SOME REMARKS ON LIOUVILLE TYPE THEOREMS

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Abstract: The goal of this note is to present elementary proofs of statements related to the Liouville theorem.

1. Introduction

We denote by $A(x) = (a_{ij}(x))$ a $(k \times k)$-matrix where the functions a_{ij}, $i, j = 1, \ldots, k$ are bounded measurable functions defined on $\mathbb{R}^k$ and which satisfy, for some $\lambda,\ \Lambda > 0$, the usual uniform ellipticity condition:

$$\lambda|\xi|^2 \leq (A(x)\xi \cdot \xi) \leq \Lambda|\xi|^2 \quad \text{a.e. } x \in \mathbb{R}^k, \quad \forall\, \xi \in \mathbb{R}^k. \tag{1.1}$$

We address here the issue of existence of solutions to the equation:

$$-\nabla \cdot (A(x)\nabla u(x)) + a(x)u(x) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k), \tag{1.2}$$

where $a \in L^\infty_{\text{loc}}(\mathbb{R}^k)$ and $a \geq 0$. When $a \neq 0$ and $\nabla \cdot (A\nabla) = \Delta$, the usual Laplace operator, the above equation is the so called stationary Schrödinger equation for which a vast literature is available (see [16], [20]). When $a = 0$ it is well known that every bounded solution to (1.2) has to be constant (see e.g. [5], [11], [12] and also [4], [19] for some nonlinear versions). The case where $a \neq 0$, and $k \geq 3$ is very different and in this case non trivial bounded solutions might exist.

Many of the results in this paper are known in one form or another (see for instance [1], [2], [3], [10], [9], [14], [16], [17]) but we have tried to develop here simple self-contained pde techniques which do not make use of probabilities, semigroups or potential theory as is sometimes the case (see e.g. [2], [3], [8], [17], [18]). One should note that some of our proofs extend also to elliptic systems.

This note is divided as follows. In the next section we introduce an elementary estimate which is used later. In Section 3 we present some Liouville type results, i.e., we show that under some conditions on a, (1.2) does not admit nontrivial bounded solutions. Finally in the last section we give an almost sharp criterion for the existence of nontrivial solutions.

2. A preliminary estimate

Let us denote by Ω a bounded open subset of $\mathbb{R}^k$ with Lipschitz boundary and starshaped with respect to the origin. For any $r \in \mathbb{R}$ we set

$$\Omega_r = r\Omega. \tag{2.1}$$

Let us denote by ϱ a smooth function such that

$$0 \le \varrho \le 1, \quad \varrho = 1 \text{ on } \Omega_{1/2}, \quad \varrho = 0 \text{ outside } \Omega, \tag{2.2}$$

$$|\nabla \varrho| \le c_\varrho, \tag{2.3}$$

where c_ϱ denotes some positive constant.

Lemma 2.1. *Suppose that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^k)$ satisfies (1.2) with $A(x)$ satisfying (1.1). Then there exists a constant C independent of r such that*

$$\begin{aligned}\int_{\Omega_r} &\{|\nabla u|^2 + au^2\}\varrho^2\Big(\frac{x}{r}\Big)\,\mathrm{d}x \\ &\le \frac{C}{r}\Big\{\int_{\Omega_r\setminus\Omega_{r/2}} |\nabla u|^2\varrho^2\Big(\frac{x}{r}\Big)\,\mathrm{d}x\Big\}^{\frac{1}{2}}\Big\{\int_{\Omega_r\setminus\Omega_{r/2}} u^2\,\mathrm{d}x\Big\}^{\frac{1}{2}},\end{aligned} \tag{2.4}$$

where $|\cdot|$ denotes the usual euclidean norm in $\mathbb{R}^k$.

Proof. By (1.2) we have for every $v \in H^1_0(\Omega_r)$

$$\int_{\Omega_r} A\nabla u \cdot \nabla v + auv\,\mathrm{d}x = 0. \tag{2.5}$$

Taking

$$v = u\varrho^2\Big(\frac{x}{r}\Big) = u\varrho^2 \tag{2.6}$$

yields

$$\int_{\Omega_r} A\nabla u \cdot \nabla\{u\varrho^2\} + au^2\varrho^2\,\mathrm{d}x = 0. \tag{2.7}$$

Since

$$\nabla \varrho^2 = \frac{2\varrho}{r} \nabla \varrho\Big(\frac{x}{r}\Big)$$

we obtain

$$\begin{aligned}\int_{\Omega_r} \{A\nabla u \cdot \nabla u\}\varrho^2 + au^2\varrho^2 \,\mathrm{d}x &= -\frac{1}{r}\int_{\Omega_r\setminus\Omega_{r/2}} A\nabla u \cdot \nabla\varrho \cdot 2\varrho u \,\mathrm{d}x \\ &\le \frac{C_1}{r}\int_{\Omega_r\setminus\Omega_{r/2}} |\nabla u||u|\varrho \,\mathrm{d}x,\end{aligned}$$

where C_1 is a constant depending on a_{ij} and c_ρ only. Using the ellipticity condition (1.1) it follows easily that

$$\min(1,\lambda)\int_{\Omega_r} \{|\nabla u|^2 + au^2\}\varrho^2 \,\mathrm{d}x \le \frac{C_1}{r}\int_{\Omega_r\setminus\Omega_{r/2}} |\nabla u||u|\varrho \,\mathrm{d}x.$$

By the Cauchy–Schwarz inequality we have

$$\begin{aligned}&\int_{\Omega_r} \{|\nabla u|^2 + au^2\}\varrho^2 \,\mathrm{d}x \\ &\qquad \le \frac{C_1}{r\min(1,\lambda)}\bigg\{\int_{\Omega_r\setminus\Omega_{r/2}} |\nabla u|^2\varrho^2 \,\mathrm{d}x\bigg\}^{1/2}\bigg\{\int_{\Omega_r\setminus\Omega_{r/2}} u^2 \,\mathrm{d}x\bigg\}^{1/2}.\end{aligned}$$

This completes the proof of the lemma. □

3. Some Liouville type results

3.1. *The case where the growth of u is controled*

In this case we have

Theorem 3.1. *Under the asumptions of Lemma 2.1, let u be solution to* (1.2) *such that for r large,*

$$\frac{1}{r^2}\int_{\Omega_r\setminus\Omega_{r/2}} u^2 \,\mathrm{d}x \le C' \tag{3.1}$$

where C' is a constant independent of r, then $u =$ constant and if $a \not\equiv 0$ or $k \ge 3$ one has $u = 0$.

Proof. From (2.4) we derive that

$$\int_{\Omega_r} |\nabla u|^2\varrho^2 \,\mathrm{d}x \le \frac{C}{r}\bigg\{\int_{\Omega_r} |\nabla u|^2\varrho^2 \,\mathrm{d}x\bigg\}^{1/2}\bigg\{\int_{\Omega_r\setminus\Omega_{r/2}} u^2 \,\mathrm{d}x\bigg\}^{1/2}$$

and thus

$$\int_{\Omega_{r/2}} |\nabla u|^2 \,\mathrm{d}x \le \int_{\Omega_r} |\nabla u|^2 \varrho^2 \,\mathrm{d}x \le \frac{C}{r^2} \int_{\Omega_r \setminus \Omega_{r/2}} u^2 \,\mathrm{d}x \le CC'.$$

It follows that the nondecreasing function

$$r \mapsto \int_{\Omega_r} |\nabla u|^2 \,\mathrm{d}x$$

is bounded and has a limit when $r \to +\infty$. Going back to (2.4) we have

$$\int_{\Omega_{r/2}} |\nabla u|^2 \,\mathrm{d}x \le \frac{c}{r} \left\{ \int_{\Omega_r \setminus \Omega_{r/2}} |\nabla u|^2 \,\mathrm{d}x \right\}^{\frac{1}{2}} \cdot r$$

for some constant c. This implies

$$\int_{\Omega_{r/2}} |\nabla u|^2 \,\mathrm{d}x \le c \left\{ \int_{\Omega_r} |\nabla u|^2 \,\mathrm{d}x - \int_{\Omega_{r/2}} |\nabla u|^2 \,\mathrm{d}x \right\}^{\frac{1}{2}} \to 0$$

as $r \to +\infty$ and the result follows. □

Remark 3.1. When $k \le 2$ condition (3.1) is satisfied if u is bounded, and in this case the only bounded solution of (1.2) is $u = 0$. **Therefore we will assume throughout the rest of this paper that $k \ge 3$.**

We denote by λ_r the first eigenvalue of the Neumann problem associated to the operator $-\nabla \cdot A\nabla + a$ in $\Omega_r \setminus \Omega_{r/2}$, i.e., we set

$$\lambda_r = \operatorname{Inf}\left\{ \int_{\Omega_r \setminus \Omega_{r/2}} A\nabla u \cdot \nabla u + au^2 \,\mathrm{d}x : u \in H^1(\Omega_r \setminus \Omega_{r/2}), \right. \\ \left. \int_{\Omega_r \setminus \Omega_{r/2}} u^2 \,\mathrm{d}x = 1 \right\}. \tag{3.2}$$

One remarks easily that if u is a minimizer of (3.2) so is $|u|$. One can show then that the first eigenvalue is simple. Moreover we have

Theorem 3.2. *Under the assumptions of Lemma 2.1, suppose that for some constants $C_0 > 0$, $\beta < 2$, one has*

$$\lambda_r \ge C_0 / r^\beta \tag{3.3}$$

for r sufficiently large, then the only bounded solution of (1.2) *is $u = 0$.*

Proof. From the definition of λ_r we have

$$\int_{\Omega_r\setminus\Omega_{r/2}} u^2\,\mathrm{d}x \le \frac{1}{\lambda_r}\Big\{\int_{\Omega_r\setminus\Omega_{r/2}} A\nabla u\cdot\nabla u + au^2\,\mathrm{d}x\Big\} \quad \forall\, u\in H^1(\Omega_r\setminus\Omega_{r/2}). \tag{3.4}$$

Going back to (2.4) we find

$$\int_{\Omega_r}(|\nabla u|^2+au^2)\varrho^2\Big(\frac{x}{r}\Big)\,\mathrm{d}x \le \frac{C}{r}\Big\{\int_{\Omega_r}(|\nabla u|^2+au^2)\varrho^2\Big(\frac{x}{r}\Big)\,\mathrm{d}x\Big\}^{1/2}\Big\{\int_{\Omega_r\setminus\Omega_{r/2}} u^2\,\mathrm{d}x\Big\}^{1/2},$$

which leads to

$$\int_{\Omega_r}(|\nabla u|^2+au^2)\varrho^2\Big(\frac{x}{r}\Big)\,\mathrm{d}x \le \frac{C}{r^2}\int_{\Omega_r\setminus\Omega_{r/2}} u^2\,\mathrm{d}x$$

for some constant C independent of r. Using in particular (2.2) we obtain

$$\int_{\Omega_{r/2}} |\nabla u|^2+au^2\,\mathrm{d}x \le \frac{C}{r^2}\int_{\Omega_r\setminus\Omega_{r/2}} u^2\,\mathrm{d}x, \quad \forall\, r>0. \tag{3.5}$$

From (3.3) and (3.4) we derive that, for some constant C,

$$\int_{\Omega_{r/2}} |\nabla u|^2+au^2\,\mathrm{d}x \le \frac{C}{r^{2-\beta}}\int_{\Omega_r\setminus\Omega_{r/2}} |\nabla u|^2+au^2\,\mathrm{d}x \le \frac{C}{r^{2-\beta}}\int_{\Omega_r} |\nabla u|^2+au^2\,\mathrm{d}x \quad \forall\, r>0. \tag{3.6}$$

Iterating p-times this formula leads to

$$\int_{\Omega_{r/2}} |\nabla u|^2+au^2\,\mathrm{d}x \le \frac{C^p}{r^{(2-\beta)p}}\int_{\Omega_{2^{p-1}r}} |\nabla u|^2+au^2\,\mathrm{d}x.$$

By (3.5) it follows that it holds

$$\int_{\Omega_{r/2}} |\nabla u|^2+au^2\,\mathrm{d}x \le \frac{C_p}{r^{(2-\beta)p+2}}\int_{\Omega_{2^p r}} u^2\,\mathrm{d}x,$$

for some constant C_p independent of r. If now u is supposed to be bounded by M we get

$$\int_{\Omega_{r/2}} |\nabla u|^2+au^2\,\mathrm{d}x \le \frac{C_p}{r^{(2-\beta)p+2}}M^2|\Omega_{2^p r}| = \frac{C_p|\Omega|M^2(2^p r)^k}{r^{(2-\beta)p+2}}.$$

($|\Omega_{2^p r}|$ denotes the Lebesgue measure of the set $\Omega_{2^p r}$). Choosing $(2-\beta)p+2>k$ the result follows by letting $r\to+\infty$. □

Remark 3.2. Under the assumption of Theorem 3.2 we have obtained in fact that (1.2) can not admit a nontrivial solution with polynomial growth. Of course this result is optimal since $Re(e^z) = e^{x_1}\cos x_2$ is harmonic in $\mathbb{R}^k$ for any $k \geq 2$. One should note that Theorem 3.2 applies also to systems satisfying the Legendre condition when auv is replaced by a nonnegative bilinear form $a(u, v)$ (see [6], [7]).

We now discuss some conditions on a which imply (3.3). We have

Theorem 3.3. *Suppose that for $|x|$ large enough*

$$a(x) \geq \frac{c}{|x|^\beta}, \qquad \beta < 2, \tag{3.7}$$

then (3.3) *holds.*

Proof. We denote by π_r the first eigenfunction corresponding to λ_r, i.e., a minimizer of (3.2). We can assume without loss of generality that

$$\pi_r > 0.$$

We have

$$\int_{\Omega_r \setminus \Omega_{r/2}} A\nabla\pi_r \cdot \nabla v + a\pi_r v \,\mathrm{d}x = \lambda_r \int_{\Omega_r \setminus \Omega_{r/2}} \pi_r v \,\mathrm{d}x \quad \forall\, v \in H^1(\Omega_r \setminus \Omega_{r/2}).$$

Taking $v = 1$ yields

$$\int_{\Omega_r \setminus \Omega_{r/2}} a(x)\pi_r \,\mathrm{d}x = \lambda_r \int_{\Omega_r \setminus \Omega_{r/2}} \pi_r \,\mathrm{d}x.$$

Using (3.7) we derive, for some constant C',

$$\frac{C'}{r^\beta} \int_{\Omega_r \setminus \Omega_{r/2}} \pi_r \,\mathrm{d}x \leq \lambda_r \int_{\Omega_r \setminus \Omega_{r/2}} \pi_r \,\mathrm{d}x$$

and the result follows. □

We now consider other cases where (3.3) holds, in particular when no decay is imposed to a. We are interested for instance in the case where at infinity a has enough mass locally. We start with the following lemma:

Lemma 3.1. *Let (for instance) $Q = (0, 1)^k$ be the unit cube in $\mathbb{R}^k$. For any $\varepsilon > 0$ and $\mu > 0$ there exists $\delta = \delta(\varepsilon, \mu)$ such that if the function a satisfies*

$$0 \leq a \leq \mu \ \textit{a.e.}\ x \in Q, \qquad \int_Q a \,\mathrm{d}x \geq \varepsilon, \tag{3.8}$$

then

$$\delta \int_Q v^2 \,\mathrm{d}x \le \int_Q |\nabla v|^2 + a v^2 \,\mathrm{d}x \qquad \forall\, v \in H^1(Q). \tag{3.9}$$

Proof. If not, there exists ε, μ and a sequence of functions a_n, v_n such that a_n satisfies (3.8) and $v_n \in H^1(Q)$ is such that

$$\frac{1}{n} \int_Q v_n^2 \,\mathrm{d}x \ge \int_Q |\nabla v_n|^2 + a_n v_n^2 \,\mathrm{d}x. \tag{3.10}$$

Dividing by $|v_n|_2$ the L^2-norm of v_n we can assume without loss of generality that

$$\int_Q v_n^2 \,\mathrm{d}x = 1. \tag{3.11}$$

By (3.10), (3.11) we have then

$$\int_Q |\nabla v_n|^2 \,\mathrm{d}x \le \frac{1}{n}, \qquad \int_Q v_n^2 \,\mathrm{d}x = 1 \tag{3.12}$$

and v_n is uniformly bounded in $H^1(Q)$. Therefore

$$v_n \to 1 \text{ in } H^1(Q). \tag{3.13}$$

From (3.10) we have

$$\int_Q a_n v_n^2 \,\mathrm{d}x \le \frac{1}{n}. \tag{3.14}$$

Thus

$$\begin{aligned} \varepsilon \le \int_Q a_n \mathrm{d}x &= \int_Q a_n v_n^2 \,\mathrm{d}x + \int_Q a_n (1 - v_n^2) \,\mathrm{d}x \\ &\le \frac{1}{n} + \mu |1 - v_n|_2 |1 + v_n|_2 \to 0 \end{aligned}$$

when $n \to +\infty$. Impossible. This completes the proof of the lemma. □

With the notation of Section 2 we set

$$\Omega = (-1, 1)^k. \tag{3.15}$$

We consider the lattice generated by $Q = (0,1)^k$ – i.e., the cubes

$$Q_i = Q_{z_i} = z_i + Q \quad \forall\, z_i \in \mathbb{Z}^k.$$

Then we have

Theorem 3.4. *Suppose that for n large enough,*

$$\int_{Q_i} a(x)\,\mathrm{d}x \geq \varepsilon \quad \forall\, Q_i \subset \mathbb{R}^k \setminus \Omega_n, \tag{3.16}$$

then

$$\lambda_{2n} \geq \delta \Big/ \Big(\frac{1}{\lambda} \vee 1\Big) \quad \forall\, n \tag{3.17}$$

where δ is defined in Lemma 3.1 and $\vee$ denotes the maximum of two numbers.

Proof. Indeed by Lemma 3.1 after a simple translation from Q_i into Q we have

$$\delta \int_{Q_i} u^2\,\mathrm{d}x \leq \int_{Q_i} |\nabla u|^2 + au^2\,\mathrm{d}x \quad \forall\, Q_i \subset \mathbb{R}^k \setminus \Omega_n \quad \forall\, u \text{ in } H^1(Q_i).$$

This leads clearly to

$$\begin{aligned}
\delta & \int_{\Omega_{2n}\setminus\Omega_n} u^2\,\mathrm{d}x \\
&\leq \int_{\Omega_{2n}\setminus\Omega_n} |\nabla u|^2 + au^2\,\mathrm{d}x \\
&\leq \int_{\Omega_{2n}\setminus\Omega_n} \frac{1}{\lambda} A\nabla u \cdot \nabla u + au^2\,\mathrm{d}x \\
&\leq \Big(\frac{1}{\lambda} \vee 1\Big) \int_{\Omega_{2n}\setminus\Omega_n} A\nabla u \cdot \nabla u + au^2\,\mathrm{d}x \quad \forall\, u \in H^1(\Omega_{2n} \setminus \Omega_n).
\end{aligned}$$

The result follows then from (3.2). □

Remark 3.3. Combining Theorems 3.2 and 3.4 it follows that (1.2) cannot have a nontrivial bounded solution (or of polynomial growth) when (3.16) holds. This is the case when at infinity

$$a \geq a_0 > 0$$

or more generally

$$a \geq a_p \tag{3.18}$$

where a_p is a periodic function with period Q.

In the case when (3.3) holds with $\beta = 2$ the technique of Theorem 3.2 cannot be applied. However, we will show that the non existence of nontrivial solutions can be established in this case too – i.e., condition (3.3) is not sharp if we impose certain growth condition on $\{a_{ij}(x)\}$. Before turning to this let us prove some general comparison result. For simplicity we will denote also by $\tilde{A}$ the operator $\nabla \cdot A\nabla u = \partial_{x_i}(a_{ij}\partial_{x_j})$.

Proposition 3.1. *Suppose that O is a bounded open subset of $\mathbb{R}^k$. Let a_1, a_2 be two bounded functions satisfying*

$$a_1 \geq a_2 \geq 0 \quad \textit{a.e. in } O. \tag{3.19}$$

Let $u_1, u_2 \in H^1(O)$ be such that

$$\begin{cases} -\tilde{A}u_2 + a_2u_2 \geq -\tilde{A}u_1 + a_1u_1 \geq 0 & \textit{in } O, \\ u_2 \geq (u_1 \vee 0) & \textit{on } \partial O, \end{cases} \tag{3.20}$$

then

$$u_2 \geq (u_1 \vee 0) \quad \textit{in } O. \tag{3.21}$$

Proof. The inequality

$$-\tilde{A}u + au \geq 0 \quad \text{in } O$$

means

$$\int_O a_{ij}\partial_{x_j}u\partial_{x_i}v + auv \,\mathrm{d}x \geq 0 \quad \forall\, v \in H_0^1(O), v \geq 0.$$

Considering $v = u_2^-$ and $-\tilde{A}u_2 + a_1u_2 \geq 0$ leads to $u_2 \geq 0$. Next considering $v = (u_1 - u_2)^+ \in H_0^1(O)$ and (3.20) we obtain

$$\begin{aligned} \int_O a_{ij}\partial_{x_j}u_1\partial_{x_i}(u_1 - u_2)^+ + a_1u_1(u_1 - u_2)^+ \,\mathrm{d}x \\ \leq \int_O a_{ij}\partial_{x_j}u_2\partial_{x_i}(u_1 - u_2)^+ + a_2u_2(u_1 - u_2)^+ \,\mathrm{d}x. \end{aligned}$$

Hence

$$\int_O a_{ij}\partial_{x_j}(u_1 - u_2)\partial_{x_i}(u_1 - u_2)^+ + (a_1u_1 - a_2u_2)(u_1 - u_2)^+ \,\mathrm{d}x \leq 0.$$

Now on $u_1 \geq u_2$ one has $a_1u_1 \geq a_1u_2 \geq a_2u_2$ and it follows that $(u_1 - u_2)^+ = 0$. □

Next we prove

Theorem 3.5. *Assume that there exists R, large enough, such that*

$$a(x) \geq \frac{c_0}{r^2} \quad \forall |x| \geq R > 0$$

where c_0 is a positive constant. In addition to (1.1), suppose that $a_{ij}(x) \in C^1(\mathbb{R}^k \backslash B(0,R))$ satisfies for some positive D:

$$\partial_{x_i}(a_{ij}(x))x_j \le D \quad \forall |x| > R.$$

(In the above inequality we make the summation convention of repeated indices). Then the equation

$$-\partial_{x_i}(a_{ij}(x)\partial_{x_j}u) + a(x)u = 0 \tag{3.22}$$

cannot have nontrivial bounded solution.

Proof. Let u_n be the solution to

$$\begin{cases} -\partial_{x_i}(a_{ij}(x)\partial_{x_j}u_n) + a(x)u_n = 0 & \text{in } B(0,n), \\ u_n = 1 \quad \text{on } \partial B(0,n), \end{cases} \tag{3.23}$$

where $B(0,n)$ denotes the ball of center 0 and radius n. From Proposition 3.1 we obtain that u, solution to (3.23), is such that:

$$-|u|_\infty u_n \le u \le |u|_\infty u_n, \tag{3.24}$$

($|u|_\infty$ denotes the L^∞-norm of u). Denote by v_n the function defined as

$$v_n = \begin{cases} c_1 & \text{in } B(0,R) \\ c_2 r^{\beta_1} + c_3 r^{\beta_2} & \text{in } B(0,n)\backslash B(0,R), \end{cases} \tag{3.25}$$

where

$$\beta_1 = -\frac{1}{2}\{(k-2) + \sqrt{(k-2)^2 + 4c'}\} < 0,$$
$$\beta_2 = -\frac{1}{2}\{(k-2) - \sqrt{(k-2)^2 + 4c'}\} > 0$$

and

$$c_1 = c_2 R^{\beta_1} + c_3 R^{\beta_2}$$
$$c_2 = \frac{-\beta_2 R^{\beta_2 - 1}}{n^{\beta_2}\beta_1 R^{\beta_1 - 1} - n^{\beta_1}\beta_2 R^{\beta_2 - 1}}$$
$$c_3 = \frac{\beta_1 R^{\beta_1 - 1}}{n^{\beta_2}\beta_1 R^{\beta_1 - 1} - n^{\beta_1}\beta_2 R^{\beta_2 - 1}}.$$

In the above setting, c' is a positive constant small enough that we will determine later.

We remark that c_2 and c_3 are both positive and that β_1, β_2 are the two roots to the second order equation

$$\beta^2 + (k-2)\beta - c' = 0. \tag{3.26}$$

The choice of $c_i, i = 1, 2, 3$ is such that v_n is a C^1 function and $v_n = 1$ on $\partial B(0, n)$.

Now we want to show that v_n, in fact, is a supersolution to (3.23).

It is easy to see that

$$-\partial_{x_i}(a_{ij}(x)\partial_{x_j} v_n) + a(x)v_n \geq 0 \quad \text{in } B(0, R).$$

For any constant β one derives also that

$$\begin{aligned}
&\partial_{x_i}(a_{ij}(x)\partial_{x_j}(r^\beta)) \\
&= \partial_{x_i}(a_{ij}(x)\beta r^{\beta-2} x_j) \\
&= \partial_{x_i}(a_{ij}(x))\beta r^{\beta-2} x_j + a_{ij}(x)\beta(\beta - 2)r^{\beta-4} x_i x_j \\
&\quad + a_{ij}(x)\beta r^{\beta-2}\delta_{ij}.
\end{aligned}$$

Therefore in $B(0, n)\backslash B(0, R)$ this leads to

$$\begin{aligned}
&-\partial_{x_i}(a_{ij}(x)\partial_{x_j} v_n) + a(x)v_n \\
&\geq -\partial_{x_i}(a_{ij}(x))x_j\{c_2\beta_1 r^{\beta_1-2} + c_3\beta_2 r^{\beta_2-2}\} \\
&\quad - a_{ij}(x)x_i x_j\{c_2\beta_1(\beta_1 - 2)r^{\beta_1-4} + c_3\beta_2(\beta_2 - 2)r^{\beta_2-4}\} \\
&\quad - a_{ij}(x)\delta_{ij}\{c_2\beta_1 r^{\beta_1-2} + c_3\beta_2 r^{\beta_2-2}\} + \frac{c_0}{r^2}\{c_2 r^{\beta_1} + c_3 r^{\beta_2}\} \\
&= \left\{-\partial_{x_i}(a_{ij}(x))x_j\{c_2\beta_1 r^{\beta_1-2} + c_3\beta_2 r^{\beta_2-2}\}\right\} \\
&\quad + c_2\left\{-a_{ij}(x)x_i x_j\beta_1(\beta_1 - 2)r^{\beta_1-4} - a_{ij}(x)\delta_{ij}\beta_1 r^{\beta_1-2} + c_0 r^{\beta_1-2}\right\} \\
&\quad + c_3\left\{-a_{ij}(x)x_i x_j\beta_2(\beta_2 - 2)r^{\beta_2-4} - a_{ij}(x)\delta_{ij}\beta_2 r^{\beta_2-2} + c_0 r^{\beta_2-2}\right\}.
\end{aligned}$$

We notice that

$$c_2\beta_1 r^{\beta_1-2} + c_3\beta_2 r^{\beta_2-2} = \frac{\beta_1\beta_2 r^{\beta_2-2}R^{\beta_2-1}\{R^{\beta_1-\beta_2} - r^{\beta_1-\beta_2}\}}{n^{\beta_2}\beta_1 R^{\beta_1-1} - n^{\beta_1}\beta_2 R^{\beta_2-1}} > 0,$$

and thus

$$\begin{aligned}
&-\partial_{x_i}(a_{ij}(x)\partial_{x_j} v_n) + a(x)v_n \\
&\geq \left\{-D\{c_2\beta_1 r^{\beta_1-2} + c_3\beta_2 r^{\beta_2-2}\}\right\} \\
&\quad + c_2\left\{-a_{ij}(x)x_i x_j\beta_1(\beta_1 - 2)r^{\beta_1-4} - a_{ij}(x)\delta_{ij}\beta_1 r^{\beta_1-2} + c_0 r^{\beta_1-2}\right\} \\
&\quad + c_3\left\{-a_{ij}(x)x_i x_j\beta_2(\beta_2 - 2)r^{\beta_2-4} - a_{ij}(x)\delta_{ij}\beta_2 r^{\beta_2-2} + c_0 r^{\beta_2-2}\right\}.
\end{aligned}$$

Taking into account (3.26) –i.e., replacing $\beta_i(\beta_i-2)$ by $c'-k\beta_i$, yields

$$\begin{aligned}
&-\partial_{x_i}(a_{ij}(x)\partial_{x_j}v_n)+a(x)v_n\\
&\geq c_2r^{\beta_1-2}\left\{[ka_{ij}(x)\frac{x_ix_j}{r^2}-a_{ii}(x)-D]\beta_1-c'a_{ij}(x)\frac{x_ix_j}{r^2}+c_0\right\}\\
&\quad+c_3r^{\beta_2-2}\left\{[ka_{ij}(x)\frac{x_ix_j}{r^2}-a_{ii}(x)-D]\beta_2-c'a_{ij}(x)\frac{x_ix_j}{r^2}+c_0\right\}\\
&\geq c_2r^{\beta_1-2}\left\{[ka_{ij}(x)\frac{x_ix_j}{r^2}-a_{ii}(x)-D]\beta_1-c'\Lambda+c_0\right\}\\
&\quad+c_3r^{\beta_2-2}\left\{[ka_{ij}(x)\frac{x_ix_j}{r^2}-a_{ii}(x)-D]\beta_2-c'\Lambda+c_0\right\}.
\end{aligned}$$

We can select a D large enough such that the term

$$ka_{ij}(x)\frac{x_ix_j}{r^2}-a_{ii}(x)-D$$

is negative (and bounded). By noticing that $\beta_2\to 0^+$ when $c'\to 0$ we can then always choose c' small enough such that

$$[ka_{ij}(x)\frac{x_ix_j}{r^2}-a_{ii}(x)-D]\beta_i-c'\Lambda+c_0>0.$$

Hence we derive that

$$-\partial_{x_i}(a_{ij}(x)\partial_{x_j}v_n)+a(x)v_n\geq 0,$$

and by Proposition 3.1,

$$u_n\leq v_n.$$

For any bounded subset $\Omega\subset B(0,d)$ in $\mathbb{R}^k$, one has clearly

$$0\leq v_n\leq \mathrm{Max}\{c_1,c_2d^{\beta_1}+c_3d^{\beta_2}\}\to 0\qquad \text{on } \Omega$$

when $n\to+\infty$ since

$$n^{\beta_2}\beta_1R^{\beta_1-1}-n^{\beta_1}\beta_2R^{\beta_2-1}\to-\infty$$

as $n\to+\infty$. From (3.24) we have also on $B(0,n)$

$$-|u|_\infty v_n\leq u\leq |u|_\infty v_n$$

for any n. Letting $n\to\infty$ leads to that

$$u=0.$$

□

Remark 3.4. The above result holds true for an operator in non-divergence form, i.e., under the assumption of Theorem 3.5 the equation

$$-a_{ij}(x)\partial^2_{x_ix_j}u-b_i(x)\partial_{x_i}u+a(x)u=0$$

with

$$(b, x) \leq D \quad \forall |x| > R$$

cannot have a nontrivial bounded solution.

4. The case of the Laplace operator

In this section we analyze the existence or nonexistence of nontrivial bounded solutions to (1.2) in the case of the Laplacian. Due to the results of the previous section it is clear that existence of nontrivial solutions will impose some kind of decay a. So, let us assume

$$a(x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^k), a \geq 0, a \not\equiv 0$$

and

$$\int_{|x|>1} a(x)|x|^{-k+2}\,\mathrm{d}x < \infty \tag{4.1}$$

with

$$k \geq 3.$$

Under the above assumptions we can show

Theorem 4.1. *(Grigor'yan* [8]*, see also* [2]*,* [3]*,* [15]*) Assume (*(4.1)*). Then there exists a function u such that*

$$0 < u < 1 \quad \textit{in } \mathbb{R}^k \tag{4.2}$$

satisfying

$$-\Delta u + au = 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}^k). \tag{4.3}$$

Proof. Let u_n be the solution of

$$\begin{cases} -\Delta u_n + a u_n = 0 \quad \text{in } B(0,n), \\ u_n = 1 \quad \text{on } \partial B(0,n). \end{cases} \tag{4.4}$$

By the maximum principle

$$0 \leq u_n \leq 1 \tag{4.5}$$

and

$$u_{n+1} \leq u_n \quad \text{in } B(0,n). \tag{4.6}$$

Thus $u_n \to u$ which satisfies (4.3). Moreover

$$0 \leq u \leq 1.$$

By the strong maximum principle (and since $a \not\equiv 0$) we have

$$u < 1 \quad \text{in } \mathbb{R}^k.$$

Once more, by the strong maximum principle, it suffices to prove that

$$u \not\equiv 0.$$

Assume, by contradiction, that

$$u \equiv 0. \tag{4.7}$$

Fix a function $\zeta \in C^\infty(\mathbb{R}^k)$, $0 \le \zeta \le 1$ such that

$$\zeta(x) = \begin{cases} 0 & |x| < R \\ 1 & |x| \ge R+1 \end{cases}$$

and R will be determined later. Multiplying (4.4) by $\frac{\zeta(x)}{|x|^{k-2}}$ yields for $n > R+1$, $\nu =$ being the outward unit normal vector:

$$-\int_{|x|=n} \frac{\partial u_n}{\partial \nu} \cdot \frac{1}{n^{k-2}} \,\mathrm{d}\sigma - \int_{B(0,n)} u_n \Delta(\frac{\zeta}{|x|^{k-2}}) \,\mathrm{d}x + \int_{|x|=n} \frac{\partial}{\partial \nu}(\frac{\zeta}{|x|^{k-2}}) \,\mathrm{d}\sigma + \int_{B(0,n)} a u_n \frac{\zeta}{|x|^{k-2}} \,\mathrm{d}x = 0. \tag{4.8}$$

By (4.5)

$$\frac{\partial u_n}{\partial \nu} \ge 0 \quad \text{on } \partial B(0,n) \tag{4.9}$$

and

$$\frac{\partial}{\partial \nu}(\frac{\zeta}{|x|^{k-2}}) = \frac{2-k}{n^{k-1}} \quad \text{on } \partial B(0,n)$$

so that

$$\int_{|x|=n} \frac{\partial}{\partial \nu}(\frac{\zeta}{|x|^{k-2}}) \,\mathrm{d}\sigma = (2-k)\sigma_k \tag{4.10}$$

where σ_k denotes the area of S^{k-1}. From (4.8), (4.9) and (4.10) we have

$$-\int_{B(0,n)} u_n \Delta(\frac{\zeta}{|x|^{k-2}}) \,\mathrm{d}x + \int_{B(0,n)} a u_n \frac{\zeta}{|x|^{k-2}} \,\mathrm{d}x \ge (k-2)\sigma_k. \tag{4.11}$$

Notice that $\Delta(\frac{\zeta}{|x|^{k-2}})$ has compact support (in $R < |x| < R+1$) since $\Delta(\frac{1}{|x|^{k-2}}) = c\delta_0$ where δ_0 denotes the Dirac measure at 0. Therefore one has

$$\lim_{n\to\infty} \int_{B(0,n)} u_n \Delta(\frac{\zeta}{|x|^{k-2}}) \,\mathrm{d}x = 0 \tag{4.12}$$

by assumption (4.7). On the other hand

$$\int_{B(0,n)} au_n \frac{\zeta}{|x|^{k-2}}\,\mathrm{d}x \le \int_{R<|x|} \frac{a(x)}{|x|^{k-2}}\,\mathrm{d}x. \tag{4.13}$$

By (4.11), (4.12) and (4.13) we have

$$\int_{R<|x|} \frac{a(x)}{|x|^{k-2}}\mathrm{d}x \ge (k-2)\sigma_k.$$

Choosing R sufficiently large and using assumption (4.1) yields a contradiction. This completes the proof of $u \not\equiv 0$. □

Remark 4.1. With the same proof and under the assumption (4.1) one can show that (1.2) admits a non trivial solution provided

$$\lim_{R\to\infty} \int_{|x|\ge R} \partial_{x_i}(a_{ij}(x)\partial_{x_j}|x|^{2-k}) < \lambda(k-2)\sigma_k.$$

This is in particular the case when $a_{ij}(x) = \delta_{ij}$ for $|x|$ large.

In the radially symmetric case we can say more.

Theorem 4.2. *Suppose that the solutions to* (1.2) *are radially symmetric, then they do not change sign and are multiple of each other.*

Proof. Let u be a radially symmetric solution to (1.2). Let us first prove that u does not change sign. Changing u into $-u$ we can suppose

$$u(0) \ge 0.$$

We argue by contradiction and assume that u changes sign. If $u(0) > 0$, there exists a r_0 such that

$$u(r_0) = 0.$$

Then

$$\int_{B(0,r_0)} \nabla u \cdot \nabla v + auv\,\mathrm{d}x = 0 \qquad \forall v \in H_0^1(B(0,r_0)). \tag{4.14}$$

Taking $v = u$ we obtain that $u \equiv 0$ in $B(0,r_0)$ and a contradiction. If $u(0) = 0$ then changing u in $-u$ there is a component of the set

$$\{x \big| u(x) > 0\}$$

which is an annulus A. But then we get (4.14) with $B(0,r_0)$ replaced by A and a contradiction as above.

Consider now u, v two solutions to (1.2). If $u \equiv 0$, $u = 0 \cdot v$, or else we have by the first part of the theorem (after changing u into $-u$ if needed):

$$u(0) > 0.$$

Then $w = v - \frac{v(0)}{u(0)} u$ is a solution to (1.2) such that $w(0) = 0$. Since it does not change sign, 0 is a minimum or a maximum and by the maximum principle $w \equiv 0$. This completes the proof of the theorem. $\square$

In the case of radially symmetric solutions we also have:

Proposition 4.1. *Suppose that $a = a(r)$, $r = |x|$ and let u be a bounded positive radially symmetric solution to*

$$-\Delta u + au = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^k).$$

We have

$$u(0) > 0,\ u = u(r) \text{ is nondecreasing on } (0, +\infty),\ \lim_{r\to\infty} u(r) = u(\infty) < +\infty.$$

Proof. $u(0) > 0$ results from the previous theorem. In addition we have

$$-u'' - \frac{k-1}{r} u' + au = 0$$

$$\Longrightarrow rau = ru'' + (k-1)u' = \frac{(r^{k-1}u')'}{r^{k-2}} \geq 0. \tag{4.15}$$

Thus $r^{k-1}u'$ is nondecreasing. Since it vanishes at 0 we have $u' \geq 0$ and u is nondecreasing. Hence u has a limit at ∞ since u is bounded. $\square$

As a consequence we have the following property for the solution u that we constructed in the Theorem 4.1.

Theorem 4.3. *Suppose that for $|x| \geq R_0$*

$$a(x) \leq a_0(|x|) \quad \text{with} \int^{+\infty} ra(r)\,\mathrm{d}r < +\infty. \tag{4.16}$$

Then the solution u constructed in Theorem 4.1 verifies

$$\lim_{|x|\to\infty} u(x) = 1.$$

Proof. We introduce

$$\tilde{a} = \begin{cases} |a|_\infty & \text{for } |x| < R_0, \\ a_0(r) & \text{for } |x| \geq R_0. \end{cases}$$

Let $\tilde{u}_n$ be the solution to

$$\begin{cases} -\Delta \tilde{u}_n + \tilde{a}\tilde{u}_n = 0 & \text{in } B(0,n), \\ \tilde{u}_n = 1 & \text{on } \partial B(0,n). \end{cases}$$

By Proposition 3.1 we have

$$0 \leq \tilde{u}_{n+1} \leq \tilde{u}_n \leq u_n \leq 1. \tag{4.17}$$

Of course since $\tilde{a}$ is radially symmetric, so is $\tilde{u}_n$ and it converges to a radially symmetric function $\tilde{u}$ which is a nontrivial solution to (see Theorem 4.1)

$$-\Delta \tilde{u} + \tilde{a}\tilde{u} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k).$$

From Proposition 4.1 we have

$$0 < \lim_{|x|\to\infty} \tilde{u} = \tilde{u}(\infty) \leq 1.$$

Suppose that $\tilde{u}(\infty) < 1$. Consider $\tilde{v}_n$ the solution of

$$\begin{cases} -\Delta \tilde{v}_n + \tilde{a}\tilde{v}_n = 0 & \text{in } B(0,n) \\ \tilde{v}_n = 1 - \tilde{u}(\infty) & \text{on } \partial B(0,n). \end{cases}$$

One has

$$\begin{cases} -\Delta(\tilde{v}_n + \tilde{u}) + \tilde{a}(\tilde{v}_n + \tilde{u}) = 0 & \text{in } B(0,n), \\ \tilde{v}_n + \tilde{u} = 1 + \tilde{u} - \tilde{u}(\infty) \leq 1 & \text{on } \partial B(0,n). \end{cases}$$

Thus, by the maximum principle,

$$\tilde{v}_n + \tilde{u} \leq \tilde{u}_n \quad \text{in } B(0,n).$$

Now clearly $\tilde{v}_n = (1 - \tilde{u}(\infty))\tilde{u}_n$ and thus

$$(1 - \tilde{u}(\infty))\tilde{u}_n + \tilde{u} \leq \tilde{u}_n.$$

Passing to the limit in n we obtain

$$(1 - \tilde{u}(\infty))\tilde{u} + \tilde{u} \leq \tilde{u}$$

which contradicts $\tilde{u}(\infty) < 1$. Thus we have $\tilde{u}(\infty) = 1$.

Now from (4.17) we derive, passing to the limit,

$$\tilde{u} \leq u \leq 1.$$

Since, we already know that $\lim_{|x|\to\infty} \tilde{u}(x) = 1$ the result follows. This completes the proof of the theorem. □

We prove now that condition (4.16) is sharp within the class of radial functions. This was observed in [3] with a different technique (see also [10], [9]). More recently (R. Pinsky [18]) established the sharpness of condition (4.16) in the class of functions a satisfying the additional assumption

$$a(x) \leq \frac{C}{(1+|x|)^2}. \tag{4.18}$$

So, let $a(r)$ be a function such that

$$\int^{+\infty} ra(r)\,\mathrm{d}r = +\infty. \tag{4.19}$$

Lemma 4.1. *Under the assumption* (4.19) *there does not exist a bounded nontrivial radially symmetric solution to*

$$-\Delta u + a(r)u = 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}^k). \tag{4.20}$$

Proof. Suppose that (4.20) admits a nontrivial bounded positive solution $u(r)$ (see Theorem 4.3). Integrating the first equality of (4.15) we find

$$\int_0^r sa(s)u(s)\,\mathrm{d}s = \int_0^r su''(s)\,\mathrm{d}s + (k-1)\int_0^r u'(s)\,\mathrm{d}s$$
$$= ru'(r) + (k-2)\{u(r) - u(0)\} = (ru)' + (k-3)u(r) - (k-2)u(0).$$

Integrating again in r yields

$$ru(r) = \int_0^r (\int_0^s \xi a(\xi)u(\xi)\,\mathrm{d}\xi)\mathrm{d}s - (k-3)\int_0^r u(s)\,\mathrm{d}s + (k-2)u(0)r \tag{4.21}$$

For $s \geq \frac{r}{2}$ we have

$$\int_0^s \xi a(\xi)u(\xi)\,\mathrm{d}\xi \geq u(0)\int_0^{\frac{r}{2}} \xi a(\xi)\,\mathrm{d}\xi.$$

Thus from (4.21) we easily obtain

$$u(r) \geq \frac{1}{r}\int_{\frac{r}{2}}^r (\int_0^s \xi a(\xi)u(\xi)\,\mathrm{d}\xi)\mathrm{d}s - (k-3)u(\infty) + (k-2)u(0)$$
$$\geq \frac{1}{2}\int_0^{\frac{r}{2}} \xi a(\xi)\,\mathrm{d}\xi \cdot u(0) - (k-3)u(\infty) + (k-2)u(0).$$

By (4.19) the left-hand side of this inequality goes to $+\infty$ with r. This contradicts the boundedness of u and completes the proof of the lemma. □

As a consequence we can now show:

Theorem 4.4. *Suppose that for $|x|$ large*

$$a(x) \geq \bar{a}(r) = \bar{a}(|x|) \tag{4.22}$$

where $\bar{a}$ satisfies (4.19) *then the problem*

$$-\Delta u + au = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k) \tag{4.23}$$

cannot admit nontrivial bounded solutions.

Proof. Suppose that (4.22) holds for $|x| \geq R$. Then define

$$\tilde{a} = \begin{cases} 0 & \text{when } |x| \leq R, \\ \bar{a}(r) & \text{when } |x| > R. \end{cases}$$

$\tilde{a}$ is a radially symmetric function satisfying (4.19). Let u be a bounded solution to (4.23). Let u_n, v_n be the solution of

$$-\Delta u_n + \tilde{a} u_n = 0 \quad \text{in } B(0,n), u_n = |u|_\infty \quad \text{on } \partial B(0,n), \tag{4.24}$$

$$-\Delta v_n + a v_n = 0 \quad \text{in } B(0,n), v_n = |u|_\infty \quad \text{on } \partial B(0,n), \tag{4.25}$$

where $|u|_\infty$ denotes the L^∞-norm of u. It follows from Proposition 3.1 and the maximum principle that

$$u < v_n \leq u_n, \quad 0 \leq u_{n+1} \leq u_n \leq |u|_\infty. \tag{4.26}$$

Changing u into $-u$ one if needed, can assume that the set

$$\{u > 0\} = \{x \in \mathbb{R}^k \mid u(x) > 0\}$$

has a positive measure. Now, clearly, by the uniqueness of the solution to (4.24), u_n is radially symmetric. By (4.26) u_n converges to u_∞ solution of

$$-\Delta u_\infty + \tilde{a} u_\infty = 0 \quad \text{in } \mathbb{R}^k$$

and u_∞ is radially symmetric. By Lemma 4.1 this implies that $u_\infty = 0$. Hence from (4.26) we get

$$u \leq 0$$

which contradicts the fact that $\{u > 0\}$ is of positive measure. □

Remark 4.2. Theorem 4.4 applies for instance when

$$a(x) = \frac{C_0}{|x|^2}$$

for a constant C_0 and $|x|$ large enough. Let λ_r be given by (3.2). Then for r large enough we have

$$\frac{c}{r^2} \le \lambda_r \le \frac{C}{r^2} \tag{4.27}$$

for some constants c, C. In other words the technique of Theorem 3.2 cannot work in this case. To show (4.27), recall the definition (3.2) and use the constant function

$$u = 1/|\Omega_r \setminus \Omega_{r/2}|^{1/2} \in H^1(\Omega_r \setminus \Omega_{r/2})$$

($|\cdot|$ is the Lebesgue measure); we obtain

$$\lambda_r \le \frac{1}{|\Omega_r \setminus \Omega_{r/2}|} \int_{\Omega_r\setminus\Omega_{r/2}} a(x)\,\mathrm{d}x = \frac{C_0}{|\Omega_r \setminus \Omega_{r/2}|} \int_{\Omega_r\setminus\Omega_{r/2}} \frac{\mathrm{d}x}{|x|^2} \le \frac{C}{r^2}$$

for r large enough. To obtain the left-hand side inequality of (4.27) we remark (see Theorem 3.3) that for r large enough

$$\begin{aligned}
\lambda_r &= \int_{\Omega_r\setminus\Omega_{r/2}} a\pi_r\,\mathrm{d}x \Big/ \int_{\Omega_r\setminus\Omega_{r/2}} \pi_r\,\mathrm{d}x \\
&= \int_{\Omega_r\setminus\Omega_{r/2}} C_0\pi_r/|x|^2\,\mathrm{d}x \Big/ \int_{\Omega_r\setminus\Omega_{r/2}} \pi_r\,\mathrm{d}x \\
&\ge \frac{c}{r^2} \int_{\Omega_r\setminus\Omega_{r/2}} \pi_r\,\mathrm{d}x \Big/ \int_{\Omega_r\setminus\Omega_{r/2}} \pi_r\,\mathrm{d}x = \frac{c}{r^2},
\end{aligned}$$

with $c = C_0$ for $\Omega_r = B(0,r)$. This completes the proof of (4.27).

We conclude this note with the following result.

Theorem 4.5.

Suppose that (4.3) *admits a bounded solution, then it admits a positive solution.*

If

$$\int_{|x|>1} a(x)|x|^{-k+2}\,\mathrm{d}x = \infty \tag{4.28}$$

then (4.3) *cannot admit nontrivial bounded solution such that*

$$0 < c \le u. \tag{4.29}$$

Proof. We first prove the existence of a positive solution. If $u < 0$, $-u$ is a positive solution. So, we can assume that u changes sign. Then introduce u_n solution of

$$-\Delta u_n + a u_n = 0 \quad \text{in } B(0,n), u_n = |u|_\infty \quad \text{on } \partial B(0,n). \tag{4.30}$$

One has

$$0 < u_{n+1} \le u_n \le |u|_\infty \tag{4.31}$$

and u_n converges to some fonction u_∞ for instance in $L^1_{\mathrm{loc}}(\mathbb{R}^k)$. Then u_∞ is a solution of (4.3). Moreover by the maximum principle one has $u \le u_n$ on $B(0,n)$ and thus $u \le u_\infty$. u_∞ cannot vanish identically and is the positive solution we are looking for.

Suppose now that u is a nonnegative bounded solution to

$$-\Delta u = -au.$$

Set $U(r) = \int_{\partial B_1} u(r\sigma)\,\mathrm{d}\sigma$ where B_1 denotes the unit ball of $\mathbb{R}^k$. Then

$$-(r^{k-1}U')' = -r^{k-1}\int_{\partial B_1} a(r\sigma)u(r\sigma)\,\mathrm{d}\sigma \tag{4.32}$$

hence

$$-r^{k-1}U' = -\int_0^r s^{k-1}\int_{\partial B_1} a(r\sigma)u(s\sigma)\,\mathrm{d}\sigma\mathrm{d}s \tag{4.33}$$

and $U(r) = \int_{\partial B_1} u(r\sigma)\,\mathrm{d}\sigma$ is nondecreasing. Moreover U is a solution of the second order differential equation (4.32). A particular solution is given (see (4.33)) by

$$U = \int_0^r \frac{1}{s^{k-1}}\int_0^s t^{k-1}\int_{\partial B_1} a(t\sigma)u(t\sigma)\,\mathrm{d}\sigma\mathrm{d}t\mathrm{d}s.$$

The solution of the homogeneous equation is given by

$$\frac{A}{r^{k-2}} + B.$$

Thus we have

$$U(r) = \frac{A}{r^{k-2}} + B + \int_0^r \frac{1}{s^{k-1}}\int_0^s t^{k-1}\int_{\partial B_1} a(t\sigma)u(t\sigma)\,\mathrm{d}\sigma\mathrm{d}t\mathrm{d}s. \tag{4.34}$$

Since u is bounded, so is U and necessarily $A = 0, B \ge 0$. From (4.34) we derive

$$U(r) = B + \int_0^r \frac{1}{s^{k-1}}\int_0^s t^{k-1}\int_{\partial B_1} a(t\sigma)u(t\sigma)\,\mathrm{d}\sigma\mathrm{d}t\mathrm{d}s. \tag{4.35}$$

Integrating by parts we get

$$\begin{aligned} U(r) &= B - \frac{1}{(k-2)r^{k-2}} \int_0^r t^{k-1} \int_{\partial B_1} a(t\sigma)u(t\sigma)\,\mathrm{d}\sigma\mathrm{d}t \\ &\quad + \int_0^r \frac{1}{(k-2)t^{k-2}} t^{k-1} \int_{\partial B_1} a(t\sigma)u(t\sigma)\,\mathrm{d}\sigma\mathrm{d}t \\ &= B + \frac{1}{k-2} \int_0^r \int_{\partial B_1} ta(t\sigma)(1 - \frac{t^{k-2}}{r^{k-2}})u(t\sigma)\,\mathrm{d}\sigma\mathrm{d}t. \end{aligned} \tag{4.36}$$

When

$$\int_{|x|>1} \frac{a(x)}{|x|^{k-2}}\,\mathrm{d}x = \int_1^{+\infty} \int_{\partial B_1} ta(t\sigma)\,\mathrm{d}\sigma\mathrm{d}t = +\infty,$$

then the equation (4.3) cannot have a solution such that

$$0 < c \le u \le C.$$

Indeed from (4.36) we would get

$$U(r) \ge B + \frac{1}{k-2} \int_1^{\frac{r}{2}} \int_{\partial B_1} ta(t\sigma)\,\mathrm{d}\sigma\mathrm{d}t(1 - \frac{1}{2^{k-2}})c \to +\infty$$

which contradicts the fact that u and U are bounded. □

Remark 4.3. Using this result one recovers easily the Lemma 4.1.

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