Well-Conditioned Boundary Element Formulation for Scattering at Partly Impenetrable Objects

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Outline

- Acoustic Scattering at Composite Objects
- Definitions
- Classical First-Kind Approach
- Experiment I
- Second-Kind Formulation
- Experiment II
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Acoustic Scattering at Composite Objects

Find U s.t. for all $i \in \{0, 1, \ldots, M\}$ we have $U|_{\Omega_i} \in H^1_{\text{loc}}(\Delta, \overline{\Omega}_i)$ and

$$(-\Delta - \kappa_i^2)U(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_i, \quad (1)$$

+ transmission conditions at $\partial\Omega_i \setminus \partial\Omega_{\blacksquare}$:

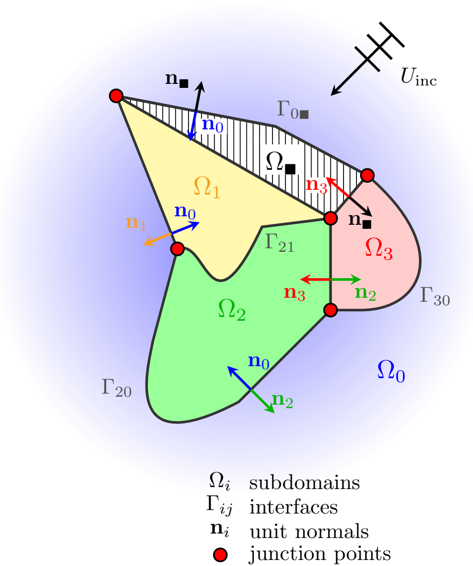
$$\begin{cases} U\big|_{\partial\overline{\Omega}_i^c} - U\big|_{\partial\Omega_i} & = 0 \\ -\mathbf{n_i} \cdot (\nabla U)|_{\partial\overline{\Omega}_i^c} + \mathbf{n_i} \cdot (\nabla U)|_{\partial\Omega_i} & = 0 \end{cases}$$

+ Dirichlet conditions at $\partial\Omega_{\blacksquare}$: $\quad U\big|_{\partial\overline{\Omega}_{\blacksquare}^c} = 0$,

+ Sommerfeld radiation condition at ∞ for $U - U_{\text{inc}}$.

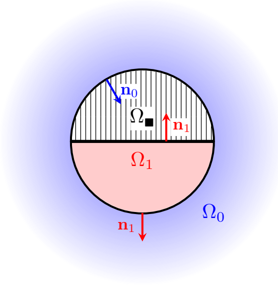
U_{inc}: incident wave, satisfying (1) for $\kappa_i \equiv \kappa_0$ on $\mathbb{R}^d$.

$\kappa_i \in \mathbb{R}_+$: constant wave number on Ω_i.



Traces, Potentials, Operators: Conventions and Notation

- Dirichlet trace: $\gamma_D^i : \; H^1_{\text{loc}}(\Omega_i) \to H^{\frac{1}{2}}(\partial\Omega_i)$,
  $\gamma_D^i V := V|_{\partial\Omega_i}$.
- Neumann trace: $\gamma_N^i : \; H^1_{\text{loc}}(\Delta, \overline{\Omega_i}) \to H^{-\frac{1}{2}}(\partial\Omega_i)$,
  $\gamma_N^i Q := \mathbf{n_i} \cdot (\nabla Q)|_{\partial\Omega_i}$.
  normal field $\mathbf{n_i}$ pointing outside of Ω_i.



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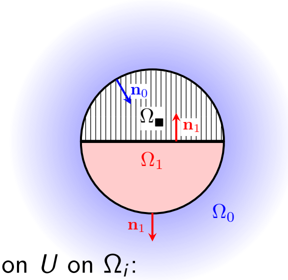

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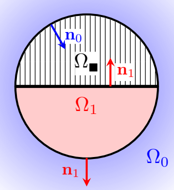
  normal field $\mathbf{n_i}$ pointing outside of Ω_i.
- i^{th} local representation formula (iRF) of the solution U on Ω_i:

$$U|_{\Omega_i} = \underbrace{\mathbb{S}_i[\kappa_i]\{\gamma_N^i U\}}_{\text{single layer potential}} - \underbrace{\mathbb{D}_i[\kappa_i]\{\gamma_D^i U\}}_{\text{double layer potential}} =: \underbrace{\mathbb{G}_i[\kappa_i](\gamma_D^i U, \gamma_N^i U)}_{\text{i}^{\text{th}}\text{ local total potential}},$$



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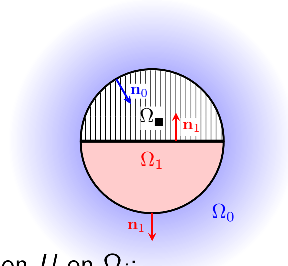
$$\mathbb{S}_i[\kappa_i](\gamma_N^i U)(\mathbf{x}) := \int_{\partial\Omega_i} \gamma_{D,\mathbf{y}}^i \mathcal{G}_{\kappa_i}(\mathbf{x},\mathbf{y}) \gamma_N^i U(\mathbf{y})\, dS(\mathbf{y}),\ \mathbf{x} \in \mathbb{R}^d \backslash \Gamma,$$

$$\mathbb{D}_i[\kappa_i](\gamma_D^i U)(\mathbf{x}) := \int_{\partial\Omega_i} \gamma_{N,\mathbf{y}}^i \mathcal{G}_{\kappa_i}(\mathbf{x},\mathbf{y}) \gamma_D^i U(\mathbf{y})\, dS(\mathbf{y}),\ \mathbf{x} \in \mathbb{R}^d \backslash \Gamma,$$

$\mathcal{G}_{\kappa_i}(\mathbf{x},\mathbf{y})$ is the fundamental solution of $(-\Delta - {\kappa_i}^2)$.

Traces, Potentials, Operators: Conventions and Notation



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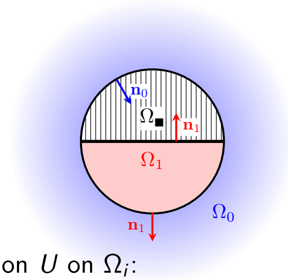
Remark: Representation formula is true for $i \in \{1, ..., M\}$.
For $i = 0$ take $(U - U_{\text{inc}})$ instead of U.



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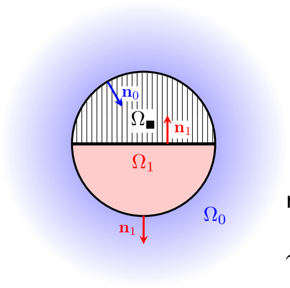
- Taking i^th traces of (iRF) gives the Calderón identity:

$$\begin{pmatrix} \gamma_D^i U \\ \gamma_N^i U \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2}\mathsf{Id} - K_i[\kappa_i] & V_i[\kappa_i] \\ W_i[\kappa_i] & \frac{1}{2}\mathsf{Id} + K_i'[\kappa_i] \end{pmatrix}}_{\text{Calderón projector}} \begin{pmatrix} \gamma_D^i U \\ \gamma_N^i U \end{pmatrix} .$$



weakly singular operator
double layer operator
adjoint double layer op.
hypersingular operator

restricting to $\partial\Omega_i$

$$\gamma_D^i U = \gamma_D^i \mathbb{S}_i[\kappa_i]\{\gamma_N^i U\} - \gamma_D^i \mathbb{D}_i[\kappa_i]\{\gamma_D^i U\},$$
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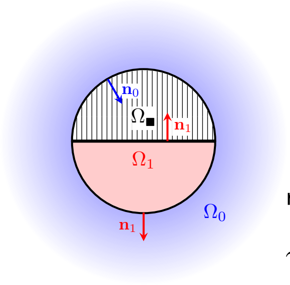

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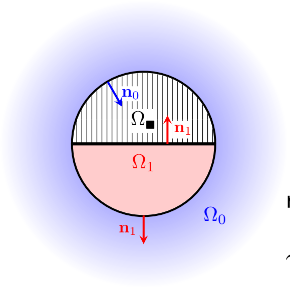

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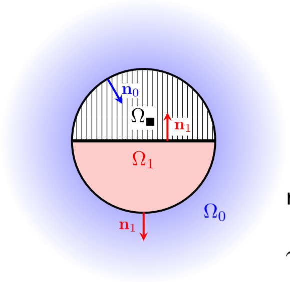

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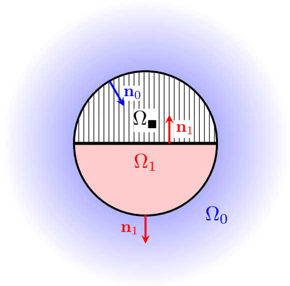
For the toy example (M=1) the relevant Calderón identities are

$$\left[\frac{1}{2}\mathrm{Id} - \underbrace{\begin{pmatrix} -K_0[\kappa_0] & V_0[\kappa_0] \\ W_0[\kappa_0] & K_0'[\kappa_0] \end{pmatrix}}_{=:\mathcal{C}_0[\kappa_0]}\right] \begin{pmatrix} \gamma_D^0 U - \gamma_D^0 U_{inc} \\ \gamma_N^0 U - \gamma_N^0 U_{inc} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\left[\frac{1}{2}\mathrm{Id} - \underbrace{\begin{pmatrix} -K_1[\kappa_1] & V_1[\kappa_1] \\ W_1[\kappa_1] & K_1'[\kappa_1] \end{pmatrix}}_{=:\mathcal{C}_1[\kappa_1]}\right] \begin{pmatrix} \gamma_D^1 U \\ \gamma_N^1 U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We couple them by using appropriate spaces incorporating the transmission conditions (TC) and boundary conditions (BC).



We define the single trace space $\mathcal{ST}(\Gamma)$ using $\mathcal{ST}_D(\Gamma)$, $\mathcal{ST}_N(\Gamma)$:

$$\mathcal{ST}_D(\Gamma) := \left\{ (v_0,\, v_1,\, \dots,\, v_M) \in \prod_{i=0}^{M} H^{\frac{1}{2}}(\partial\Omega_i) \,\middle|\, \exists V \in H_0^1(\mathbb{R}^d \setminus \Omega_{\blacksquare}) \text{ s.t. } \gamma_D^i V = v_i,\ \forall i \in \{0, 1, \dots, M\} \right\},$$

where $H_0^1(\mathbb{R}^d \setminus \Omega_{\blacksquare})$ incorporates the homogeneous Dirichlet boundary conditions at $\Omega_{\blacksquare}$.

$$\mathcal{ST}_N(\Gamma) := \left\{ (q_0,\, q_1,\, \dots,\, q_M) \in \prod_{i=0}^{M} H^{-\frac{1}{2}}(\partial\Omega_i) \,\middle|\, \exists Q \in H^1_{\text{loc}}(\Delta, \mathbb{R}^d) \text{ s.t. } \gamma_N^i Q = q_i,\ \forall i \in \{0, 1, \dots, M\} \right\},$$

$$\mathcal{ST}(\Gamma) := \left\{ (v_i, q_i)_{0 \le i \le M} \in \prod_{i=0}^{M} \left(H^{\frac{1}{2}}(\partial\Omega_i) \times H^{-\frac{1}{2}}(\partial\Omega_i) \right) \,\middle|\, (v_0, v_1, \dots, v_M) \in \mathcal{ST}_D(\Gamma),\ (q_0, q_1, \dots, q_M) \in \mathcal{ST}_N(\Gamma) \right\}.$$

- $\mathcal{ST}(\Gamma) \subset \mathcal{MT}(\Gamma) := \prod_{i=0}^{M} \left(H^{\frac{1}{2}}(\partial\Omega_i) \times H^{-\frac{1}{2}}(\partial\Omega_i) \right),$
  multi trace space with bilinear form for $\vec{u}, \vec{v} \in \mathcal{MT}(\Gamma)$ given by

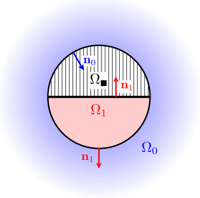
$$\begin{aligned} \langle \vec{u},\, \vec{v} \rangle &= \langle (u_0, p_0, \ldots, u_M, p_M),\, (v_0, q_0, \ldots, v_M, q_M) \rangle \\ &:= \sum_{i=0}^{M} \int_{\partial\Omega_i} u_i\, q_i - v_i\, p_i \; dS\,. \end{aligned}$$

- Polarity property of $\mathcal{ST}(\Gamma)$:
  For $\vec{u} \in \mathcal{MT}(\Gamma)$ we have the following equivalence:

$$\vec{u} \in \mathcal{ST}(\Gamma) \quad \Leftrightarrow \quad \langle \vec{u}, \vec{v} \rangle = 0 \;\forall \vec{v} \in \mathcal{ST}(\Gamma).$$

Idea of Classical Single Trace Formulation [von Petersdorff '1989]

- $$\Bigg[\tfrac{1}{2}\mathrm{Id}_0 - \underbrace{\begin{pmatrix} -K_0[\kappa_0] & V_0[\kappa_0] \\ W_0[\kappa_0] & K_0'[\kappa_0] \end{pmatrix}}_{=:\mathcal{C}_0[\kappa_0]}\Bigg] \begin{pmatrix} \gamma_D^0 U - \gamma_D^0 U_{inc} \\ \gamma_N^0 U - \gamma_N^0 U_{inc} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
  $$\Bigg[\tfrac{1}{2}\mathrm{Id}_1 - \underbrace{\begin{pmatrix} -K_1[\kappa_1] & V_1[\kappa_1] \\ W_1[\kappa_1] & K_1'[\kappa_1] \end{pmatrix}}_{=:\mathcal{C}_1[\kappa_1]}\Bigg] \begin{pmatrix} \gamma_D^1 U \\ \gamma_N^1 U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$



- $\vec{u} := (\gamma_D^0 U, \gamma_N^0 U, \gamma_D^1 U, \gamma_N^1 U) \in \mathcal{ST}(\Gamma) \Leftrightarrow \vec{u}$ satisfies TC and BC,

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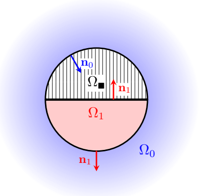
$$\left[\tfrac{1}{2}\mathrm{Id}_1 - \underbrace{\begin{pmatrix} -K_1[\kappa_1] & V_1[\kappa_1] \\ W_1[\kappa_1] & K_1'[\kappa_1] \end{pmatrix}}_{=:\mathcal{C}_1[\kappa_1]}\right] \begin{pmatrix} \gamma_D^1 U \\ \gamma_N^1 U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

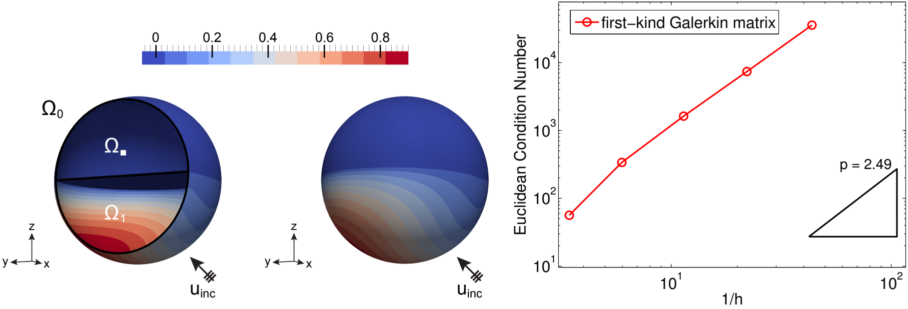


- $\vec{u} := (\gamma_D^0 U, \gamma_N^0 U, \gamma_D^1 U, \gamma_N^1 U) \in \mathcal{ST}(\Gamma) \Leftrightarrow \vec{u}$ satisfies TC and BC,
- Polarity property: $\vec{u}, \vec{v} \in \mathcal{ST}(\Gamma) \Rightarrow \langle \vec{u}, \vec{v} \rangle = 0.$

Find $\vec{u} \in \mathcal{ST}(\Gamma)$:
$\langle -\mathbf{C}\vec{u}, \vec{v} \rangle = \langle (\tfrac{1}{2} Id - \mathbf{C})\vec{\tilde{u}}_{\mathrm{inc}}, \vec{v} \rangle, \quad \forall\, \vec{v} \in \mathcal{ST}(\Gamma)$

$\mathbf{C} := diag(\mathcal{C}_0[\kappa_0], \mathcal{C}_1[\kappa_1]),\ \vec{\tilde{u}}_{\mathrm{inc}} = (\gamma_D^0 U_{\mathrm{inc}}, \gamma_N^0 U_{\mathrm{inc}}, 0, 0).$

Experiment I: Condition Numbers of Classical STF



Radius: $r = 0.5$, wave numbers: $(\kappa_0, \kappa_1) = (2, 4)$,
incident plane wave U_{inc}: direction of propagation $\mathbf{d} = \frac{1}{\sqrt{2}}(0, 1, 1)^T$,
cont. pw. linear elements for Dirichlet, pw. constant for Neumann data.

The first-kind solution was implemented based on a library by L. Kielhorn (BETL2).

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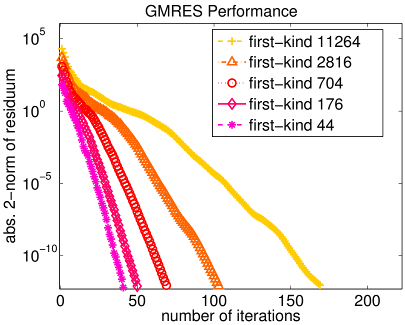

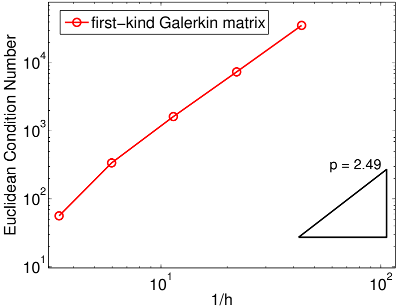

- Ill-conditioned Galerkin matrices.
- No suitable preconditioner available for the non-homogeneous case.

The first-kind solution was implemented based on a library by L. Kielhorn (BETL2).

New Idea: Global Potential

Null field property: If U solves the acoustic scattering problem, then for $W = U - \chi_{\Omega_0} U_{\text{inc}}$

$$\mathbb{G}_i[\kappa_i](\gamma_D^i W, \gamma_N^i W)(\mathbf{x}) = \begin{cases} W(\mathbf{x}), & \mathbf{x} \in \Omega_i \\ 0 & \mathbf{x} \in \mathbb{R}^d \backslash \overline{\Omega_i \cup \Omega_{\text{[redacted]}}} \end{cases}.$$

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$\Rightarrow$**New global ansatz:** Sum over all local total potentials!

Global total potential: $\mathbb{M}_\Gamma : \mathcal{MT}(\Gamma) \to H^1_{\text{loc}}(\Delta, \mathbb{R}^d \setminus \Gamma)$

$\mathbb{M}_\Gamma \vec{v} := \sum_{i=0}^{M} \mathbb{G}_i[\kappa_i](v_i, q_i), \;\; \vec{v} = (v_0, q_0, \dots, v_M, q_M).$

For the toy problem we have

$$\mathbb{M}_\Gamma \vec{v} = \mathbb{G}_0[\kappa_0](v_0, q_0) + \mathbb{G}_1[\kappa_1](v_1, q_1) .$$

Second-kind STF [Claeys '2011, Greengard/Lee '2012, Claeys/Hiptmair/Spindler '2015]

Apply the same technique as for the local total potentials to the multi-potential:

- restrict the global total potential to each boundary $\partial\Omega_j$, $j \in \{0,1\}$:

$$\begin{pmatrix} \gamma_D^0(U-U_{\mathrm{inc}}) \\ \gamma_N^0(U-U_{\mathrm{inc}}) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2}Id_0 - K_{0,0}[\kappa_0] & V_{0,0}[\kappa_0] \\ W_{0,0}[\kappa_0] & \frac{1}{2}Id_0 + K'_{0,0}[\kappa_0] \end{pmatrix}}_{=:\mathcal{A}_{0,0}} \begin{pmatrix} \gamma_D^0(U-U_{\mathrm{inc}}) \\ \gamma_N^0(U-U_{\mathrm{inc}}) \end{pmatrix} + \underbrace{\begin{pmatrix} -\frac{1}{2}Id_1 - K_{0,1}[\kappa_1] & V_{0,1}[\kappa_1] \\ W_{0,1}[\kappa_1] & -\frac{1}{2}Id_1 + K'_{0,1}[\kappa_1] \end{pmatrix}}_{=:\mathcal{A}_{0,1}} \begin{pmatrix} \gamma_D^1 U \\ \gamma_N^1 U \end{pmatrix}$$

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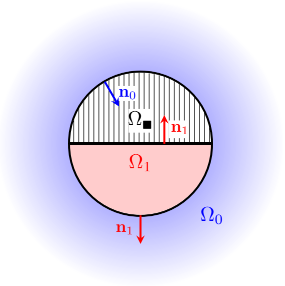
- couple the equations using the TC and BC:
  Search for
  $\vec{u} := (\gamma_D^0 U, \gamma_N^0 U, \gamma_D^1 U, \gamma_N^1 U) \in \mathcal{ST}(\Gamma)$.
- test with $\vec{v} \in \mathcal{MT}(\Gamma)$.



Second-Kind STF [Claeys '2011, Greengard/Lee '2012, Claeys/Hiptmair/Spindler '2015]

Search for $\vec{u} \in \mathcal{ST}(\Gamma)$: $\langle (Id - \mathcal{A})\vec{u}, \vec{v} \rangle$
$= \langle \vec{u}_{\text{inc}}, \vec{v} \rangle, \ \forall \vec{v} \in \mathcal{MT}(\Gamma).$

$$\mathcal{A} := \begin{pmatrix} \mathcal{A}_{0,0} & \mathcal{A}_{0,1} \\ \mathcal{A}_{1,0} & \mathcal{A}_{1,1} \end{pmatrix},$$

where $\mathcal{A}_{i,j}$, $j, i \in \{0, 1\}$, are the generalized Calderón projectors from the previous slide.

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We have

- $\mathcal{A}\vec{u} \in \mathcal{ST}(\Gamma) \ \forall \vec{u} \in \mathcal{ST}(\Gamma)$ (using jump relations),
- Polarity property of $\mathcal{ST}(\Gamma)$.
  For $\vec{u} \in \mathcal{MT}(\Gamma)$ the following equivalence holds:
  $\vec{u} \in \mathcal{ST}(\Gamma) \quad \Leftrightarrow \quad \langle \vec{u}, \vec{v} \rangle = 0 \ \forall \vec{v} \in \mathcal{ST}(\Gamma).$

$\Rightarrow$ equation is trivial for $\vec{v} \in \mathcal{ST}(\Gamma)$.
$\Rightarrow$ use a complement space $\mathcal{ST}^c(\Gamma)$ of $\mathcal{ST}(\Gamma) \subset \mathcal{MT}(\Gamma)$.

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$$\mathcal{E}(\vec{v}) = (v_\blacksquare, q_\blacksquare) := (\gamma_D^{\blacksquare,c} V, -\gamma_N^{\blacksquare,c} Q) = (0, -\gamma_N^{\blacksquare,c} Q),$$

where V, Q are the functions from the definition of $\mathcal{ST}(\Gamma)$. Thus, $\gamma_\Gamma(\mathbb{M}_\Gamma + \mathbb{G}_\blacksquare[\kappa] \circ \mathcal{E})\big|_{\mathcal{ST}(\Gamma)} \equiv 0$, i.e.

$$\left(\underbrace{\begin{pmatrix} \mathcal{A}_{0,0}[\kappa] & \mathcal{A}_{0,1}[\kappa] \\ \mathcal{A}_{1,0}[\kappa] & \mathcal{A}_{1,1}[\kappa] \end{pmatrix}}_{=:\mathcal{A}[\kappa]} + \underbrace{\begin{pmatrix} \begin{pmatrix} V_{0,\blacksquare}[\kappa] \\ -\frac{1}{2} Id_0 + K'_{0,\blacksquare}[\kappa] \end{pmatrix} \\ \begin{pmatrix} V_{1,\blacksquare}[\kappa] \\ -\frac{1}{2} Id_1 + K'_{1,\blacksquare}[\kappa] \end{pmatrix} \end{pmatrix}}_{=:\mathcal{B}[\kappa]} \mathcal{E} \right)\Bigg|_{\mathcal{ST}(\Gamma)} \equiv 0.$$

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- $\left(\mathcal{A} - \mathcal{A}[\kappa]\right)$ is a compact operator on $\mathcal{MT}_{L^2}(\Gamma) := \prod_{i=0}^{M} \left(L^2(\partial\Omega_i) \times L^2(\partial\Omega_i)\right)$ [Claeys/Hiptmair/Spindler '2015] ,
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- For pure transmission problems: Gårding inequality holds.
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  $\Rightarrow$ any L^2-stable set of basis functions can be used for discretization.
- Presence of $\Omega_{\blacksquare}$: Additionally, we need to assume that discrete inf-sup conditions hold.
- Uniqueness of the formulation is still an open question. But numerical experiments show evidence.

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- For $\mathcal{ST}^c_{L^2}(\Gamma)$, we choose the orthogonal complement of $\mathcal{ST}_{L^2}(\Gamma)$ in $\mathcal{MT}_{L^2}(\Gamma)$. It is obtained by interchanging the role of Dirichlet and Neumann data of $\mathcal{ST}_{L^2}(\Gamma)$.

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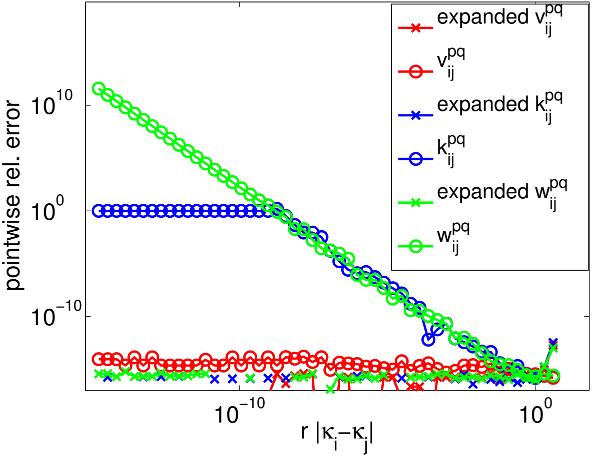
- For an implementation, it is useful to consider the formulation interface-wise to avoid cancellation due to occuring differences of kernels. On $\partial\Omega_1 \cap \partial\Omega_0$, we obtain a kernel of the form

$$(\mathcal{G}_{\kappa_1} - \mathcal{G}_{\kappa}) - (\mathcal{G}_{\kappa_0} - \mathcal{G}_{\kappa}) = \mathcal{G}_{\kappa_1} - \mathcal{G}_{\kappa_0}.$$

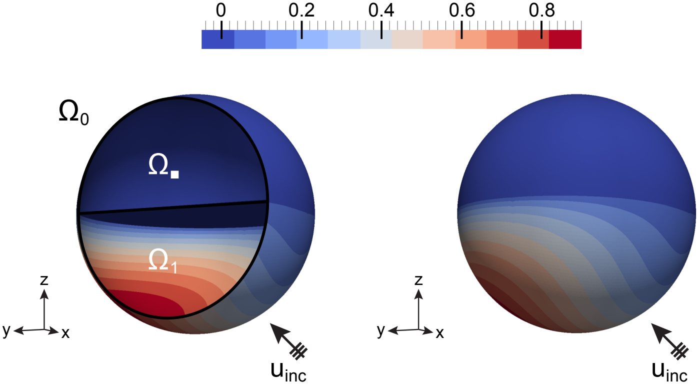
  This kernel has to be treated carefully when getting close to the diagonal $\mathbf{x} = \mathbf{y}$. We use Taylor expansions to avoid cancellation.

Second-Kind STF: Importance of series expansion



$\kappa_0 = 3,\ \kappa_1 = 5,\ \alpha(\mathbf{n}_0(\mathbf{y}), \mathbf{x}-\mathbf{y}) = 45^\circ,\ \alpha(\mathbf{n}_1(\mathbf{x}), \mathbf{x}-\mathbf{y}) = 120^\circ,\ N_{\text{trunc}} = 30.$

Experiment II: Second-Kind STF vs. First-Kind STF



Radius: $r = 0.5$, wave numbers: $\kappa_0 = 2, \kappa_1 = 4$,

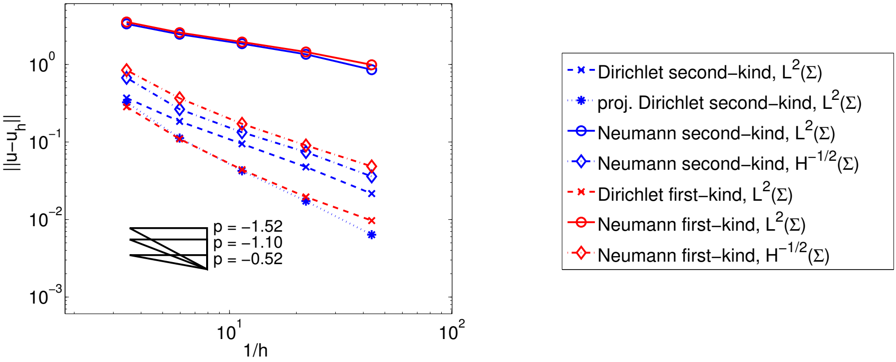
incident field U_{inc}: plane wave, direction of propagation $\mathbf{d} = \frac{1}{\sqrt{2}}(0,1,1)^T$,

First-kind: cont. pw. lin. elements for Dirichlet, pw. const. for Neumann data.

Second-kind: pw. constant elements for both Dirichlet and Neumann data.

The solutions were all implemented based on a library by L. Kielhorn (BETL2).
The reference solution is the second-kind solution of highest resolution.

Experiment II: Error Convergence



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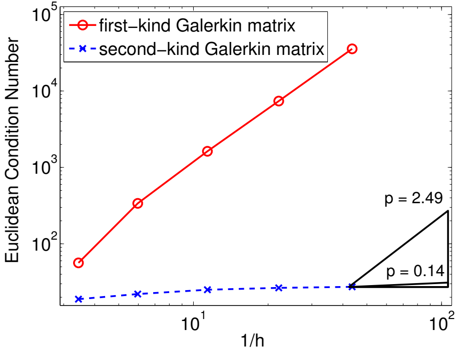
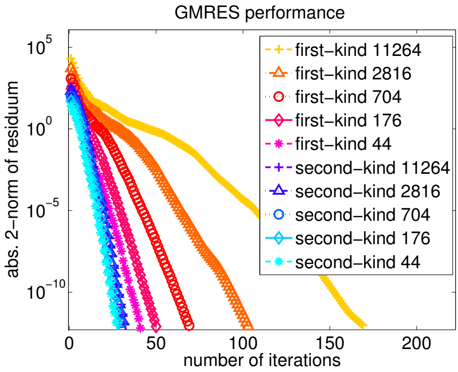
Experiment II: Condition Numbers and Iterative Solver





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Conclusion

Advantages of new second-kind formulation:

- Less restrictions on basis functions.
- Well-conditioned Galerkin matrices on fine meshes for the second-kind ansatz lead to fast convergence of GMRES.
- Competitive accuracy for post-processed traces (projected onto continuous piecewise linear boundary element space).
- A combined field integral approach is available to avoid spurious resonances due to the impenetrable part [Claeys/Hiptmair/Spindler '2015].

Outlook

Ongoing/Future work:

- The uniqueness result for the second-kind formulation is still a pending question. Nevertheless, numerical tests provide evidence.
- The extension of the second-kind formulation to vectorial Helmholtz equation is work in progress. First numerical experiments show promising results.
- Extension of the second-kind formulation to pure diffusion problems, i.e. for piecewise constant second-order coefficients $\mu_i > 0$ in

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Thank you for your attention!

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