SELECTED TOPICS IN CLASSICAL AND QUANTUM GEOMETRY

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Herbstsemester 2011

1. Symplectic linear algebra

(1) Symplectic forms and presymplectic forms
(2) Normal form theorem
(3) Weak and strong infinite-dimensional symplectic spaces
(4) The symplectic orthogonal space
(5) Symplectic, isotropic, coisotropic, and Lagrangian subspaces
(6) Linear symplectic reduction
(7) Canonical relations, their composition and the extended linear symplectic category.
(8) Kähler structures

2. Symplectic manifolds

(1) Generalities (definitions, symplectomorphisms, Liouville's volume form)
(2) Lagrangian and Hamiltonian mechanics; the Legendre transformation of hyperregular Lagrangians
(3) Moser's trick
(4) Darboux and Darboux–Weinstein theorems
(5) Classification of compact symplectic surfaces
(6) Presymplectic (sub)manifolds and reduction
(7) Symplectic, isotropic, coisotropic, and Lagrangian submanifolds
(8) Reduction of Lagrangian submanifolds intersecting coisotropic submanifolds
(9) Canonical relations; composition thereof; the extended symplectic "category"
(10) Generating functions and Morse families
(11) Example: (deformed) conormal bundles in cotangent bundles
(12) (Almost) Kähler structures

3. Reduction and symmetry

(1) Algebraic description of reduction

(2) Symplectic and Hamiltonian vector fields
(3) Symplectic actions of Lie groups
(4) Symplectic, Hamiltonian and Poisson actions of Lie algebras; moment maps; obstructions (Digression: Lie algebra cohomology)
(5) Marsden–Weinstein reduction
(6) Noether's Theorem
(7) Infinite-dimensional examples: BF theories, Chern–Simons theory, Maxwell's equations

4. Poisson geometry

(1) Motivations, definitions and examples; Poisson algebras
(2) Digression: The Schouten–Nijenhuis bracket
(3) Poisson cohomology and its interpretation up to degree two
(4) Canonical actions of Lie groups and quotients
(5) Poisson and Hamiltonian vector fields
(6) Coisotropic submanifolds and reduction; algebraic description ("coisotropes"); examples

5. Canonical quantization

(1) Schrödinger's quantization of $T^*\mathbb{R}^n$
(2) Schrödinger's equation; position and momentum operators; the ordering problem
(3) Expectation values; Ehrenfest's Theorem; the "correspondence principle"
(4) Heisenberg's uncertainty principle
(5) Schrödinger's and Heisenberg's pictures
(6) The momentum description
(7) Problems of canonical quantization; "quantization as a functor"; Dirac's dream; the Groenewald–van Howe Theorem

6. Geometric quantization

(1) Prequantization and the integrality condition (Digression: line bundles, connection, curvature, Chern class)
(2) The integral Hall effect
(3) Prequantization and representations of Lie algebras
(4) Prequantization of the Poisson algebra of functions
(5) Polarizations

7. Path-integral quantization

(1) Motivations
(2) From the Schrödinger equation to the path integral
(3) From the path integral to the Schrödinger equation
(4) The semiclassical limit
(5) Perturbation theory
(6) Introduction to field theory, functional integrals, regularization, and renormalization
(7) Degenerate critical points

8. Deformation quantization

(1) Motivations and definitions
(2) The Moyal product
(3) Introduction to the A-model and to the Poisson sigma model
(4) The star-Schrödinger equation

9. Graded linear algebra

(1) Superspaces, graded vector spaces, filtered vector spaces
(2) Morphisms and graded morphisms
(3) Supertrace and superdeterminant
(4) Super, graded and filtered algebras
(5) Graded Lie algebras [GLAs], graded Poisson algebras [GPAs], n-Poisson algebras, Gerstenhaber algebras; examples
(6) The graded symmetric algebra
(7) The associated graded algebra of a filtered algebra
(8) Left and right derivations
(9) The Moyal product on a graded vector space with constant Poisson structure
(10) Differential GLAs [DGLAs] and GPAs [DGPAs]
(11) L_∞-algebras

10. The BRS method (after Kostant and Sternberg)

(1) Koszul resolution
(2) The BRS differential and its cohomology
(3) Strategies for quantization
(4) The Clifford algebra as quantization of the exterior algebra of a vector space with scalar product
(5) The action of the orthogonal group
(6) Creation and annihilation operators

(7) Quantization of a finite-dimensional quadratic Lie algebra and the anomaly
(8) Anomaly-free quantization of a finite-dimensional Lie algebra plus its dual
(9) Modules of the Clifford algebra
(10) The infinite-dimensional case
(11) Deformation quantization description
(12) The BVF method (after Stasheff and Schätz)

11. Graded manifolds

(1) Definitions
(2) Graded vector fields
(3) The graded Euler vector field
(4) Cohomological vector fields, differential graded manifolds, reinterpretation of BRS and BVF methods; geometrical interpretation of L_∞-algebras
(5) Grassmann integration
(6) The Berezinian bundle
(7) The divergence operator
(8) Change of variables and the Berezinian of a transformation
(9) Graded differential forms; the de Rham differential
(10) Integral forms (see next Section)
(11) The GLA of multivector fields
(12) Graded manifolds of maps

12. The BRST method

(1) Integration of invariant functions on principal bundles and the Faddeev–Popov determinant
(2) Reinterpretation as integrals on graded manifolds (ghosts, antighosts, Lagrange multipliers)
(3) BRST cohomology and gauge-fixing-independent integration
(4) Infinite dimensions

13. Graded symplectic geometry

(1) Graded symplectic linear algebra; structure theorems
(2) Graded symplectic manifolds; Darboux coordinates
(3) Main facts on symplectic forms and Hamiltonian vector fields
(4) Normal form of odd symplectic manifolds
(5) Symplectic manifolds of degree -1 and their induced cohomology
(6) Reinterpretation of the Berezinian

(7) Integral forms (after Manin and Ševera); integration on submanifolds

14. The BV method

(1) Symplectic manifolds of degree -1 and the canonical BV operator
(2) BV cohomology and integration on Lagrangian submanifolds
(3) The quantum master equation (QME)
(4) The classical master equation (CME)
(5) Reinterpretation of the BRST method
(6) Application to functional integrals
(7) The BV-BFV method for field theories on manifolds with boundary

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