[MAT121] Analysis I
Homework 12

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Deadline: Friday, 15.12.2017, 12:00
Remember to write your name and your assistant's name

(1) [10p] Chain and product rules

Let $f, g : \mathbb{R} \to \mathbb{R}$ be k-times differentiable, $0 < k \in \mathbb{N}$. Then prove that both fg and $f \circ g$ are k-times differentiable.

(You may use the results of **Satz 8.1.3** and **Proposition 8.1.5**)

(2) [10p] Convex functions

Let $f :]a, \infty[\longrightarrow \mathbb{R}$, $a \in \mathbb{R}$, be *convex* (see §**8.5** of the Script for a definition of convexity). Show that

$$\lim_{x\to\infty} \frac{f(x)}{x} = C \in \mathbb{R} \cup \{\infty\} \ .$$

In addition, show that

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} f'(x^+) = \lim_{x\to\infty} f'(x^-) \ .$$

Recall that $f'(x^+) = \lim_{h\downarrow 0} \frac{f(x+h)-f(x)}{h}$ and $f'(x^-) = \lim_{h\downarrow 0} \frac{f(x)-f(x-h)}{h}$ are the derivatives from the right and from the left, respectively.

(3) [10p] Finite differences and derivatives

Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, be a function differentiable infinitely many times. Prove that

$$\lim_{h\to 0} \delta_h\big(f(x)\big) := \lim_{h\downarrow 0} \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h} = f'(x) \qquad (\forall x \in D) \ .$$

Also, prove that $\delta_h\big(f(\cdot)\big)$ approximates $f'(\cdot)$ better, as h becomes small, than $\frac{f(\,\cdot\,+h)-f(\cdot)}{h}$ and $\frac{f(\cdot)-f(\,\cdot\,-h)}{h}$:

$$\lim_{h\to 0} \frac{\frac{f(x+h)-f(x)}{h} - f'(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x)-f(x-h)}{h} - f'(x)}{h} = \frac{f''(x)}{2} \qquad (\forall x \in D) \ ,$$

$$\lim_{h\to 0} \frac{\delta_h\big(f(x)\big) - f'(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x+\frac{h}{2})-f(x-\frac{h}{2})}{h} - f'(x)}{h} = 0 \qquad (\forall x \in D) \ ,$$

$$\lim_{h\to 0} \frac{\delta_h\big(f(x)\big) - f'(x)}{h^2} = \frac{f'''(x)}{24} \qquad (\forall x \in D) \ .$$

Using the $O(\cdot)$ notation for the order of convergence, the above could be rewritten as

$$\delta_h(f(\cdot)) = f'(\cdot) + O(h^2) \, .$$

(*Hint.* Taylor expansions may be useful)

(4) [20p] Functions' graphs

Characterize the graph $(\,\cdot\,, f(\cdot)), (\,\cdot\,, g(\cdot)) \subset \mathbb{R}^2$ of the following functions

$$f(x) = \frac{x}{x^2+1} \quad [\mathbf{5p}], \qquad\qquad g(x) = |x|^x \quad [\mathbf{15p}].$$

You should study the behavior of the functions close to possibly ill-defined points (*e.g.*, $0, \pm\infty$), and find the critical points, if any, and decide whether they are local maxima or minima or none of the two. Represent the relevant features of the graphs as a curve on the Cartesian plane. Is g differentiable in zero?

(5) [10p] Limits with Taylor expansions

Let f, g, h be infinitely differentiable functions, and denote by $f^{(k)}, g^{(k)}, h^{(k)}$ their k-th derivatives, $k \in \mathbb{N}$ (with the convention $f^{(0)} = f, g^{(0)} = g, h^{(0)} = h$). In addition, suppose that h is different from zero in all points of some neighborhood of zero, not including the latter. Determine *all* values of $f^{(k)}(0), g^{(k)}, h^{(k)}(0)$, $k \in \mathbb{N}$, for which the following limit is either zero or infinity:

$$\lim_{x \downarrow 0} \frac{f(x) - g(2x)}{h(3x)} \, .$$