[MAT121] Analysis I, Probeprüfung Mat 121

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The justification of arguments used in proofs is important.

Exercice 1
Study the sequence defined by

$$x_0 = 0, \qquad x_{n+1} = \frac{1}{1+x_n}.$$

Does it converge, if so, towards which limit ?

To do so:

1. Show that f is monotone decreasing,
2. Show that for any n, $0 \leq x_n \leq 1$,
3. Identify the potential limit l, why can it be a limit
4. Show that
$$\left|x_{n+1} - l\right| = \frac{|x_n - l|}{(l+1)(x_n+1)}$$
and prove the convergence.

Exercice 2
Let $(u_n)_{n\in\mathbb{N}}$ a sequence that converges towards l. Consider the sequence

$$v_n = \frac{u_0 + u_1 + \ldots + u_n}{n}.$$

Show that v_n converges towards l

Give an example of a sequence for which v_n converges but u_n does not converge

Exercise 3
Show that

$$\cosh(a+b) = \cosh a \cosh b + \sinh a + \sinh b, \qquad \sinh(a+b) = \cosh a \sinh b + \cosh b \sinh a$$

$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b}$$

Exercise 4
Show that the derivative of arcsh is

$$\frac{1}{\sqrt{1+t^2}}.$$

Justify why this function is a differentiable using arguments developed in the lectures.

Exercise 5

Consider φ defined by:

$$\varphi(t) = \begin{cases} \exp\left(-\frac{1}{(1+t)^2} - \frac{1}{(1-t)^2} \right) & \text{if } x \in]-1, 1[\\ 0 & \text{else.} \end{cases}$$

1. Show that φ is differentiable on $\mathbb{R}$, compute its derivative and show that it is continuous (hint. The values $t = \pm 1$ play a special role)

2. Show that the m-th derivative $\varphi^{(m)}$ exist and is continuous over $\mathbb{R}$. To to so, show by induction that

$$\varphi^{(m)}(t) = P_m(t)$$

Exercise 6

1. Show that the logarithm is a concave function (compute its second derivative).

2. Using the concavity of the logarithm, show that for any $u, v > 0$, $p, q \geq 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

that

$$u^{1/p} v^{1/q} \leq \frac{u}{p} + \frac{v}{q}.$$

Solution

Exercice 1

1. f is decreasing because $x \mapsto 1/x$ is decreasing
2. By induction. Since $x_0 = 0$, this is true for $n = 0$. Assume it is true for n: $0 \leq x_n \leq 1$. f is decreasing so $f(x_n) \leq f(0) = 1$ and $f(x_n) \geq f(1) = 1//2 \leq 1$ so that
$$0 \leq x_{n+1} \leq 1.$$
3. f is continuous so if the sequence has a limit, $l = \lim x_n = \lim f(x_n) = f(\lim x_n) = f(l)$. So
$$l = \frac{1}{1+l}$$
and we know that $l \in [0,1]$. The only possibility is $l = \dfrac{-1+\sqrt{5}}{2}$ because l is a root of $l(l+1) = 1$.
4. The we compute $x_{n+1} - l$ and get
$$x_{n+1} - l = \frac{l - x_n}{(1+l)(1+x_n)}.$$
Since $0 \leq x_n \leq 1$, we have
$$|x_{n+1} - l| \leq \frac{1}{1+l}|x_n - l|$$
and since $l + 1 = \dfrac{1+\sqrt{5}}{2} > 1$, this shows that
$$x_{n+1} - l| \leq k^n |x_0 - l| = k^n l$$
with $k = \dfrac{2}{1+\sqrt{5}} < 1$, and the sequence converges.

Exercice 2

Let $\varepsilon > 0$. There exists N such that if $n > N$, $|u_n - l| < \varepsilon/2$.

We have $v_n - l = \frac{1}{n}\sum_{i=1}^{n}(u_i - l)$, so that
$$|v_n - l| \leq \frac{1}{n}\sum_{i=1}^{n}|u_i - l|.$$

Since u_n converges, u_n is bounded, say by M. We can write
$$|v_n - l| \leq \frac{1}{n}\sum_{i=1}^{N-1}|u_i - l| + \frac{1}{n}\sum_{i=N}^{n}|u_i - l| \leq \frac{NM}{n} + \frac{1}{n}\sum_{i=N}^{n}|u_i - l|$$

If $n > \frac{2}{\varepsilon}(NM)$, then
$$\frac{1}{n}\sum_{i=1}^{N-1}|u_i - l| \leq \varepsilon/2$$

Remember that N depends on ε only, so if we take $n > \max(\frac{2}{\varepsilon}(NM), N)$, we have

$$|v_n - l| \leq \frac{\varepsilon}{2} + \frac{1}{n}\sum_{i=N}^{n} |u_i - l| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

because

$$\frac{1}{n}\sum_{i=N}^{n} |u_i - l| \leq \frac{n-N}{n}\frac{\varepsilon}{2} \leq \varepsilon/2.$$

A counter example is $u_n = (-1)^n$. We have $v_{2n} = 0$ and $v_{2n+1} = -\frac{1}{2n+1}$ so v_n tends to 0 but u_n has no limit.

Exercice 3
We have

$$\sinh(a+b) = \frac{e^{a+b} - e^{-a-b}}{2} = \frac{e^a e^b - e^{-a}e^{-b}}{2} = e^a \sinh b + e^{-b}\sinh a = e^b \sinh a + e^{-a}\sinh b,$$

so

$$\sinh(a+b) = \frac{e^a + e^{-a}}{2}\sinh b + \frac{e^b + e^{-b}}{2}\sinh a = \cosh a \sinh b + \cosh b \sinh a.$$

For $\cosh(a+b)$ we do the same.

For

$$\tanh(a+b) = \frac{sh(a+b)}{\cosh(a+b)} = \frac{\cosh a \sinh b + \cosh b \sinh a}{\cosh a \cosh b + \sinh a \sinh b} = \frac{\frac{\cosh a \sinh b + \cosh b \sinh a}{\cosh a \cosh b}}{\frac{\cosh a \cosh b + \sinh a \sinh b}{\cosh a \cosh b}} = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b}.$$

Exercice 4
The sinh is defined from $\mathbb{R}$ to $\mathbb{R}$ and is monotone increasing:

$$\sinh'(x) = \frac{e^x + e^{-x}}{2} = \cosh(x) \geq 0$$

from the lectures (chapter on derivatives). So the inverse function exists and is differentiable 9from the lectures). To compute the derivative, we can either do it by writhing that

$$\operatorname{arcsh}(x) = \ln(x + \sqrt{x^2+1}),$$

then

$$\operatorname{arcsh}'(x) = \frac{1}{x + \sqrt{x^2+1}}(1 + \frac{2x}{2\sqrt{x^2+1}}) = \frac{1}{\sqrt{x^2+1}}$$

or we use the result of the lecture:

$$\operatorname{arcsh}'(x) = \frac{1}{\sinh'(\operatorname{arcsh}(x))} = \frac{1}{\cosh(\operatorname{arcsh}(x))} = \frac{2}{e^{\operatorname{arcsh}(x)} + e^{-\operatorname{arcsh}(x)}} = \frac{2}{x + \sqrt{x^2+1} + \frac{1}{x+\sqrt{x^2+1}}}$$

$$= \frac{2(x + \sqrt{x^2+1}}{(x+\sqrt{x^2+1})^2} = \frac{1}{\sqrt{x^2+1}}$$

Exercice 5
In order to simplify the notations, we write $\varphi(x) = e^{Q(x)}$ with

$$Q(x) = -\frac{1}{(1-t)^2} - \frac{1}{(1+t)^2}.$$

We have in $]0,1[$ that φ has a derivative and

$$\varphi'(x) = Q'(x)e^{Q(x)} = Q'\varphi(x)$$

If $x \notin [-1,1]$, we have also $\varphi'(x) = 0$. To show that φ is differentiable on $x = \pm$, we need to show that $\lim_{x\to\pm 1} Q'(x)e^{Q(x)} = 0$ and using the theorem 8.1.4, we can conclude.

For $x = 1$, we set $X = 1/(x-1)$ and see that

$$Q'(x)e^{Q(x)} = (h(X) - X^3)e^{-p(X)-X^2}$$

with h and p two functions that have a limit when $X \to -\infty$ (since $x < 1$). We have seen in the lecture that $\lim_{X\to-\infty} X^\alpha e^X = 0$ for any $\alpha \in \mathbb{R}$ so

$$\lim_{X\to-\infty} (h(X) - X^3)e^{-p(X)-X^2} = 0$$

The same argument is used for $x = -1$, and then φ' is defined and continuous for any $x \in \mathbb{R}$

By induction we see that in $]-1,1[$,

$$\varphi^{(m+1})(x) = Q_{m+1}(x)\varphi(x)$$

with

$$Q_{m+1} = Q'_m + Q_m\left(-\frac{1}{(1-t)^2} - \frac{1}{(1+t)^2}\right)$$

outside of $[-1,1]$, $\varphi^{(m)} = 0$. Using exactly the same arguments we see that all the derivatives are continuous accross $x = \pm 1$

Exercice 6

- We know that $\ln'(x) = \frac{1}{x}$ for $x > 0$. This function is differentiable, so that $\ln''(x) = \left(\frac{1}{x}\right)' = -\frac{1}{x^2} < 0$, so the logarithm is a concave function.
- We have $\ln(u^{1/p}v^{1/q}) = \frac{1}{p}\ln u + \frac{1}{q}\ln v \leq \ln\left(\frac{1}{p}u + \frac{1}{q}v\right)$. Taking the exponential which is monotone increasing, we get:
$$u^{1/p}v^{1/q} \leq \frac{u}{p} + \frac{v}{q}.$$