[MAT121] Analysis I
Solutions to Extra Exercises

Prof. Dr. Rémi Abgrall

December 22, 2017

(1) Taylor series

Consider the function $f(x) := \log(\frac{1+x}{1-x})$ for $x \in (-1, 1)$.

(a) Determine the Taylor series of f around the point $x_0 = 0$.

(b) Determine the convergence radius ρ of this Taylor series.

(c) Show that for $|x| < \rho$ the Taylor series coincides with the function f, i.e. the series of Taylor polynomials converges to $f(x)$ for every x with $|x| < \rho$.

Solution:

(a)

$$f'(x) = \frac{1-x}{1+x} \cdot \Big(\frac{1+x}{1-x}\Big)' = \frac{1-x}{1+x} \cdot \Big(-1 - \frac{2}{(x-1)}\Big)' = \frac{2}{1-x^2}.$$

We notice that

$$g(x) := \frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}.$$

Then it is easy to calculate all the derivatives:

$$g^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} + \frac{(-1)^n n!}{(1+x)^{n+1}} = f^{(n+1)}(x); \tag{1}$$

in particular, for n odd we have

$$\frac{f^{(n)}(0)}{n!} = \frac{g^{(n-1)}(0)}{n!} = \frac{(n-1)!}{n!} = \frac{1}{n},$$

and zero otherwise. Therefore, the Taylor expansion is

$$2 \cdot \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)}.$$

(b) We just need to calculate

$$\lim_{k\to\infty} \frac{2k+1}{2k+3} = 1.$$

So $|x^2| < 1$ and it is equivalent to $|x| < 1$. Therefore, the radius of convergence ρ is 1.

(c) Here we need to prove that the remainder term $R_k(x) := f(x) - p_k(x)$ goes to zero as $k \to \infty$. Let $x > 0$. We can use the Cauchy form of the remainder term:

$$R_k(x) := \frac{f^{(k+1)}(\xi)}{k!}(x-\xi)^k \cdot x,$$

where $0 < \xi < x$ is some number. In our case using (1) we obtain

$$R_k(x) = \frac{g^{(k)}(\xi)}{k!}(x-\xi)^k \cdot x = \Big(\frac{(x-\xi)^k x}{(1-\xi)^{k+1}} + \frac{(x-\xi)^k x}{(1+\xi)^{k+1}}\Big).$$

Observe that $\frac{x-\xi}{1+\xi} < 1$ as well as $\frac{x-\xi}{1-\xi} < 1$. Therefore, we have the sum of expressions of the form

$$\alpha^k \cdot const,$$

which goes to zero as $k \to \infty$, since $\alpha < 1$.
The case $x < 0$ can be done similarly.

(2) Riemann sums

Use the definition of the integral, i.e. Riemann sums, to determine the following integrals

(a) $\int_a^b e^{2t} dt$ for $a, b \in \mathbb{R}$ and $a < b$.

(b) $\int_a^b \frac{1}{t^2} dt$ for $a, b \in \mathbb{R}$ and $0 < a < b$.

It can be assumed that the functions are integrable and you may use Proposition 9.1.4.

Solution:

(a) So we use Proposition 9.1.4 and for each n choose a particulary nice division of the interval $[a; b]$ into the intervals of the form $[a + \frac{k}{h}; a + (k+1)h]$, where $h := \frac{b-a}{n}$ and $k = 0 \ldots n-1$. Then the lower Riemann sum is

$$\sum_{k=0}^{n-1} e^{2(a+kh)} h = e^{2a} \frac{e^{2(b-a)} - 1}{e^{2h} - 1} h = \frac{e^{2b} - e^{2a}}{2} \frac{2h}{e^{2h} - 1}.$$

Now we observe that $\lim_{h \to 0} \frac{2h}{e^{2h}-1} = 1$, hence $\int\limits_a^b e^{2t} \mathrm{d}t = \frac{e^{2b} - e^{2a}}{2}$.

(b) We put $\rho := \sqrt[n]{b/a}$ and take the division into the intervals of the form $Q_k := [t_{k-1}; t_k]$, where $t_k := a \cdot \rho^k$ and $k = 1 \ldots n$. For each k, the length of Q_k is $a\rho^{k-1}(\rho - 1)$. Take the upper Riemann sum:

$$\bar{S}_n = \sum_{k=1}^{n} \frac{1}{a^2 \rho^{2k-2}} a\rho^{k-1}(\rho - 1) = \frac{\rho - 1}{a} \sum_{k=0}^{n-1} \rho^{-k} = \frac{1}{a} \frac{(\rho - 1)(1 - \rho^{1-n})}{1 - \rho^{-1}} = \frac{1}{a}\rho(1 - \rho^{1-n}).$$

But $\rho^{-n} = a/b$; also $\lim_{n \to \infty} \rho = 1$, hence

$$\lim_{n \to \infty} \bar{S}_n = \frac{1}{a}\Big(1 - \frac{a}{b}\Big) = \frac{1}{a} - \frac{1}{b},$$

as one can check using primitive integral formulae.

(3) Partial fraction decomposition

Let $a, b \in \mathbb{R}$ and $2 < a < b$. Determine the following integrals

(a)

$$\int_a^b \frac{7t+1}{t^2+t-6} dt.$$

Hint: Determine A, B such that $\frac{7t+1}{t^2+t-6} = \frac{A}{t-2} + \frac{B}{t+3}$.

(b)

$$\int_a^b \frac{-t^2+5t-2}{t(t+1)(t-1)} dt.$$

Hint: Use a similar decomposition as in **(a)**.

Solution:

(a) We can decompose $(t^2+t-6) = (t-2)(t+3)$. Therefore $\frac{7t+1}{t^2+t-6} = \frac{A}{t-2} + \frac{B}{t+3}$ implies $A(t+3) + B(t-2) = 7t+1$, i.e. $A+B = 7$ and $3A - 2B = 1$. This system of equations has a unique solution being $A = 3$ and $B = 4$. Hence $\frac{7t+1}{t^2+t-6} = \frac{3}{t-2} + \frac{4}{t+3}$, i.e.

$$\begin{aligned}\int_a^b \frac{7t+1}{t^2+t-6} dt &= \int_a^b \frac{3}{t-2} dt + \int_a^b \frac{4}{t+3} dt = [3\log(t-2) + 4\log(t+3)]_a^b \\ &= 3\log(b-2) + 4\log(b+3) - 3\log(a-2) - 4\log(a+3)\end{aligned}$$

(b) Again we want to decompose $\frac{-t^2+5t-2}{t(t+1)(t-1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{t-1}$. We get $A(t^2-1) + B(t^2-t) + C(t^2+t) = -t^2 + 5t - 2$, i.e. $A = 2$. The yields $B + C = -3$ and $C - B = 5$, which is solved by $B = -4$ and $C = 1$.

Altogether

$$\frac{-t^2+5t-2}{t(t+1)(t-1)} = \frac{2}{t} + \frac{-4}{t+1} + \frac{1}{t-1}$$

and thus

$$\int_a^b \frac{7t+1}{t^2+t-6} dt = \int_a^b \frac{2}{t} dt + \int_a^b \frac{-4}{t+1} dt + \int_a^b \frac{1}{t-1} dt = [2\log(t) - 4\log(t+1) + \log(t-1)]_a^b$$

Remark: In general a broken rational function can be decomposed into a sum of terms of the from $\frac{a_i}{(t-t_i)^j}$ with t_i the zero's of the denominator. This is called *partial fraction decomposition.*

(4) Monotone functions

Show that a monotone function $f : [a, b] \to \mathbb{R}$ is integrable.

Solution: We only show the case for a monotone increasing function, since the case for a monotone decreasing is very similar.

Take an equidistant partition T_n of the interval $[a, b]$ in n intervals, i.e. let $t_i = a + i \cdot \frac{b-a}{n}$, $i = 0, 1, \ldots, n$. Then by monotonicity it follows that $f(t_i) \le f(x) \le f(t_{i+1})$ for $x \in (t_i, t_{i+1})$. Hence

$$\underline{S}(T_n) = \sum_{i=1}^n f(t_{i-1})(t_i - t_{i-1}) = \sum_{i=1}^n f(t_{i-1}) \frac{b-a}{n}$$

and in the same way

$$\overline{S}(T_n) = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1}) = \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

Which yields

$$c_n := \overline{S}(T_n) - \underline{S}(T_n) = (\sum_{i=1}^{n} f(t_i) - \sum_{i=1}^{n} f(t_{i-1}))\frac{b-a}{n} = (f(b) - f(a))\frac{b-a}{n}$$

Clearly $\lim_{n\to\infty} c_n = 0$ and therefore by Proposition 9.1.3. f is integrable.