i_*\delta''_x, \alpha_*\delta''_x) \in T\Lambda$. Thus, by (1), we have
$$\omega(\delta'_x, \delta''_x) + \omega(i_*\delta'_x, i_*\delta''_x) = 0,$$
and (2) follows.
(3) For any $\xi, \eta \in \Gamma(A)$, $(\overrightarrow{\xi}(x), 0_y, \overrightarrow{\xi}(xy)),\ (\overrightarrow{\eta}(x), 0_y, \overrightarrow{\eta}(xy)) \in T\Lambda$. Thus
$$\omega(\overrightarrow{\xi}(x), \overrightarrow{\eta}(x)) - \omega(\overrightarrow{\xi}(xy), \overrightarrow{\eta}(xy)) = 0.$$
Hence $\omega(\overrightarrow{\xi}, \overrightarrow{\eta})$ is a right invariant function on G. Similarly, $\omega(\overleftarrow{\xi}, \overleftarrow{\eta})$ is a left invariant function on G.
(4) By considering the vectors $(\overrightarrow{\xi}(x), 0_{\beta(x)}, \overrightarrow{\xi}(x))$ and $(0_x, \overleftarrow{\eta}(\beta(x)), \overleftarrow{\eta}(x)) \in T\Lambda$, we obtain $\omega(\overrightarrow{\xi}(x), \overleftarrow{\eta}(x)) = 0$.
(5) follows from (2) and the fact that $i_*\overrightarrow{\xi} = -\overleftarrow{\xi}$.

□

Define a section $\gamma \in \Gamma(\wedge^2 A^*)$ and a bundle map: $\lambda\colon A \to T^*M$ by

$$\omega(\overrightarrow{\xi}, \overrightarrow{\eta}) = \alpha^*\gamma(\xi, \eta), \quad \forall \xi, \eta \in \Gamma(A), \tag{2.1}$$

and

$$\langle \lambda(\xi), v \rangle = \omega(\overrightarrow{\xi}(m), v), \quad \forall \xi \in A|_m, v \in T_mM \tag{2.2}$$

Lemma 2.3.

(1) $\omega(\overleftarrow{\xi}, \overleftarrow{\eta}) = -\beta^*\gamma(\xi, \eta), \quad \forall \xi,\ \eta \in \Gamma(A)$;
(2) for all $\xi, \eta \in \Gamma(A)$,
$$\gamma(\xi, \eta) = \langle \rho(\xi), \lambda(\eta) \rangle; \tag{2.3}$$
(3) $\lambda\colon A \to T^*M$ is a vector bundle isomorphism.

Proof.

(1) follows from Proposition 2.2 (5).

(2) We have

$$\omega(\overrightarrow{\xi}, \overrightarrow{\eta}) = \omega(\overrightarrow{\xi} - \overleftarrow{\xi}, \overrightarrow{\eta}) = \omega(\overrightarrow{\eta}, \rho(\xi)) = \langle \rho(\xi), \lambda(\eta) \rangle$$

(3) Assume that $\lambda(\xi) = 0$. That is, $\omega(\overrightarrow{\xi}(m), v) = 0,\ \forall v \in T_m M$, which implies that $\overrightarrow{\xi}(m) \lrcorner \omega = 0$ by Proposition 2.2 (4). Hence $\xi = 0$ since ω is non-degenerate. This means that λ is injective. On the other hand, assume that $v \in (\lambda(A|_m))^{\perp}$. Then $\omega(\overrightarrow{\xi}(m), v) = 0,\ \forall \xi \in A|_m$. Thus $v \lrcorner \omega = 0$ using Proposition 2.2 (1), which implies that $v = 0$. Therefore λ is surjective.

□

Lemma 2.4. For any $f \in C^{\infty}(M)$,

$$\overrightarrow{\lambda^{-1}(\mathrm{d}f)} = X_{\alpha^* f}; \quad \overleftarrow{\lambda^{-1}(\mathrm{d}f)} = X_{\beta^* f}. \tag{2.4}$$

Proof. First, one shows that $X_{\alpha^* f}$ is a right invariant vector field on G and $X_{\beta^* f}$ is a left invariant vector field. This can be shown using the same argument as in the case of symplectic groupoids [6]. Namely the multiplicativity of ω together with dimension counting implies that the graph Λ is coisotropic with respect to $(\pi_G, \pi_G, -\pi_G)$. The later implies that $X_{\alpha^* f}$ is a right invariant vector field on G and $X_{\beta^* f}$ is a left invariant vector field.

Second, for any $v \in T_m M$, we have

$$\omega(X_{\alpha^* f}(m), v) = \langle \alpha^* \mathrm{d}f(m), v \rangle = \langle \mathrm{d}f(m), \alpha_* v \rangle = \langle \mathrm{d}f(m), v \rangle$$

It thus follows that $\lambda(X_{\alpha^* f}) = \mathrm{d}f$, or $\overrightarrow{\lambda^{-1}(\mathrm{d}f)} = X_{\alpha^* f}$. The other equation can be proved similarly. □

By pulling back the 2-form $\gamma \in \Gamma(\wedge^2 A^*)$ via λ^{-1}, one obtains a bivector field $\pi \in \Gamma(\wedge^2 TM)$. We introduce a bracket and Hamiltonian vector fields by the usual definitions; i.e., $\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g)$ and $X_f = \pi^{\#}(\mathrm{d}f)$.

Corollary 2.5.

$$\alpha_* \pi_G = \pi; \quad \beta_* \pi_G = -\pi; \tag{2.5}$$

or equivalently

$$\alpha_* X_{\alpha^* f} = X_f; \quad \beta_* X_{\beta^* f} = -X_f, \quad \forall f \in C^{\infty}(M). \tag{2.6}$$

Proof. For any $f,\ g \in C^\infty(M)$,

$$\{\alpha^* f, \alpha^* g\} = \omega(X_{\alpha^* f}, X_{\alpha^* g}) = \omega(\overrightarrow{\lambda^{-1}(\mathrm{d}f)}, \overrightarrow{\lambda^{-1}(\mathrm{d}g)}) = \\ = \alpha^*(\pi(\mathrm{d}f, \mathrm{d}g)) = \alpha^*\{f, g\}.$$

Similarly, we have $\{\beta^* f, \beta^* g\} = -\beta^*\{f, g\}$. □

We are now ready to prove the main result of the section.

Theorem 2.6.

(1) *π is a ϕ-Poisson tensor in the sense of* [14]*—i.e., it satisfies* (1.1).

(2) *The bundle map $\lambda\colon A \to T^*M$ establishes a Lie algebroid isomorphism, where the Lie algebroid on T^*M is induced by the twisted Poisson tensor π as given by Eq. (1.2).*

Proof. Let $\Omega = \alpha^*\phi - \beta^*\phi$. Thus $\forall f,\ g \in C^\infty(M)$

$$\begin{aligned} (X_{\alpha^* f} \wedge X_{\alpha^* g}) \lrcorner \Omega &= (X_{\alpha^* f} \wedge X_{\alpha^* g}) \lrcorner \alpha^*\phi \\ &= \alpha^*[(\alpha_* X_{\alpha^* f} \wedge \alpha_* X_{\alpha^* g}) \lrcorner \phi] \\ &= \alpha^*[X_f \wedge X_g \lrcorner \phi]. \end{aligned}$$

Thus by Eq. (1.4)

$$\begin{aligned} [X_{\alpha^* f}, X_{\alpha^* g}] - X_{\{\alpha^* f, \alpha^* g\}} &= \pi_G^\#(\Omega(X_{\alpha^* f}, X_{\alpha^* g}, \bullet) \\ &= \pi_G^\#(\alpha^*\phi(X_f, X_g, \bullet)) \end{aligned}$$

Thus it follows that

$$\lambda[X_{\alpha^* f}, X_{\alpha^* g}] = \mathrm{d}\{f, g\} + \phi(X_f, X_g, \bullet).$$

Note that λ intertwines the anchors: $\pi^\# \circ \lambda = \rho$, according to Eq. (2.3). Therefore, using Lie algebroid properties, one shows that the push forward Lie algebroid on T^*M via λ is given by Eq. (1.2). This forces, by the Jacobi identity, π to be ϕ-Poisson, and λ is a Lie algebroid isomorphism between A and $(T^*M)_{\pi,\phi}$. □

3. Integration of $T^*M_{(\pi,\phi)}$

We briefly describe the integration procedure for Lie algebroids of [3, 13], adapted to the case of $T^*M_{(\pi,\phi)}$. First one defines the manifold $P(T^*M_{(\pi,\phi)})$ of C^1-Lie algebroid morphisms $TI \to T^*M_{(\pi,\phi)}$, where I is the interval $[0, 1]$ and TI is given its canonical Lie algebroid structure. An element of $PT^*M_{(\pi,\phi)}$ consists of a C^2-path $X\colon I \to M$ together with a section η of $T^*I \otimes X^*T^*M$ satisfying

$$\mathrm{d}X = \pi^\#(X)\eta.$$

On this manifold one may consider as equivalent two elements which are related by a Lie algebroid morphism $T(I \times I) \to T^*M_{(\pi,\phi)}$ that fixes the endpoints. The quotient space $G(T^*M_{(\pi,\phi)})$ may be given a groupoid structure. For our purposes it is however better to use a different description of $G(T^*M_{(\pi,\phi)})$, i.e., as the leaf space of a foliation. Namely, let $P_0\Gamma(T^*M_{(\pi,\phi)})$ be the space of C^2-paths in the Lie algebra of sections of $T^*M_{(\pi,\phi)}$ with endpoints at zero. We give this space the structure of a Lie algebra by the pointwise Lie bracket. One may then define an infinitesimal action of this Lie algebra on $P(T^*M_{(\pi,\phi)})$. To describe it, we prefer to introduce local coordinates $\{x^i\}$ on M (alternatively, one may use a torsion-free connection). Since $\{\mathrm{d}x^i\}$ is a local basis of sections of $T^*M_{(\pi,\phi)}$, we may define structure functions f by

$$\left[\, \mathrm{d}x^i \,,\, \mathrm{d}x^j \,\right] = f^{ij}_k \, \mathrm{d}x^k,$$

where a sum over repeated indices is understood. If we write locally $\pi = \pi^{ij}\partial_i\partial_j$ and $\phi = \phi_{ijk}\mathrm{d}x^i\mathrm{d}x^j\mathrm{d}x^k$, we may compute:

$$f^{ij}_k = \partial_k \pi^{ij} + \pi^{mi}\pi^{nj}\phi_{mnk}.$$

The action is then as follows. To $B \in P_0\Gamma(T^*M_{(\pi,\phi)})$ we associate a vector field ξ_B on $P(T^*M_{(\pi,\phi)})$. We can always write $\xi_B = \xi^h_B + \xi^v_B$ with $\xi^h_B(X,\eta) \in \Gamma(I, X^*TM)$ and $\xi^v_B(X,\eta) \in \Gamma(I, T^*I \otimes X^*T^*M)$. We set then

$$(\xi^h_B(X,\eta))^i = -\pi^{ij}(X)\,(B_X)_j, \tag{3.1a}$$

$$(\xi^v_B(X,\eta))_i = -\mathrm{d}(B_X)_i - f^{rs}_i(X)\,\eta_r\,(B_X)_s, \tag{3.1b}$$

where B_X is the section of X^*T^*M defined by $B_X(t) = B(t)(X(t))$.

Thus, the infinitesimal action of $P_0\Gamma(T^*M_{(\pi,\phi)})$ defines a foliation on $P(T^*M_{(\pi,\phi)})$ and $G(T^*M_{(\pi,\phi)})$ is its quotient space. Let us briefly recall its groupoid structure. The target map α associates to a class of morphisms (X,η) the value of X at 0, while the source map β associates to it the value of X at 1 (observe that the infinitesimal action preserves the endpoints of X). The identity section associates to a point m in M the class $\epsilon(m)$ of the constant path at m with $\eta = 0$. The product is obtained by joining the base paths and restricting the fiber maps consequently (the product is more precisely defined on smooth representatives such that η vanishes with its derivatives at the endpoints).

4. Quasi-symplectic reduction

In this section we describe how to obtain $G(T^*M_{(\pi,\phi)})$ by some sort of symplectic reduction, though our replacement for a symplectic form will be a non-degenerate but not necessarily closed 2-form.

Let T^*PM denote the manifold of C^1-bundle maps $TI \to T^*M$ (over C^2-maps). This space is morally a cotangent bundle and as such it has a canonical symplectic structure Ω_0. Explicitly, a point in T^*PM is a pair (X, η) where X is a C^2-path $I \to M$ and η is a C^1-section of $T^*I \otimes X^*T^*M$. The tangent space at (X, η) is the direct sum of $T^h_{(X,\eta)}T^*PM = \Gamma(I, X^*TM)$ and $T^v_{(X,\eta)}T^*PM = \Gamma(I, T^*I \otimes X^*T^*M)$. Using this splitting, we write

$$\Omega_0(X,\eta)(\xi_1 \oplus e_1, \xi_2 \oplus e_2) = \int_I \langle\, e_1\,,\,\xi_2\,\rangle - \langle\, e_2\,,\,\xi_1\,\rangle\,, \tag{4.1}$$

where $\langle\ \ ,\ \ \rangle$ denotes the canonical pairing between tangent and cotangent fibers of M.

Using the 3-form ϕ on M we may also define a second 2-form on T^*PM:

$$\Omega_1(X,\eta)(\xi_1 \oplus e_1, \xi_2 \oplus e_2) = \frac{1}{2}\int_I \phi(X)(\pi^\#(X)\eta, \xi_1, \xi_2). \tag{4.2}$$

The 2-form $\Omega = \Omega_0 + \Omega_1$ is still non-degenerate but no longer closed.

The manifold $P(T^*M_{(\pi,\phi)})$ introduced in the previous Section may be regarded as a submanifold of T^*PM. If we introduce "momentum maps" $H\colon T^*PM \to P_0\Gamma(T^*M_{(\pi,\phi)})^*$ by

$$H_B(X,\eta) = \int_I \langle\, B_X\,,\,\mathrm{d}X - \pi^\#(X)\eta\,\rangle\,,$$

then $P(T^*M_{(\pi,\phi)})$ is $H^{-1}(0)$. One may check that $\mathrm{d}H_B$ lies in the image of Ω for any $B \in P_0\Gamma(T^*M_{(\pi,\phi)})$; so, since Ω is non-degenerate, one may define a map $B \to \hat{\xi}_B$ that associates a vector field $\hat{\xi}_B$ on T^*PM to B by

$$\iota_{\hat{\xi}_B}\Omega = \mathrm{d}H_B. \tag{4.3}$$

One may easily check that the restriction of $\hat{\xi}_B$ to $P(T^*M_{(\pi,\phi)})$ is tangent to it. More to the point, one may check that the vector field on $P(T^*M_{(\pi,\phi)})$ so obtained is precisely the ξ_B of (3.1) which defines the infinitesimal action of $P_0\Gamma(T^*M_{(\pi,\phi)})$ on $P(T^*M_{(\pi,\phi)})$.

5. Proof of the Theorem

In the setting of the previous Section, we want to prove that the restriction $\underline{\Omega}$ of Ω to $P(T^*M_{(\pi,\phi)})$ is basic w.r.t. to the projection $p\colon P(T^*M_{(\pi,\phi)}) \to G(T^*M_{(\pi,\phi)})$, viz., $\underline{\Omega} = p^*\omega$; moreover, we want to prove that ω satisfies all the required conditions.

Observe that $\underline{\Omega}$ is automatically horizontal by (4.3). On the other hand, unlike the usual symplectic case, it is not clear that $\underline{\Omega}$ is also invariant; in fact, at first, we may only see that $\mathrm{L}_{\xi_B}\underline{\Omega} = \iota_{\xi_B}\mathrm{d}\underline{\Omega} = \iota_{\xi_B}\mathrm{d}\underline{\Omega}_1$, where $\underline{\Omega}_1$ denotes the restriction of Ω_1 to $P(T^*M_{(\pi,\phi)})$. To proceed, we must understand $\underline{\Omega}_1$ better.

Let PM be the manifold of C^2-paths in M. Let $\mathrm{ev}\colon I\times PM \to M$ be the evaluation map and $\mathrm{pr}\colon I\times PM \to PM$ the projection to the second factor. Define $\Phi = \mathrm{pr}_*\,\mathrm{ev}^*\phi \in \Omega^2(PM)$, where pr_* denotes integration along the fiber. If we finally denote by $q\colon P(T^*M_{(\pi,\phi)}) \to PM$ the map that retains only the base map of the Lie algebroid morphism, we realize immediately that

$$\underline{\Omega}_1 = q^*\Phi.$$

By the generalized Stokes' Theorem and the fact that ϕ is closed, we obtain $\mathrm{d}\Phi = \underline{\alpha}^*\phi - \underline{\beta}^*\phi$, where $\underline{\alpha}$ and $\underline{\beta}$ are the maps $PM \to M$ that assign to a path its values at 0 and at 1, respectively. Thus,

$$\mathrm{d}\underline{\Omega} = q^*(\underline{\alpha}^*\phi - \underline{\beta}^*\phi).$$

Since the vector field ξ_B does not move the endpoints, we conclude that $\iota_{\xi_B}\mathrm{d}\underline{\Omega} = 0$, viz., that $\underline{\Omega}$ is invariant as well. We write then $\underline{\Omega} = p^*\omega$ as at the beginning of the Section. The 2-form ω on $G(T^*M_{(\pi,\phi)})$ is clearly multiplicative since the product is defined by joining the paths and Ω is defined as an integral. Moreover, recalling the definition of the source and target map β and α, we observe that $\underline{\alpha}\circ q = \alpha\circ p$ and $\underline{\beta}\circ q = \beta\circ p$. So we may write the equation above as

$$\mathrm{d}\underline{\Omega} = p^*(\alpha^*\phi - \beta^*\phi).$$

Since $\mathrm{d}\underline{\Omega} = p^*\mathrm{d}\omega$ and p is a surjection, this shows that ω is relatively ϕ-closed.

Finally, we need to prove that the 2-form ω is non-degenerate. It is clear from the construction that ω is non-degenerate along the identity M. The claim thus follows from the following:

Lemma 5.1. *A multiplicative 2-form $\omega \in \Omega^2(G)$ on a Lie groupoid $G \rightrightarrows M$ is non-degenerate if and only if it is non-degenerate along the identity M.*

Proof. First of all, note that for any $\delta_x \in T_xG$, and $\xi \in \Gamma(A)$, we have

$$\omega(\overleftarrow{\xi}(x), \delta_x) = \omega(\overleftarrow{\xi}(v), \beta_*\delta_x) \tag{5.1}$$

$$\omega(\overrightarrow{\xi}(x), \delta_x) = \omega(\overrightarrow{\xi}(u), \alpha_*\delta_x), \tag{5.2}$$

where $u = \alpha(x)$ and $v = \beta(x)$. Eq. (5.1), for instance, follows from the fact that both $(\delta_x, \delta_x, \beta_*\delta_x)$, and $(0, \overleftarrow{\xi}(x), \overleftarrow{\xi}(v))$ are tangent to the graph of the groupoid multiplication $\Lambda \subset G\times G\times G$. Eq. (5.2) can be proved similarly. Now assume that $\delta_x \in \ker\omega_x$. It follows from Eq. (5.1) that $\beta_*\delta_x \in \ker\omega_v$ since M is isotropic with respect to ω.

Therefore $\beta_*\delta_x = 0$ by assumption. Hence $\delta_x = \overrightarrow{\eta}(x)$. On the other hand, according to Eq. (5.2), one has $\omega(\overrightarrow{\eta}(u), T_uM) = 0$ since α is a submersion. This implies that $\overrightarrow{\eta}(u) \in \ker \omega_u$. Therefore $\overrightarrow{\eta}(u) = 0$ by assumption. This implies that $\delta_x = \overrightarrow{\eta}(x) = 0$. This concludes the proof. □

We need now to prove that the correspondence between ϕ-twisted Poisson structures and twisted symplectic groupoids is a bijection. The proof is divided into two steps.

Step 1. By construction (see [2, 3]) the Lie algebroid of $G(T^*M_{(\pi,\phi)})$ is $T^*M_{(\pi,\phi)}$. As discussed in Sect. 2, the relatively ϕ-closed, multiplicative, non-degenerate 2-form ω determines an automorphism λ of T^*M and a bivector field γ on M as in Eqs. (2.1) and (2.2). We have to show that λ is the identity and that $\gamma = \pi$. First of all we observe that it is enough to consider (2.1) at the unit element $\epsilon(m) \in G(T^*M_{(\pi,\phi)})$ corresponding to $m \in M$:

$$\omega(\epsilon(m))(\overrightarrow{\xi_1}(\epsilon(m)), \overrightarrow{\xi_2}(\epsilon(m))) = \gamma(m)(\xi_1, \xi_2), \quad \forall \xi_1, \xi_2 \in A|_m.$$

By construction $\epsilon(m)$ is the equivalence class of the path $X(t) = m$, $\eta(t) = 0$, $\forall t \in I = [0,1]$. The vector field $\overrightarrow{\xi_i}$, $i = 1, 2$, evaluated at $\epsilon(m)$ is the projection to $T_{\epsilon(m)}G(T^*M_{(\pi,\phi)})$ of the vector $\widehat{\xi_i} \in T_{(m,0)}P(T^*M_{(\pi,\phi)})$ defined by $\widehat{\xi_i}(t) = (\pi^\#(m)\xi_i\, t, \xi_i \mathrm{d}t)$. Observing then that for $\eta = 0$ the 2-form Ω_1 of Eq. (4.2) vanishes, we get, also using (4.1),

$$\begin{aligned}\omega(\epsilon(m))(\overrightarrow{\xi_1}(\epsilon(m)), \overrightarrow{\xi_2}(\epsilon(m))) = \Omega_0(m,0)(\widehat{\xi_1}, \widehat{\xi_2}) = \\ = 2\int_0^1 \pi(m)(\xi_1,\xi_2)\, t\, \mathrm{d}t = \pi(m)(\xi_1, \xi_2),\end{aligned}$$

which shows $\gamma = \pi$. As for (2.2), observe that $\omega(\epsilon(m))(\overrightarrow{\xi_1}(\epsilon(m)), v)$ is just $\Omega_0(m,0)(\widehat{\xi_1}, \widehat{v})$ with $\widehat{v}(t) = (v, 0)$. As a consequence,

$$\omega(\epsilon(m))(\overrightarrow{\xi_1}(\epsilon(m)), v) = \int_0^1 \langle \xi_1, v\rangle\, \mathrm{d}t = \langle \xi_1, v\rangle,$$

which shows that λ is the identity.

Step 2. Assume that $(G \rightrightarrows M, \omega + \phi)$ is an α-simply connected non-degenerate twisted symplectic groupoid. Let π be its induced ϕ-twisted Poisson structure on M. Then the above integration process integrates the Lie algebroid $T^*M_{(\pi,\phi)}$ into a Lie groupoid, which is known to be isomorphic to $G \rightrightarrows M$, and a multiplicative 2-form ω' on that groupoid. By identifying this groupoid with $G \rightrightarrows M$, therefore one may think ω'

as a multiplicative 2-form on G. One needs to show that $\omega' = \omega$. By Step 1, we conclude that ω' and ω must coincide along the identity space M. Let $\tilde{\omega} = \omega - \omega'$. Then $\tilde{\omega}$ is a multiplicative closed 2-form on G and $\tilde{\omega}|_M = 0$. Given any $\xi \in \Gamma(A)$, it is easy to see that $(\overrightarrow{\xi}(\alpha(x)), 0, \overrightarrow{\xi}(x))$ is tangent to the graph Λ of groupoid multiplication. On the other hand, for any $\delta_x \in T_xG$, it is also clear that $(\alpha_*\delta_x, \delta_x, \delta_x) \in T\Lambda$. It thus follows that

$$\tilde{\omega}(\overrightarrow{\xi}(\alpha(x)), \alpha_*\delta_x) - \tilde{\omega}(\overrightarrow{\xi}(x), \delta_x) = 0.$$

Therefore we have $\overrightarrow{\xi} \lrcorner \tilde{\omega} = 0$. Thus

$$L_{\overrightarrow{\xi}}\tilde{\omega} = (di_{\overrightarrow{\xi}} + i_{\overrightarrow{\xi}}d)\tilde{\omega} = 0,$$

which implies that $\tilde{\omega} = 0$ since any point in G can be reached by a product of (local) bisections generated by $\overrightarrow{\xi}$. This concludes the proof.

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