ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics SAM

Ridgelets: An Optimally Adapted Representation System for Solving Transport Equations

Philipp Grohs, Axel Obermeier[1]

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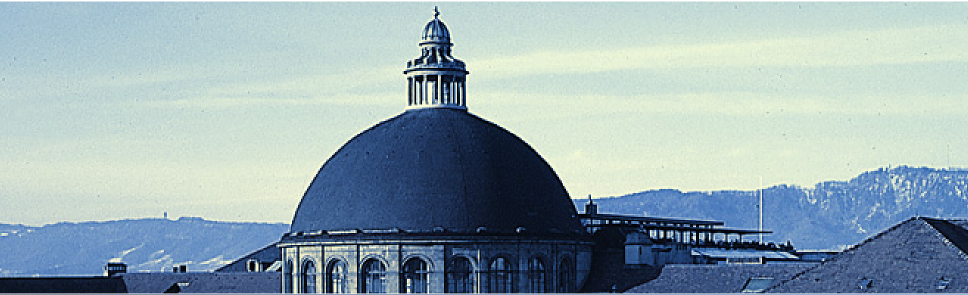

[1] Funded by SNF Project 146356

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Motivation

Historical Background

- Long after wavelets were introduced in imaging science, it was realised that they can be used to solve PDEs as well.
- Similarly, the application of directional dictionaries to solve PDEs has only begun long after their introduction in imaging.

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Selling Point

- With wavelets, it was possible to prove *optimal* rates for *both* convergence and complexity of the algorithm in the context of elliptic PDEs, see e.g. [CDD].
- We want to translate this machinery to a different class of PDEs, where wavelets and FE methods struggle.

[CDD] A. Cohen, W. Dahmen, R. DeVore, Adapt. wavelet meth. for elliptic op. eq.: conv. rates, Math. Comp., 2001

CDD-Schemes

Necessary Ingredients

(I) Variational formulation $a(u, v) = \ell(v)$ for all $v \in H$, with H-bounded and coercive bilinear form a, i.e.

$$a(v, v) \sim \|v\|_H^2 \quad \text{and} \quad a(u, v) \lesssim \|u\|_H \|v\|_H$$

for a Hilbert space H.

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(II) Frame property of (scaled version of) $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda}$ for Hilbert space H, i.e.

$$\|u\|_H \sim \left\| \left(w_\lambda \langle \phi_\lambda, u \rangle_H \right)_{\lambda\in\Lambda} \right\|_{\ell^2}$$

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for some weight $\mathbf{W} = \mathrm{diag}\big((w_\lambda)_\lambda\big)$.

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(III) Compressibility of Galerkin matrix $\mathbf{A}$

(IV) Typical solutions u can be optimally approximated by frame Φ

Compressibility

Definition [CDD]

A matrix $\mathbf{A}$ is σ^*-compressible if for every $\sigma < \sigma^*$ and $i \in \mathbb{N}$, there exists a matrix $\mathbf{A}^{[i]}$ such that

- $\mathbf{A}^{[i]}$ has at most $\alpha_i 2^i$ nonzero entries in each column
- the inequality $\left\| \mathbf{A} - \mathbf{A}^{[i]} \right\| \leq C_i$ holds
- the sequences $(\alpha_i)_i$ and $(C_i 2^{\sigma i})_i$ are both summable

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Benefit

- Makes it possible to build routine

$$\mathbf{APPLY}[\varepsilon, \mathbf{A}, \mathbf{u}] \to \mathbf{v}_\varepsilon$$

  that calculates $\mathbf{Au}$ (up to accuracy ε) in *linear complexity* (in terms of the number of non-zero entries in $\mathbf{u}$, say N).
- Compare with normal case, which needs $\mathcal{O}(N^2)$ operations.

CDD-Schemes, cont.

SOLVE

Ingredients (I-III) allow us to directly use results from [Ste] to construct an algorithm **SOLVE**$[\varepsilon, \mathbf{A}, \mathbf{f}] \rightarrow \mathbf{u}_\varepsilon$ which computes an approximate solution $\mathbf{u}_\varepsilon$ of the linear system $\mathbf{Au} = \mathbf{f}$ up to an error ε in optimal complexity.

[Ste] R. Stevenson, Adaptive solution of operator equations using wavelet frames, SIAM J. Numer. Anal., 2003

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Optimal Complexity

This means that if $u \in H$ is a solution which has an N-term approximation rate of order $\sigma < \sigma^*$, i.e. there exist v_N such that

$$\|u - v_N\|_H \lesssim N^{-\sigma},$$

then the algorithm **SOLVE** produces approximands u_N with the same asymptotic rate in order N flops:

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Kinetic Transport Equations

Main Equation

$$\vec{s} \cdot \nabla u + \kappa u = f + Q(u), \quad (\vec{x}, \vec{s}) \in \Omega \times \mathbb{S}^{d-1},$$

- $\Omega \subset \mathbb{R}^d$
- κ absorption coefficient
- f source term
- Q scattering operator – e.g. $Q(u)(\vec{x}, \vec{s}) = \int_{\mathbb{S}^{d-1}} K(\vec{s}, \vec{s}')u(\vec{x}, \vec{s}')\,\mathrm{d}\vec{s}'$

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Describes

Stationary distribution of a phase-space density u whose evolution is governed by:

- free transport
- absorption
- external sources
- interaction with the surrounding medium via a scattering operator

Kinetic Transport Equations

Difficulties

- "Curse of dimensionality": Problem is $(2d - 1)$-dimensional
- Line singularities along rays may appear
- The equation is not H^1-elliptic – wavelet and FE discretisations do not lead to well-conditioned linear systems
- Anisotropic meshes are impractical since they need to be combined for different directions.

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Wish List

- Multiscale system to alleviate curse of dimensionality
- "Blindness" of representation system to line singularities
- Well-conditioned linear system
- One dictionary for all directions

Linear Transport Equations

Simplification – For Now

We fix the transport direction s and neglect scattering, which leads to

$$Au(\vec{x}) := \vec{s} \cdot \nabla u(\vec{x}) + \kappa(\vec{x})u(\vec{x}) = f(\vec{x}), \quad \vec{x} \in \Omega$$

with BCs $u|_{\partial\Omega_+} = 0$, where

$$\partial\Omega_+ = \left\{\vec{x} \in \partial\Omega \,:\, \vec{s} \cdot n(\vec{x}) > 0\right\}$$

and $n(\vec{x})$ is the inward pointing normal of $\partial\Omega$.

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Road Map to Full Problem

- Devise optimal discretisations for such equations – i.e. prove that ingredients (I)-(IV) can be satisfied in this context.
- Solve full kinetic transport equation by *collocation* or *tensor product* methods, i.e. solving equations of the above type for many directions $\vec{s}$.

Linear Transport Equations

Variational Formulation

We can recast the linear transport equation in variational form as follows

$$a(u, v) = \ell(v), \quad \text{for all } v \in H^{\vec{s}}(\Omega),$$

where

$$a(u, v) := \int_\Omega Au(\vec{x})Av(\vec{x})\,\mathrm{d}\vec{x}, \quad \ell(v) := \int_\Omega Av(\vec{x})f(\vec{x})\,\mathrm{d}\vec{x}$$

and

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Theorem

Assume that $f \in L^2(\Omega)$ and κ strictly positive. Then $a(\cdot, \cdot)$ is $H^{\vec{s}}$-bounded and coercive. In particular, by the Lax-Milgram lemma the variational formulation is well-posed.

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Theorem $\Longrightarrow$ Ingredient (I)

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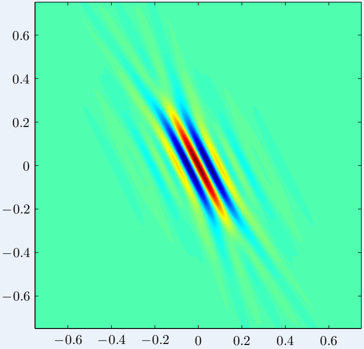
Ridgelets

Intuition (for $\Omega = \mathbb{R}^2$)

A ridgelet frame Φ ([Can,Gro]) roughly consists of functions (illustration of one for $j = 3$) which:

- have width ~ 1
- have height $\sim 2^{-j}$
- oscillate horizontally with frequency $\sim 2^j$
- are rotated around $\sim 2^j$ uniformly spaced angles



Real value of ridgelet

Skip Construction

[Can] E. Candès, Ridgelets: Theory and applications, PhD thesis, Stanford University, 1998
[Gro] P. Grohs, Ridgelet-type frame decomp. for Sobolev spaces related to lin. transport, J. Fou. Anal. Appl., 2011

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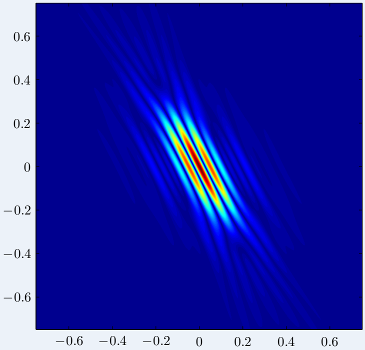

Absolute value of ridgelet

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[Can] E. Candès, Ridgelets: Theory and applications, PhD thesis, Stanford University, 1998
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Ridgelets

Construction [Gro]

We start with a partitioning

$$\{\hat{\psi}_{j,\ell}\}_{j\in\mathbb{N}_0,\,\ell\in\{1,\dots,L_j\}},$$

where $L_j \sim 2^j$, such that

$$\sum_{j,\ell} \hat{\psi}_{j,\ell}(\xi)^2 = 1,$$

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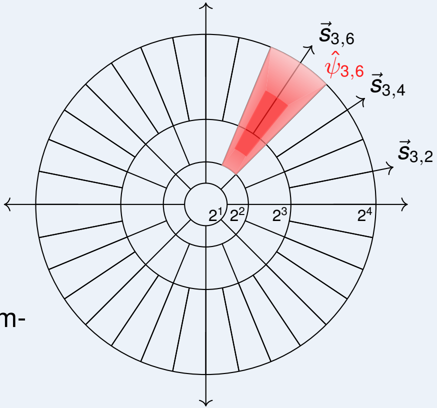

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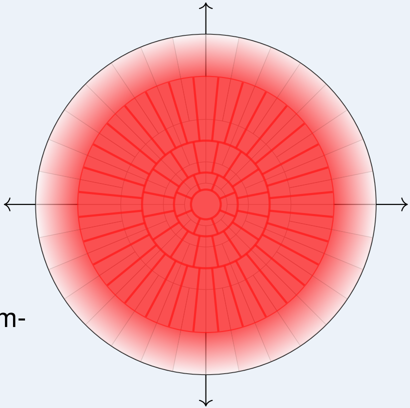

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Ridgelets

Definition

We define the ridgelet system Φ as consisting of the functions

$$\varphi_{j,\ell,\vec{k}} := 2^{-j/2} T_{U_{j,\ell}\vec{k}} \psi_{j,\ell}, \quad j \in \mathbb{N}_0,\ \ell \in \{1, \dots, L_j\},\ \vec{k} \in \mathbb{Z}^2.$$

We collect all indices $\lambda = (j, \ell, \vec{k})$ in the set Λ and write $\Phi = \{\varphi_\lambda\}_{\lambda \in \Lambda}$.

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Notation

We let $T_y f(\cdot) := f(\cdot - y)$, be the translation operator, and define

$$U_{j,\ell} := R^{-1}_{\vec{s}_{j,\ell}} D_{2^{-j}},$$

where

- $D_a := \operatorname{diag}(a,1)$ dilates the first component
- $R_{\vec{s}}$ is the rotation which takes $\vec{s} \in \mathbb{S}^1$ to $(1,0)^\top$
- $\vec{s}_{j,\ell}$ corresponds to the direction of the support of $\hat{\psi}_{j,\ell}$

Ridgelets

Other Domains

- The construction of ridgelets is not inherently limited to $\Omega = \mathbb{R}^d$
- However, it is not immediate how to preserve all their favourable properties on a bounded domain

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Other Domains

- The construction of ridgelets is not inherently limited to $\Omega = \mathbb{R}^d$
- However, it is not immediate how to preserve all their favourable properties on a bounded domain
- Very recently ([GKMP]), progress has been made in constructing shearlets frames (which are closely related) on domains
- In principle, we believe it will be possible to proceed in a similar fashion for ridgelets (work in progress!)

[GKMP] P.Grohs, G.Kutyniok, J.Ma, P.Petersen, Multiscale aniso. directional sys. on bounded domains, preprint, 2015

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Frame Property

Theorem [Gro]

With weights $w_\lambda := 1 + 2^j |\vec{s} \cdot \vec{s}_{j,\ell}|$ *and* $\mathbf{W} := \mathrm{diag}\big((w_\lambda)_\lambda\big)$, *the dictionary* Φ *forms a Gelfand frame for* $H^{\vec{s}}$, *i.e.*

$$\|\mathbf{W}\langle \Phi, u\rangle\|_{\ell^2} \lesssim \|u\|_{H^{\vec{s}}} \quad \text{and} \quad \|\Phi \mathbf{u}\|_{H^{\vec{s}}} \lesssim \|\mathbf{W}\mathbf{u}\|_{\ell^2},$$

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Theorem [Gro] $\Longrightarrow$ Ingredient (II)

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Theorem (follows from [DFR])

Let $\mathbf{A} := \mathbf{W}^{-1}\langle A\Phi, A\Phi\rangle\mathbf{W}^{-1} := \left\{\frac{1}{w_\lambda w_{\lambda'}}\int_\Omega A\varphi_\lambda(\vec{x})A\varphi_{\lambda'}(\vec{x})\,\mathrm{d}\vec{x}\right\}_{\lambda,\lambda'\in\Lambda}$.

- $\mathbf{A}$ is bounded on ℓ^2 and boundedly invertible on its range.
- With $\mathbf{f} := \mathbf{W}^{-1}\langle A\Phi, f\rangle$, $\mathbf{A}\mathbf{u} = \mathbf{f}$ is well-conditioned and can be solved with a damped Richardson iteration.
- The solution of the variational problem is given by $u = \Phi\mathbf{W}^{-1}\mathbf{u}$.

[DFR] S. Dahlke, M. Fornasier, T. Raasch, Adapt. frame methods for elliptic op. eq., Adv. Comput. Math., 2007

Achieving Compressibility

Theorem [GO]

Let $0 < p < 1$ *and* $Au := \vec{s} \cdot \nabla u + \kappa u$. *If* $\hat{\Phi}$ *and* κ *are sufficiently differentiable,* $\mathbf{A} := \mathbf{W}^{-1} \langle A\Phi, A\Phi \rangle \mathbf{W}^{-1}$ *is* $\frac{1}{2}\left(\frac{1}{p} - 1\right)$*-compressible.*

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[GO] P. Grohs, A. O., Optimal adapt. ridgelet schemes for linear advection eq., Appl. and Comp. Harm. Anal., 2015

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Projection (for the experts)

- To avoid that errors in $\ker \mathbf{A}$ accumulate during the iteration, a suitable projection **P** must be applied every few iterations.
- To maintain optimal computational complexity, **P** *must be compressible* as well!

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Projection (for the experts)

- To avoid that errors in $\ker \mathbf{A}$ accumulate during the iteration, a suitable projection **P** must be applied every few iterations.
- To maintain optimal computational complexity, **P** *must be compressible* as well!
- So far, it has not been possible to prove that the most obvious choice – an orthogonal projection – is compressible
- However, in our setting, it is possible to prove that $\mathbf{P} := \mathbf{W} \langle \Phi, \Phi \rangle \mathbf{W}^{-1}$ is compressible as well!

[GO] P. Grohs, A. O., Optimal adapt. ridgelet schemes for linear advection eq., Appl. and Comp. Harm. Anal., 2015

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Weak ℓ^p-Spaces

Definition

For $0 < p < 2$, we define the *weak ℓ^p-spaces*

$$\ell_w^p := \left\{(c_n)_{n\in\mathbb{N}} \in \ell^2 \ : \ |c_\mathbb{N}|_{\ell_w^p} := \sup_{n\in\mathbb{N}} c_n^* n^{\frac{1}{p}} < \infty\right\}$$

where c_n^* is the decreasing rearrangement of $(|c_n|)_{n\in\mathbb{N}}$.

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$$\ell^p_w := \left\{ (c_n)_{n\in\mathbb{N}} \in \ell^2 \; : \; |c_{\mathbb{N}}|_{\ell^p_w} := \sup_{n\in\mathbb{N}} c_n^* n^{\frac{1}{p}} < \infty \right\}$$

where c_n^* is the decreasing rearrangement of $(|c_n|)_{n\in\mathbb{N}}$.

Relation to N-term Approximation (e.g. [DeV])

An N-term approximation rate of order $N^{-\sigma^*}$ is equivalent to membership of the coefficient sequence in ℓ^p_w, with $\frac{1}{p} = \sigma^* + \frac{1}{2}$.

[DeV] R. DeVore, Nonlinear approximation, Acta Numerica, 1998

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Sobolev Spaces & Approximation

With the technique of *hypercube embeddings*, one can show that the best possible approximation rate for a generic $f \in H^t(\mathbb{R}^d)$ is $\sigma^* = \frac{t}{d}$ [DDMGL]. Correspondingly, we set $\frac{1}{p^*} := \frac{t}{d} + \frac{1}{2}$.

[DeV] R. DeVore, Nonlinear approximation, Acta Numerica, 1998
[DDMGL] S. Dahlke, F. De Mari, P. Grohs, D. Labate, Harm. and Appl. Anal. – From Groups to Signals, Birkhäuser, 2015

Approximation by Ridgelets

Definition

We say that a function $f \in L^2(\mathbb{R}^d)$ is *in* $H^t(\mathbb{R}^d)$ *apart from hyperplanes*, if (with H being the Heaviside step function)

- there are N hyperplanes h_i (with corresponding normalised orthogonal vectors $\vec{n}_i$ and offsets t_i),
- as well as functions $f_0, f_1, \ldots, f_N \in H^t(\mathbb{R}^d)$,
- such that f can be represented as

$$f(\vec{x}) = f_0(\vec{x}) + \sum_{i=1}^{N} f_i(\vec{x}) H(\vec{x} \cdot \vec{n}_i - t_i),$$

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Notation

- Let $\mathcal{P}_{h_i}$ denote the projection onto hyperplane h_i
- Define $\langle x \rangle := \sqrt{1 + |x|^2}$, the *regularised absolute value*

Approximation by Ridgelets

Theorem (O.-Grohs)

Assume that $f \in H^t(\mathbb{R}^d)$ apart from hyperplanes, and that along the interfaces, f decays as follows

$$\left|f_i(\mathcal{P}_{h_i}\vec{x})\right| \lesssim \left\langle \mathcal{P}_{h_i}\vec{x}\right\rangle^{-2n}, \qquad i \geq 1,$$

for some $n \in \mathbb{N}$. Then $\langle \Phi, f\rangle \in \ell_w^{p'}$, the weak ℓ^p-space with $p' = p^* + \delta^*$, where p^* is the optimal p for $f \in H^t(\mathbb{R}^d)$ even without singularities(!) *and $\delta^* \leq \frac{d}{n}$.*

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Similarly, if u is the solution to $Au = f$, satisfying the same decay condition across the interfaces[3] *for u, then $\mathbf{W}\langle \Phi, u \rangle \in \ell_w^{p'}$ with the same p' as above.*

[3] It can be shown that u has the same kind of decomposition as f. The required decay condition for the u_i follows if one demands global decay for the f_i, i.e. $\left|f_i(\vec{x})\right| \lesssim \langle \vec{x} \rangle^{-2n}$.

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Theorem (O.-Grohs) $\Longrightarrow$ Ingredient (IV)

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Main Result

Theorem [GO]

Let $f \in H^t$ apart from hyperplanes with compact support or exponential decay, and let u be the solution to $Au = f$. Then, approximands u_N with quasi-optimal approximation rate can be computed in order N flops, i.e. for arbitrary $\delta > 0$,

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- First construction of adaptive PDE solver with non-standard frames for problems of non-Laplacian-type (to our knowledge)
- It is possible to construct frames for the full kinetic transport equation using tensor product construction (work in progress).

Table of Contents

Numerical Examples

Approximation of Smooth Solution

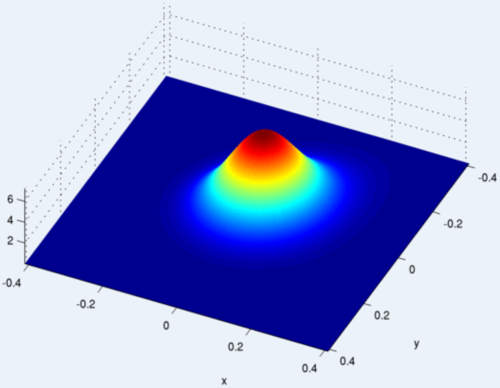

Figure: Solution of transport equation with smooth source (Gaussian)

Numerical Experiments

Approximation of Smooth Solution

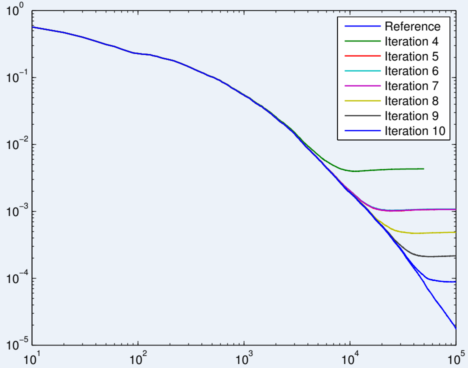

Figure: N-Term approximations of (outer) iterands;
Solution computed by **SOLVE** converges exponentially!

Numerical Experiments

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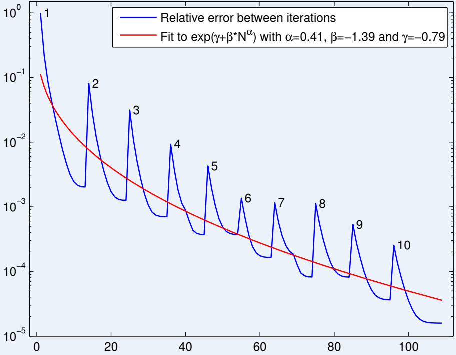

Figure: Geometric convergence of Richardson iterations (Outer iterations marked in plot)

Numerical Experiments

Approximation of Singular Solution

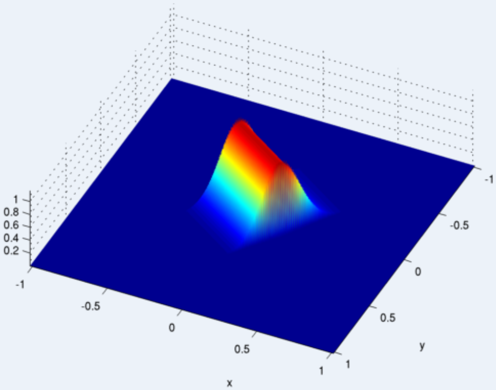

Figure: Singular solution of transport equation

Numerical Experiments

Approximation of Singular Solution

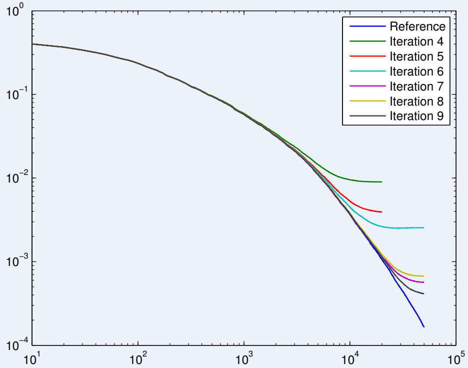

Figure: Solution computed by **SOLVE** converges exponentially, even if line discontinuities are present!

Numerical Experiments

Lessons Learned

- + Exponential convergence of N-term approximation
- + Close to theory – **APPLY** and **SOLVE** implemented as described
- + Possible because matrix entries decay as expected

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Lessons Learned

+ Exponential convergence of N-term approximation
+ Close to theory – **APPLY** and **SOLVE** implemented as described
+ Possible because matrix entries decay as expected
– Quadrature effort is substantial
– Domain $\Omega = \mathbb{R}^2$ (construction on bounded domains is in progress)
– Implementation is only “proof-of-concept”, *not* competitive

FFT-based Implementation

Key Properties [EGO]

- No adaptiveness – uses FFT like a black box
- Does not need to build the matrix, but just apply transformations to each vector (CG also works for abstract vectors and linear operations on them)

[EGO] S. Etter, P. Grohs, A. O., FFRT – A fast finite ridgelet transf. for radiative transp., Multiscale Model. Simul., 2015

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[EGO] S. Etter, P. Grohs, A. O., FFRT – A fast finite ridgelet transf. for radiative transp., Multiscale Model. Simul., 2015

Solution of Radiative Transport Equation

Including Scattering $Q(u)$

We demonstrate our solver with a simple scattering kernel:

- $Q(u)(x) = \int_{\mathbb{S}^1} \sigma u(x, s) \, \mathrm{d}s$
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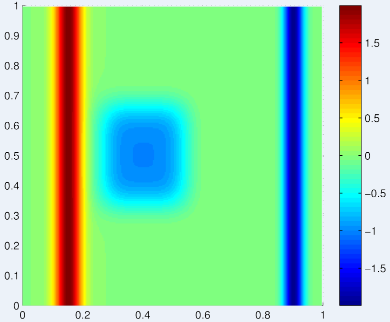

Figure: Red: Source
Blue: Absorption

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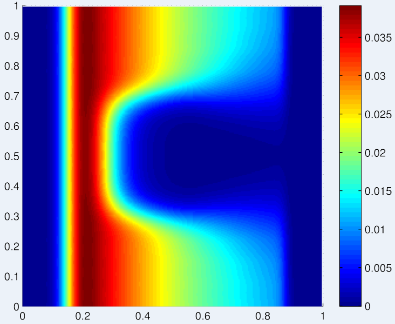

Figure: Solution without scattering

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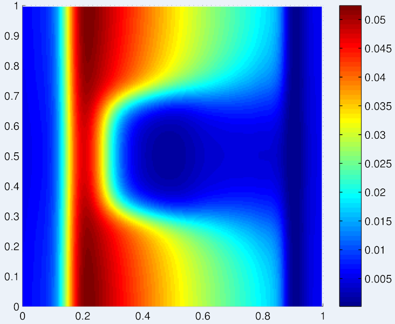

Figure: Scattering with $\sigma = 0.2$

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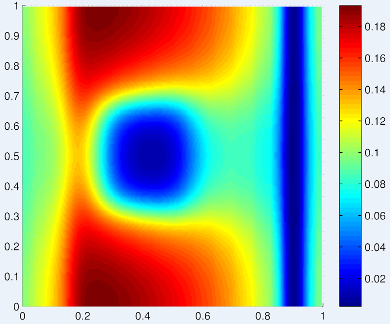

Figure: Scattering with $\sigma = 0.5$

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- Model problem: Scattering around obstacle
- Using *sparse collocation scheme* breaks curse of dimensionality [GS]

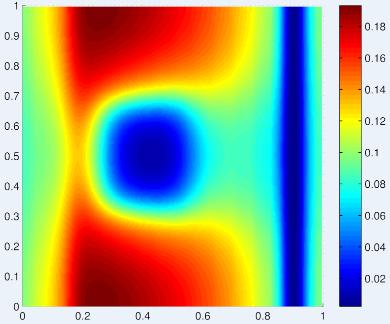

Figure: Scattering with $\sigma = 0.5$

[GS] K. Grella, Ch. Schwab, Sparse discr. ordinates meth. in radiative transfer, Comp. Meth. Appl. Math., 2012

Conclusion

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Bottom Line

"Ridgelets are for linear transport equations what wavelets are for elliptic equations."

The End

Thank you for your attention!

Questions?