Exercise Sheet 13 - Solution

Exercise 1

a) For each x_1 and $x_2 \in D(f)$ it must hold that $x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$ and for each $y \in W(f)$ there is a x sucht that $y = f(x)$.

b) The graph of $f^{-1}(x)$ can be constructed from the graph of $f(x)$ by reflection in the line $y = x$.

Exercise 2 (6 points)

a) 1) (1 point) The derivative of the function y is $y' = \frac{3}{2}(x+2)^2$. This is an increasing function in $x \in \mathbb{R}$, that is, for each x there is exactly one value $y = f(x)$. See also graphical representation.

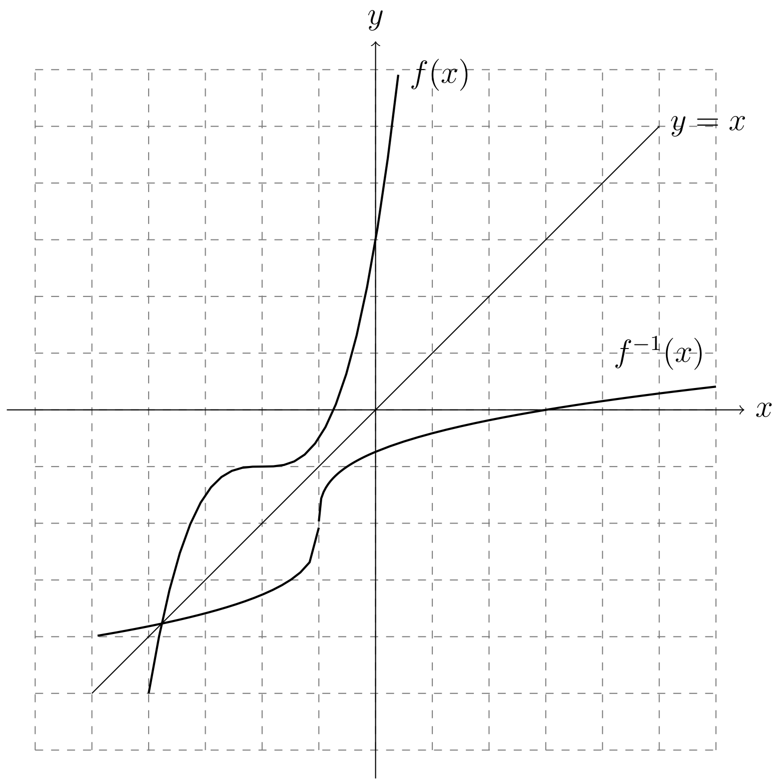
2) (1 point) We solve for x and afterwards exchange x and y.

$$\begin{aligned}
y &= \frac{1}{2}(x+2)^3 - 1 && \Big| +1 \\
y + 1 &= \frac{1}{2}(x+2)^3 && \Big| \cdot 2 \\
2y + 2 &= (x+2)^3 && \Big| \sqrt[3]{\cdots} \\
\sqrt[3]{2y+2} &= x + 2 && \Big| -2 \\
\sqrt[3]{2y+2} - 2 &= x && \Big| x \leftrightarrows y \\
y &= \sqrt[3]{2x+2} - 2 && \Big|
\end{aligned}$$

3) (1 point) Reflection of the graph of f in the line $y = x$ changes the domain of f into the image of f^{-1}. Similarly, the image of f becomes the domain of f^{-1}.

$$\begin{array}{llcll}
D(f): & -\infty < x < \infty & \Rightarrow & W(f^{-1}) = D(f): & -\infty < y < \infty \\
W(f): & -\infty < y < \infty & \Rightarrow & D(f^{-1}) = W(f): & -\infty < x < \infty
\end{array}$$

4) (1 point) Graphical representation:



b) (2 points)
We choose $f(x) = e^x$ and $g(x) = \ln(x)$.
Then $g(x)$ is the inverse function of $f(x)$. (1 point)
From this we get:

$$g'(x) = \big(\ln(x)\big)' = \frac{1}{f'(g(x))} = \frac{1}{e^{g(x)}} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$ (1 point)

Exercise 3 (5 points)

a) (1 point) In order to compute the derivative of $\arctan\left(\sqrt{\frac{x-1}{x+1}}\right)$ we need

$$\Big(\arctan(x)\Big)' = \frac{1}{1+x^2} \quad \text{and} \quad \Big(\sqrt{x}\Big)' = \frac{1}{2\sqrt{x}} \qquad \text{(1 point)}$$

Applying the chain rule yields

$$\begin{aligned}
\arctan\left(\sqrt{\frac{x-1}{x+1}}\right)' &= \frac{1}{1+\left(\sqrt{\frac{x-1}{x+1}}\right)^2} \cdot \left(\sqrt{\frac{x-1}{x+1}}\right)' \\
&= \frac{1}{1+\frac{x-1}{x+1}} \cdot \frac{1}{2\sqrt{\frac{x-1}{x+1}}} \cdot \left(\frac{x-1}{x+1}\right)' \\
&= \frac{x+1}{x+1+x-1} \cdot \frac{1}{2} \cdot \sqrt{\frac{x+1}{x-1}} \cdot \frac{x+1-x+1}{(x+1)^2} \\
&= \frac{x+1}{2x} \cdot \frac{1}{\cancel{2}} \cdot \sqrt{\frac{x+1}{x-1}} \cdot \frac{\cancel{2}}{(x+1)^2} = \frac{x+1}{2x} \cdot \sqrt{\frac{x+1}{x-1}} \cdot \frac{1}{(x+1)^2} \\
&= \frac{\cancel{x+1}}{2x} \cdot \sqrt{\frac{x+1}{x-1}} \cdot \frac{1}{(x+1)^{\cancel{2}}} = \frac{1}{2x} \cdot \sqrt{\frac{x+1}{x-1}} \cdot \frac{1}{x+1} \\
&= \frac{1}{2x} \cdot \frac{\cancel{\sqrt{x+1}}}{\sqrt{x-1}} \cdot \frac{1}{\cancel{\sqrt{x+1}} \cdot \sqrt{x+1}} = \frac{1}{2x} \cdot \frac{1}{\sqrt{x-1}} \cdot \frac{1}{\sqrt{x+1}} \\
&= \frac{1}{2x} \cdot \frac{1}{\sqrt{x^2-1}} \qquad \text{(1 point)}
\end{aligned}$$

b) (2 points) Rearranging $f(x)$ yields

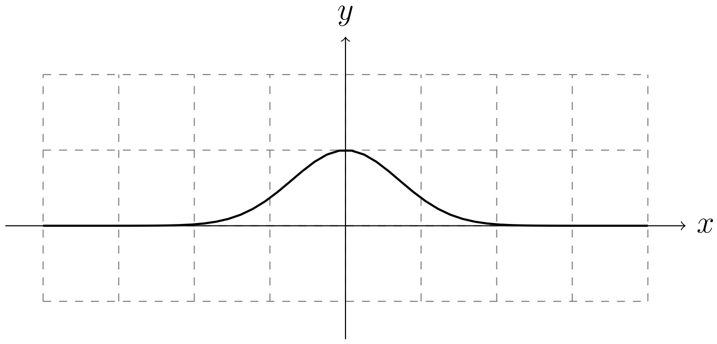
$$\begin{aligned}
f(x) &= \frac{1}{4x^2+16x+17} = \frac{1}{4x^2+16x+16+1} == \frac{1}{(2x+4)^2+1} \\
&= \frac{1}{1+(2x+4)^2} \qquad \text{(1 point)}
\end{aligned}$$

With $u(x) = 2x+4 \quad \Rightarrow \quad \frac{du}{dx} = 2 \quad \Rightarrow \quad dx = \frac{du}{2}$ we get:

$$\begin{aligned}
\int f(x)\,dx &= \int \frac{1}{1+(2x+4)^2}\,dx = \int \frac{1}{1+u^2}\frac{du}{2} = \frac{1}{2} \cdot \int \frac{1}{1+u^2}\,du \\
&= \frac{1}{2} \cdot \arctan(u) + C = \frac{1}{2} \cdot \arctan(2x+4) + C \qquad \text{(1 point)}
\end{aligned}$$

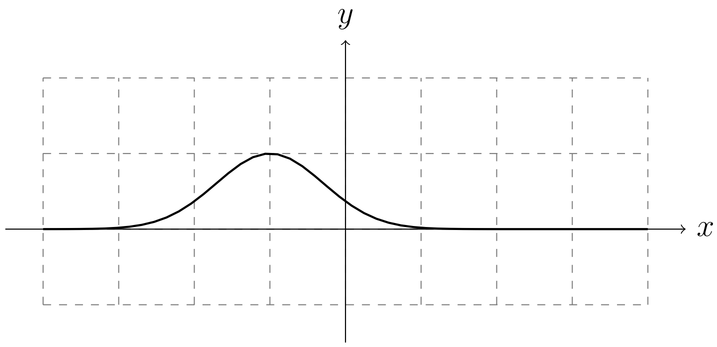
Exercise 4 (4 points)

a) (1 point) The argument $-x^2$ is always negative, so the function $y = e^{-x^2}$ only takes on values between 0 and 1! At $x = 0$ the function takes on the maximal value 1. As x increases the function value decreases. It holds $\lim_{x\to\pm\infty} \left(e^{-x^2}\right) = 0$.

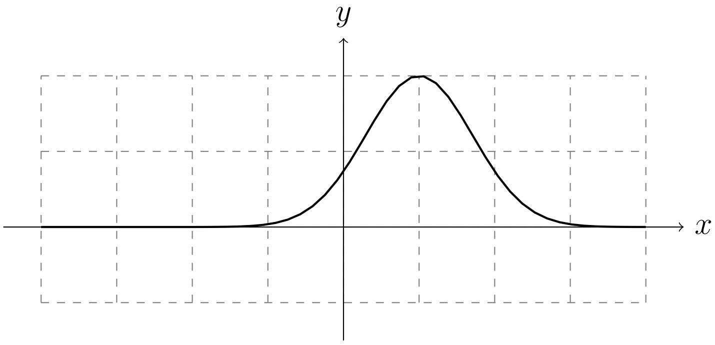


b) (1 point) We set $f(x) = e^{-x^2}$ and $g(x) = x \cdot e^{-x^2}$. It is not possible to construct $g(x)$ from $f(x)$ due to the factor x.

c) (1 point) We set $f(x) = e^{-x^2}$ and $g(x) = e^{-(x+1)^2}$.
It holds $g(x) = f(x - (-1))$. So the function $f(x)$ is shifted by one unit to the left.



d) (1 point) We set $f(x) = e^{-x^2}$ and $g(x) = 2 \cdot e^{-(x-1)^2}$.
It holds $g(x) = 2 \cdot f(x - 1)$. So the function $f(x)$ is shifted by one unit to the right and the function values are stretched by a factor of 2.



Exercise 5 (2 points)

a) (1 point) In order to compute the sought integral we apply the rule $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$ where $|x| < 1$. ($\frac{1}{2}$ point)

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{dx}{\sqrt{4(1-\frac{1}{4}x^2)}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-(\frac{x}{2})^2}}$$

We substitute:

$$u(x) = \frac{x}{2} \qquad \Rightarrow \qquad \frac{du}{dx} = \frac{1}{2} \qquad \Rightarrow \qquad dx = 2du$$

$$\frac{1}{2}\int \frac{dx}{\sqrt{1-(\frac{x}{2})^2}} = \frac{1}{2}\int \frac{2du}{\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}}$$

$$= \arcsin(u) + C = \arcsin\left(\frac{x}{2}\right) + C$$ ($\frac{1}{2}$ point)

With $|\frac{x}{2}| < 1$ we get $|x| < 2$.

b) (1 point) In order to compute the sought integral we apply the rule $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|+C$. ($\frac{1}{2}$ point)

$$\int \frac{dx}{\sqrt{x^2-4}} = \int \frac{dx}{\sqrt{x^2-2^2}} = \ln\left|x+\sqrt{x^2-4}\right| + C$$ ($\frac{1}{2}$ point)

where $|x| > 2$.