[MAT121] Analysis I, Probeprüfung Mat 121
Solutions

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Exercise 1

1. **1point** We have for $\lambda_1, \lambda_2 \geq 0$, with $\lambda_1 + \lambda_2 = 1$,
$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2).$$
Then by induction, assume this is true for n x_j, and assume that the $\lambda_j > 0$. Then
$$\sum_{j=1}^{n+1} \lambda_j x_j = \sum_{j=1}^{n} \lambda_j \big(\frac{\sum_{j=1}^n \lambda_j x_j}{\sum_{j=0}^n \lambda_j} + \lambda_{n+1} x_{n+1} = \big(\sum_{j=1}^n \lambda_j \big) x + \lambda_{n+1} x_{n+1}.$$
Then, since $\big(\sum_{j=1}^n \lambda_j \big) + \lambda_{n+1} = 1$ and $\sum_{j=1}^n \lambda_j \geq 0$,
$$f\Big(\sum_{j=1}^{n+1} \lambda_j x_j \Big) \leq \big(\sum_{j=1}^n \lambda_j \big) f\big(\sum_{j=0}^n \mu_j x_j \big) + \lambda_{n+1} f(x_{n+1})$$
with
$$\mu_j = \frac{\lambda_j}{\sum_{j=1}^n \lambda_j}$$
which are positive numbers that sum up to 1. Using the induction, we conclude.

2. (a) **2points** In [0,1], $t \mapsto t^{1/p}$ is montone increating so $0 \leq t^{1/p} \leq 1$ and then $1 - t^{1/p} \geq 0$ for $t \in [0,1]$: the function is defined.

   (b) **2points** f is the composition of $t \mapsto 1 - t^{1/p}$ which is differentiable for $t \in [0, 1[$ and $v \mapsto v^p$ whchi is also differentiable for $v > 00$. As the composition of differntiable functions, f is differentiable. A standard calculation shows that $f'(t) = -p \times \frac{1}{p} t^{1/p-1} \big(1 - t^{1/p} \big)^{p-1} = -t^{1/p-1} \big(1 - t^{1/p} \big)^{p-1}$.

   (c) **2points**To show that f is convex, it is enough to show that f' is increasing. the simplet seems to compute the second derivative that exists with the same arguments. We have:
$$\begin{aligned} f''(t) &= -(\frac{1}{p} - 1) t^{1/p-2} (1 - t^{1/p})^{p-1} + \frac{p-1}{p} t^{2/p-2} (1 - t^{1/p})^{p-2} \\ &= -(1 - t^{1/p})^{p-2} (\frac{1}{p} - 1) \Big(t^{1/p-2} (1 - t^{1/p}) - t^{2/p-2} \Big) \\ &= -(1 - t^{1/p})^{p-2} (\frac{1}{p} - 1) t^{1/p-2} \Big(1 - t^{1/p} + t^{1/p} \Big) \\ &= \big(1 - \frac{1}{p} \big) (1 - t^{1/p})^{p-2} t^{1/p-2} \geq 0 \end{aligned}$$
because $t \in [0, 1]$ and $p \geq 1$. So f is convex.

(d) **2points** We have $f(\sum_{i=1}^n \alpha_i x_i) \leq \sum_{i=1}^n \alpha_i f(x_i)$, i.e after some tiny simplifications:

$$\left(1 - \left(\frac{\sum_{i=1}^n a_i^p}{\sum_{j=1}^n (a_j + b_j)^p}\right)^{1/p}\right)^p \leq \sum_{i=1}^n \frac{b_j^p}{\sum_{j=1}^n (a_j + b_j)^p}$$

so that

$$1 - \left(\frac{\sum_{i=1}^n a_i^p}{\sum_{j=1}^n (a_j + b_j)^p}\right)^{1/p} \leq \left(\sum_{i=1}^n \frac{b_j^p}{\sum_{j=1}^n (a_j + b_j)^p}\right)^{1/p}$$

i.e.

$$\left(\sum_{j=1}^n (a_j + b_j)^p\right)^{1/p} \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}$$

Exercise 2

1. **1 points** φ is obviously continuous as the composition of continuous functions.
2. **3 points** Since $[a, b]$ is compact, it reaches a minimum.
3. **3 points** Assume that $\varphi(c) \neq 0$. Then, $c \neq f(c)$. Hence

$$|f(c) - f(f(c))| < |c - f(c)|$$

   i.e. $\varphi(f(c)) < \varphi(c)$ which is impossible: $c = f(c)$.
4. **3 points** If there are two such solution, $c_1 \neq c_2$, again we get a contradiction.

Exercise 3

1. **4 points**The functions f_1 is differentiable as the sum/difference of differentiable functions. We have

$$f_1'(t) = \tan^2 t - t^2,$$

2. **3 points** We have $f_1'(t) = (\tan t + t)(\tan t - t)$. On $[0, 1]$, $\tan t \geq 0$ so $\tan t + t \geq 0$. The derivative of $\tan t - t$ is $1 + \tan^2 t - 1 \geq 0$, so $\tan t - t \geq 0$ and then $f_1'(t) \geq 0$.
3. **3 points** $f_1(t) \leq f_1(0) = 0$

Exercise 4

1. **2 points** From the definition of the limit: for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|t| < \delta$, then

$$\left|\frac{f(2t) - f(t)}{t} - a\right| \leq \varepsilon$$

   which is precisely what is asked for.
2. **2 points** If $|t| < \delta$, then $|\frac{t}{2^n}| < \delta$ and we first have

$$|f(t) - f(t/2) - a\ \frac{t}{2}| < \varepsilon \frac{t}{2}$$

   and then

$$\left|f(\frac{t}{2^n}) - f(\frac{t}{2^{n+1}}) - a\ \frac{t}{2^{n+1}}\right| < \varepsilon \frac{t}{2^{n+1}}.$$

3. **2 points** From this we get

$$\Big|f(t)-f(\frac{t}{2^{n+1}})-a\,t\,(\sum_{j=1}^{n+1}\frac{1}{2^j})\Big| \le \sum_{j=1}^{n+1}|f(\frac{t}{2^n})-f(\frac{t}{2^{n+1}})-a\,\frac{t}{2^{n+1}}|$$

Since $\sum_{j=1}^{n+1}\frac{1}{2^j}=1-\frac{1}{2^{n+2}}$, we get

$$\Big|f(t)-f(\frac{t}{2^{n+1}})-a\,t\,(1-\frac{1}{2^{n+2}})\Big) \le \varepsilon\,t\,\big(1-\frac{1}{2^{n+2}}\big).$$

4. **2 points** Taking the limit when $n\to\infty$ and using the continuity of f at 0, we get

$$|f(t)-f(0)-a\,t| \le \varepsilon\,t$$

5. **2 points** and then f is differentiable at 0 with derivative a.

Exercise 5

1. **4 points**Setting $u(x)=x-2$ and $v'=e^{-x}$, we have $v=-e^{-x}$ and $u'=1$, so

$$\int_0^1 (x-2)e^{-x}\,dx = \Big[-(x-2)e^{-x}\Big]_0^1 - \int_0^1\Big(-e^{-x}\Big) = -e^{-1}-2+\int_0^1 e^{-x}dx = -2-\frac{1}{e}-\big(e^{-1}-1\big)$$

i.e.

$$\int_0^1 (x-2)e^{-x}\,dx = -1.$$

2. **4 points** $1/16\Big[(4x+1)^4\Big]_0^{1/2}$

3. **4 points**

$$\int_0^1 (e^x+e^{-x})\,dx = \Big[e^x-e^{-x}\Big]_0^1 = e-1-(e^{-1}-1)=e-e^{-1}$$

Exercise 6

1. **5 points** $\sum_{n=1}^{\infty}\left(\frac{5n^2+3n+1}{2n^2+2}-2\right)^n$ Here $u_n=\left(\frac{5n^2+3n+1}{2n^2+2}-2\right)^n$. Then $u_n^{1/n}=\frac{5n^2+3n+1}{2n^2+2}-2$ which tends to $\frac{5}{2}-2=\frac{1}{2}<1$, so the serie is convergent.

2. **5 points** $\sum_{n=1}^{\infty}(-1)^n\frac{n-1}{n^2+n}$. Introducing $f(x)=\frac{x-1}{x^2+x}$, we see that $\frac{n-1}{n^2+n}=f(n)$. Let us check if f is decreasing. we have $f'(x)=-\frac{x^2-2x-1}{(x^2+x)^2}=-\frac{(x-1)^2+1}{x^2(x+1)^2}<0$ so that the sequence is decreasing and the serie is convergent.